Korean J. Math. **23** (2015), No. 1, pp. 1–10
http://dx.doi.org/10.11568/kjm.2015.23.1.1

KOLMOGOROV DISTANCE FOR MULTIVARIATE NORMAL APPROXIMATION

Yoon Tae Kim and Hyun Suk Park*

Abstract. This paper concerns the rate of convergence in the multidimensional normal approximation of functional of Gaussian fields. The aim of the present work is to derive explicit upper bounds of the *Kolmogorov distance* for the rate of convergence instead of *Wasserstein distance* studied by Nourdin *et al.* [*Ann. Inst. H. Poincaré(B) Probab.Statist.* 46(1) (2010) 45-98].

1. Introduction

Let Z be a standard Gaussian random variable on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that $\{F_n\}$ is a sequence of real-valued random variables of an infinite-dimensional Gaussian field. In the paper [6] and [7], authors combine Stein's method and Malliavin calculus to derive explicit upper bounds for quantities of the type

$$|\mathbb{E}[h(F_n)] - \mathbb{E}[h(Z)]|, \tag{1}$$

Received October 14, 2014. Revised January 14, 2015. Accepted January 19, 2015.

2010 Mathematics Subject Classification: 60H07.

Key words and phrases: Malliavin calculus, Kolmogorov distance, Stein's method, multidimensional normal approximation, Wasserstein distance, fractional Brownian motion.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A4A01012783 and NRF-2013R1A1A2008478).

*Corresponding author.

where h is a suitable test function. In the paper [9], authors extend the results of [6]and [7] to the multidimensional normal approximation of functional of Gaussian fields in *Wasserstein distance*.

For a given function $f : \mathbb{R} \to \mathbb{R}$, the *Stein equation* associated with f is defined by

$$f(x) - \mathbb{E}[f(Z)] = h'(x) - xh(x) \text{ for all } x \in \mathbb{R}. \tag{2}$$

A solution to the equation (2) is a function h such that h is Lebesgue-almost everywhere differentiable and there exists a version of h' satisfying (2). If $h \in Lip(1)$, where $Lip(1)$ is the collection of all functions with Lipschitz constant bounded by 1, then the equation (2) has a soluiton h such that $\|h'\|_\infty \leq 1$ and $\|h''\|_\infty \leq 2$. Recall that *Wasserstein distance* between the laws of two real-valued random variables X and Y is defined by

$$d_W(X, Y) = \sup_{h \in Lip(1)} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|.$$

In the paper [9], authors obtain explicit upper bounds of d_W in the case when Z is a d-dimensional Gaussian vector, $F = (F^{(1)}, \ldots, F^{(d)})$ of smooth functionals of Gaussian fields, and d_W is *Wasserstein distance* probability law on $\mathbb{R}^d$.

In this paper, we consider the case when the test function h is non-smooth such as the indicator functions of Borel-measurable convex sets. The test function of the *Kolmogorov distance* is such a class. This distance is defined by

$$d_{Kol}(X, Y) = \sup_{\{h = \mathbf{1}_{(-\infty, z]} : z \in \mathbb{R}^d\}} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|.$$

For the proof of *quantitative Breuer-Major theorems* in [8], the upper bound of the *Kolmogorov distance* is obtained by using the relation (see Theorem 3.1 in [3])

$$d_{Kol}(X, Y) \leq 2\sqrt{d_W(X, Y)}. \tag{3}$$

In this paper, by using the smoothing inequality, we directly derive an explicit upper bound of the *Kolmogorov distance* for a sequence $\{F_n = (F_n^{(1)}, \ldots, F_n^{(d)}), n \geq 1\}$ As an application, we find an explicit upper bound of the *Kolmogorov distance* in the *Breuer-Major central limit theorem* for fractional Brownian motion. (For the *Wasserstein distance*, see Theorem 4.1 in [9]). We stress that our upper bound is more efficient than the upper bound obtained by the relationship (3) as our upper bound converges to zero more fast.

2. Preliminaries

In this section, we recall some basic facts about Malliavin calculus for Gaussian processes. The reader is referred to [10] for a more detailed explanation. Suppose that $\mathcal{H}$ is a real separable Hilbert space with scalar product denoted by $< \cdot, \cdot >_{\mathcal{H}}$. Let $B = \{B(h), h \in \mathcal{H}\}$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $\mathbb{E}[B(h)B(g)] = \langle h, g\rangle_{\mathcal{H}}$.

Let $\mathcal{S}$ be the class of smooth and cylindrical random variables F of the form

$$F = f(B(\varphi_1), \cdots, B(\varphi_n)), \tag{4}$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$, $i = 1, \cdots, n$. The Malliavin derivative of F with respect to B is the element of $L^2(\Omega, \mathcal{H})$ defined by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\varphi_1), \cdots, B(\varphi_n))\varphi_i, \tag{5}$$

We denote by $\mathbb{D}^{l,p}$ the closure of its associated smooth random variable class with respect to the norm

$$\|F\|_{l,p}^p = \mathbb{E}(|F|^p) + \sum_{k=1}^{l} \mathbb{E}(\|D^k F\|_{\mathcal{H}^{\otimes k}}^p).$$

We denote by δ the adjoint of the operator D, also called the *divergence operator*. The domain of δ, denoted by $\mathrm{Dom}(\delta)$, is an element $u \in \mathbb{L}^2(\Omega; \mathcal{H})$ such that

$$|\mathbb{E}(< DF, u >_{\mathcal{H}})| \leq C(\mathbb{E}|F|^2)^{1/2} \text{ for all } F \in \mathbb{D}^{l,2}.$$

If $u \in \mathrm{Dom}(\delta)$, then $\delta(u)$ is the element of $L^2(\Omega)$ defined by the duality relationship

$$E[F\delta(u)] = E[\langle DF, u\rangle_{\mathcal{H}}] \text{ for every } F \in \mathbb{D}^{1,2}.$$

Let $F \in L^2(\Omega)$ be a square integrable random variable. The operator L is defined through the projection operator J_n, $n = 0, 1, 2 \ldots$, as $L = \sum_{n=0}^{\infty} -nJ_nF$, and is called the *infinitesimal generator of the Ornstein-Uhlhenbeck semigroup*. The relationship between the operator D, δ, and L is given as follows: $\delta DF = -LF$, that is, for $F \in L^2(\Omega)$ the statement $F \in \mathrm{Dom}(L)$ is equivalent to $F \in \mathrm{Dom}(\delta D)$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}(\delta)$), and in this case $\delta DF = -LF$. We also define the operator L^{-1}, which is the *pseudo-inverse* of L, as $L^{-1}F = \sum_{n=1}^{\infty} \frac{1}{n} J_n(F)$. Note that L^{-1} is an operator with values in $\mathbb{D}^{2,2}$ and $LL^{-1}F = F - E[F]$ for all $F \in L^2(\Omega)$.

3. Main results

In this section, we derive an explicit upper bound of the *Kolmogorov distance* for normal approximation. We begin by the following simple lemma.

LEMMA 3.1. *Let*

$$f(t) = a \log\left(\frac{1}{\sqrt{1-e^{-2t}}}\right) + b\sqrt{1-e^{-2t}} \ \textit{for}\ t > 0,$$

where a and b are positive constants such that $a < b$. Then the minimum with respect to t is attained for

$$t = -\frac{1}{2}\log\left(1 - \left(\frac{a}{b}\right)^2\right),$$

and

$$\inf_{t>0} f(t) = a(\log(b) - \log(a)) + a,$$

Proof. The solution t^* of the equation $f'(t) = 0$ is given by

$$t^* = -\frac{1}{2}\log\left(1 - \left(\frac{a}{b}\right)^2\right).$$

It is clear that $\inf_{t>0} f(t) = f(t^*) = a(\log(b) - \log(a)) + a$. □

We define the following smoothing of h by $T_t h$ for small $t > 0$:

$$T_t h(x) = \mathbb{E}\Big[h(e^{-t}x + \sqrt{1-e^{-2t}}Z)\Big],$$

where $Z \sim \mathcal{N}(0, I)$. We use the following differential equation in [5] or (26.1.16) in the book [1],

$$T_t h(x) - \Phi h = \Delta\Psi_t(x) - x \cdot \nabla\Psi_t(x), \tag{6}$$

where

$$\Psi_t(x) = -\int_t^\infty \left\{ \int_{\mathbb{R}^d} \tilde{h}(e^{-s}x + \sqrt{1-e^{-2s}}y)\phi(y)dy \right\} ds.$$

Let $\mathcal{C}$ be the class of all Borel convex sets in $\mathbb{R}^d$ and $\tilde{h} = h - \int_{\mathbb{R}^d} hd\Phi$. In [1], the bound for the error arising from this smoothing is given by

$$\sup_{\{h:h=\mathbf{1}_C, C\in\mathcal{C}\}} \left|\mathbb{E}[\tilde{h}(F_n)]\right| \leq \sup_{\{h:h=\mathbf{1}_C, C\in\mathcal{C}\}} \left|\mathbb{E}[T_t\tilde{h}(F_n)]\right| + b\sqrt{1-e^{-2t}}e^t, \tag{7}$$

where b is a positive constant being independent of n. We first estimate, for small $t > 0$,

$$\mathbb{E}[T_t\tilde{h}(F_n)] = \mathbb{E}[\Delta\Psi_t(F_n) - F_n \cdot \nabla\Psi_t(F_n)], \tag{8}$$

Obviously, for $i, j = 1, \ldots, d$,

$$\begin{aligned} \frac{\partial^2}{\partial x^i \partial x^j}\Psi_t(x) &= -\int_t^\infty \left(\frac{e^{-s}}{\sqrt{1-e^{-2s}}}\right)^2 \\ &\quad \left\{ \int_{\mathbb{R}^d} \tilde{h}(e^{-s}x + \sqrt{1-e^{-2s}}y)\frac{\partial^2}{\partial y^i \partial y^j}\phi(y)dy \right\} ds. \end{aligned}$$

From the estimate $\int_{\mathbb{R}^d} \left|\frac{\partial^2}{\partial y^i \partial y^j}\phi(y)\right| dy \leq 1$ for $i, j = 1, \ldots, d$ we have

$$\sup_{x\in\mathbb{R}^d} \left|\frac{\partial^2}{\partial x^i \partial x^j}\Psi_t(x)\right| \leq \int_t^\infty \frac{e^{-2s}}{1-e^{-2s}} ds = \log\left(\frac{1}{\sqrt{1-e^{-2t}}}\right). \tag{9}$$

THEOREM 3.2. *Let Σ be a $d \times d$ be a symmetric positive-definite matrix . Suppose that $\{F_n = (F_n^{(1)}, \ldots, F_n^{(d)}), n \geq 1\}$ is a sequence of $\mathbb{R}^d$-valued centered square integrable random variables such that $F_n^{(i)} \in \mathbb{D}^{1,2}$ for every $i = 1, \ldots, d$ and $n \geq 1$. For $n \geq 1$ such that $\|\Sigma^{-1/2}\|_1^2 A_n < b$, we have that*

$$\begin{aligned} &\sup_{\{h:h=\mathbf{1}_C, C\in\mathcal{C}\}} \left|\mathbb{E}[h(F_n)] - \mathbb{E}[h(\Sigma^{1/2}Z)]\right| \\ &\leq \|\Sigma^{-1/2}\|_1^2 A_n\Big(\log(b) - \log(\|\Sigma^{-1/2}\|_1^2 A_n)\Big) + \|\Sigma^{-1/2}\|_1^2 A_n, \end{aligned} \tag{10}$$

where b is a positive constant, the norm $\|\Sigma^{-1/2}\|_1^2$ denotes the subordinate matrix norm for a matrix $\Sigma^{-1/2}$ based on ℓ_1 vector norms, and

$$A_n = \sum_{l,v=1}^{d} \sqrt{\mathbb{E}\left[\left(\Sigma_{lv} - \langle DL^{-1}F_n^{(l)}, DF_n^{(v)}\rangle_{\mathcal{H}}\right)^2\right]}.$$

Proof. Since $\mathcal{C}$ is invariant nonsingular, linear transformation, we have

$$\begin{aligned} &\sup_{\{h:h=\mathbf{1}_C, C\in\mathcal{C}\}} \left|\mathbb{E}[h(F_n)] - \mathbb{E}[h(\Sigma^{1/2}Z)]\right| \\ &= \sup_{\{h:h=\mathbf{1}_C, C\in\mathcal{C}\}} \left|\mathbb{E}[h(\Sigma^{-1/2}F_n)] - \mathbb{E}[h(Z)]\right|. \end{aligned} \tag{11}$$

Using the smoothing inequality (7) and (11) yields

$$\begin{aligned} &\sup_{\{h:h=\mathbf{1}_C, C\in\mathcal{C}\}} \left|\mathbb{E}[\tilde{h}(\Sigma^{-1/2}F_n)]\right| \\ &\le \sup_{\{h:h=\mathbf{1}_C, C\in\mathcal{C}\}} \left|\mathbb{E}[T_t\tilde{h}(\Sigma^{-1/2}F_n)]\right| + c\sqrt{1-e^{-2t}}e^t. \end{aligned} \tag{12}$$

By (8) and (9), we estimate

$$\begin{aligned} &\left|\mathbb{E}[T_t\tilde{h}(\Sigma^{-1/2}F_n)]\right| \\ &= \left|\mathbb{E}\left[\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x^i \partial x^j}\Psi_t(\Sigma^{-1/2}F_n)\delta_{i,j}\right]\right. \\ &\quad \left. - \sum_{i,l=1}^{d}\Sigma_{il}^{-1/2}\sum_{j,v=1}^{d}\Sigma_{jv}^{-1/2}\mathbb{E}\left[\frac{\partial^2}{\partial x^i \partial x^j}\Psi_t(\Sigma^{-1/2}F_n)\langle DL^{-1}F_n^{(l)}, DF_n^{(v)}\rangle_{\mathcal{H}}\right]\right| \\ &= \left|\mathbb{E}\left[\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x^i \partial x^j}\Psi_t(\Sigma^{-1/2}F_n)\sum_{l,v=1}^{d}\Sigma_{il}^{-1/2}\Sigma_{jv}^{-1/2}\sum_{r=1}^{d}\Sigma_{rl}^{1/2}\Sigma_{rv}^{1/2}\right]\right. \\ &\quad \left. - \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x^i \partial x^j}\Psi_t(\Sigma^{-1/2}F_n)\sum_{l,v=1}^{d}\Sigma_{il}^{-1/2}\Sigma_{jv}^{-1/2}\mathbb{E}\left[\langle DL^{-1}F_n^{(l)}, DF_n^{(v)}\rangle_{\mathcal{H}}\right]\right| \end{aligned} \tag{13}$$

$$
\begin{aligned}
&= \left| \mathbb{E}\left[\sum_{l,v=1}^{d} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x^i \partial x^j} \Psi_t(\Sigma^{-1/2}F_n) \Sigma_{il}^{-1/2} \Sigma_{jv}^{-1/2} \right.\right. \\
&\qquad \left.\left. \times \left(\Sigma_{lv} - \langle DL^{-1}F_n^{(l)}, DF_n^{(v)} \rangle_{\mathcal{H}} \right) \right] \right| \\
&\le \log\left(\frac{1}{\sqrt{1-e^{-2t}}} \right) \sum_{l,v=1}^{d} \sum_{i,j=1}^{d} \left| \Sigma_{il}^{-1/2} \Sigma_{jv}^{-1/2} \right| \\
&\qquad \times \mathbb{E}\left[\left| \Sigma_{lv} - \langle DL^{-1}F_n^{(l)}, DF_n^{(v)} \rangle_{\mathcal{H}} \right| \right] \qquad (14) \\
&\le \log\left(\frac{1}{\sqrt{1-e^{-2t}}} \right) \Big(\sup_{1 \le l \le d} \sum_{i=1}^{d} \left| \Sigma_{il}^{-1/2} \right| \Big)^2 \\
&\qquad \times \sum_{l,v=1}^{d} \sqrt{ \mathbb{E}\left[\left(\Sigma_{lv} - \langle DL^{-1}F_n^{(l)}, DF_n^{(v)} \rangle_{\mathcal{H}} \right)^2 \right] }. \qquad (15)
\end{aligned}
$$

Since we take $n \ge 1$ such that $\|\Sigma^{-1/2}\|_1^2 A_n < b$, it follows, from (7) and Lemma 3.1 together with (15), that

$$
\begin{aligned}
&\sup_{\{h: h = \mathbf{1}_C, C \in \mathcal{C}\}} \left| \mathbb{E}[T_t \tilde{h}(\Sigma^{-1/2}F_n)] \right| \\
&\le \|\Sigma^{-1/2}\|_1^2 A_n \Big(\log(b) - \log(\|\Sigma^{-1/2}\|_1^2 A_n) \Big) + \|\Sigma^{-1/2}\|_1^2 A_n. \qquad (16)
\end{aligned}
$$

□

REMARK 3.3. By the Cauchy-Schwartz inequality, the right-hand side in (14) can be estimated as

$$
\begin{aligned}
&\log\left(\frac{1}{\sqrt{1-e^{-2t}}} \right) \sum_{j=1}^{d} \Big(\sum_{i=1}^{d} |\Sigma_{ij}^{-1/2}| \Big)^2 \\
&\times \mathbb{E}\left[\sqrt{ \sum_{l,v=1}^{d} \Big(\left| \Sigma_{lv} - \langle DL^{-1}F_n^{(l)}, DF_n^{(v)} \rangle_{\mathcal{H}} \right| \Big)^2 } \right] \qquad (17)
\end{aligned}
$$

By a similar estimate as for (15), we have, from (17), that

$$\begin{aligned} &\sup_{\{h:h=\mathbf{1}_C, C\in\mathcal{C}\}} \left|\mathbb{E}[T_t\tilde{h}(\Sigma^{-1/2}F_n)]\right| \\ &\leq \ aB_n\Big(\log(b) - \log(aB_n)\Big) + aB_n, \end{aligned} \tag{18}$$

where $a = \sum_{j=1}^d \left(\sum_{i=1}^d |\Sigma_{ij}^{-1/2}|\right)^2$ and

$$B_n = \mathbb{E}\left[\sqrt{\sum_{l,v=1}^d \left(\left|\Sigma_{lv} - \langle DL^{-1}F_n^{(l)}, DF_n^{(v)}\rangle_{\mathcal{H}}\right|\right)^2}\right].$$

4. Applications

In this section, we use our main results in order to obtain an explicit upper bound of the *Kolmogorov distance* instead of the *Wasserstein distance* used for Theorem 4.1 in the paper [9] corresponding to Lemma 4.1 below. We recall that a fractional Brownian motion $B^H = \{B_t^H, t \geq 0\}$, with Hurst parameter H, is a centered Gaussian process with covariance

$$R(s,t) = \mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Fix an integer $q \geq 2$. We assume that $H < 1 - \frac{1}{2q}$. Let us set

$$S_n(t) = \frac{1}{\sigma\sqrt{n}} \sum_{k=0}^{[nt]-1} H_q(B_{k+1}^H - B_k^H), \text{ for } t \geq 0,$$

where H_q is the qth Hermite polynomial function and $\sigma = \sqrt{q!\sum_{r\in\mathbb{Z}}\rho^2(r)}$,

$$\rho(r) = \frac{1}{2}(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H}).$$

In the paper [2] or [4], authors prove that as $n \to \infty$,

$$S_n \xrightarrow{f.d.d} B^H,$$

where the notation $\xrightarrow{f.d.d}$ denotes convergence in the sense of finite-dimensional distributions. In the paper [9], authors obtain the multidimensional bound for the *Wasserstein distance* proved for $\{S_n(t), t \geq 0\}$.

LEMMA 4.1. *For any fixed $d \geq 1$ and $0 = t_1 < \cdots < t_d$, there exists a constant c, depending only on d, H and $(t_0, t_1, \ldots, t_d)$ such that for every $n \geq 1$*

$$d_W(F_n, Z) \leq c \times \begin{cases} n^{-1/2} & \text{for } H \in (0, \frac{1}{2}] \\ n^{H-1} & \text{for } H \in (\frac{1}{2}, \frac{2q-3}{2q-2}] \\ n^{qH-q+\frac{1}{2}} & \text{for } H \in (\frac{2q-3}{2q-2}, \frac{2q-1}{2q}] \end{cases},$$

where $Z \sim \mathcal{N}_d(0, I_d)$ and $F_n = (F_n^{(1)}, \ldots, F_n^{(d)})$,

$$F_n^{(i)} = \frac{S_n(t_i) - S_n(t_{i-1})}{\sqrt{t_i - t_{i-1}}}.$$

If we use the relation (3), then the upper bound of the *Kolmogorov distance* equals

$$d_{Kol}(F_n, Z) \leq c \times \begin{cases} n^{-1/4} & \text{for } H \in (0, \frac{1}{2}] \\ n^{\frac{H-1}{2}} & \text{for } H \in (\frac{1}{2}, \frac{2q-3}{2q-2}] \\ n^{\frac{2qH-2q+1}{4}} & \text{for } H \in (\frac{2q-3}{2q-2}, \frac{2q-1}{2q}] \end{cases}. \tag{19}$$

THEOREM 4.2. *Let F_n be a sequence given in Lemma 4.1. Then for sufficiently large $n \geq 1$, we have*

$$d_{Kol}(F_n, Z) \leq c \times \begin{cases} \log(n) n^{-1/2} & \text{for } H \in (0, \frac{1}{2}] \\ \log(n) n^{H-1} & \text{for } H \in (\frac{1}{2}, \frac{2q-3}{2q-2}] \\ \log(n) n^{qH-q+\frac{1}{2}} & \text{for } H \in (\frac{2q-3}{2q-2}, \frac{2q-1}{2q}] \end{cases}. \tag{20}$$

Proof. By ignoring terms in the upper bound (10) of Theorem 3.2 being of lower order than $\log(A_n)A_n$, we have, taking $C = (-\infty, z]$, that

$$d_{Kol}(F_n, Z) \leq c|\log(A_n)|A_n. \tag{21}$$

By the estimate (21) and Lemma 4.1, we get the results. □

REMARK 4.3. we can see that the upper bound (20) obtained by using Theorem 3.2 is more efficient than that in (19) obtained by using Lemma 4.1 in the sense that the latter one converges to zero more slowly as n tends to infinity.

Acknowledgements
We would like to thank the editor and the anonymous referees for their comments which considerably improve the paper.

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Yoon Tae Kim
Department of Finance and Information Statistics
Hallym University
Chunchon 200-702, Korea
E-mail: `ytkim@hallym.ac.kr`

Hyun Suk Park
Department of Finance and Information Statistics
Hallym University
Chunchon 200-702, Korea
E-mail: `hspark@hallym.ac.kr`