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BASE OF THE NON-POWERFUL SIGNED TOURNAMENT

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Abstract. A signed digraph S is the digraph D by assigning signs 1 or -1 to each arc of D. The base of S is the minimum number k such that there is a pair walks which have the same initial and terminal point with length k, but different signs. In this paper we show that for $n \geq 5$ the upper bound of the base of a primitive non-powerful signed tournament S_n, which is the signed digraph by assigning 1 or -1 to each arc of a primitive tournament T_n, is $\max\{2n+2, n+11\}$. Moreover we show that it is extremal except when $n = 5, 7$.

1. Introduction

A digraph $D = (V, A)$ is *primitive* if there is a positive integer k such that for each vertices u, v of D, there is a directed walk of length k from u to v. A *signed digraph* S is a digraph where each arc of S is assigned signs 1 or -1. If W is a directed walk of a signed digraph S, then the multiple of signs of all arcs in W is said to be the *sign* of W in S, denoted by $\mathrm{sgn}(W)$. If two walks W_1 and W_2 have the same initial point, the same terminal point, the same length and different signs, then we say that W_1 and W_2 are a *pair of SSSD walks.* A signed digraph S is

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powerful if S contains no pair of SSSD walks. S is *non-powerful* if it is not powerful. So every primitive non-powerful signed digraph contains a pair of SSSD walks. From now on we assume that S is a primitive non-powerful signed digraph. For each pair of vertices u, v of S, we define the *local base* $l_S(u, v)$ from u to v by the smallest integer l such that if $k \geq l$, then there is a pair of SSSD walks of length k in S from u to v. We define the *base* $l(S)$ of S by $\max\{l_S(u, v) | u, v \in V(S)\}$.

A square matrix with its entries in the sign set $\{1,\ 0,\ -1\}$ is said to be the *sign pattern matrix*. In computing the powers of A, we use the usual arithmetic rules of signs such that $1 + 1 = 1$, $-1 + (-1) = -1$ and $1 \cdot 1 = -1 \cdot (-1) = 1$ and $1 \cdot (-1) = -1$. Sometimes we contact the ambiguous situations such that $1 + (-1)$ or $(-1) + 1$. As in [3], in this case we use the symbol $\sharp$ as follows:

$$(-1) + 1 = 1 + (-1) = \sharp; \quad a + \sharp = \sharp + a = \sharp \text{ for any } a \in \{1,\ -1,\ \sharp,\ 0\}$$

$$0 \cdot \sharp = \sharp \cdot 0 = 0; \quad b \cdot \sharp = \sharp \cdot b = \sharp \text{ for any } b \in \{1,\ -1,\ \sharp\}.$$

When the power of a sign pattern matrix contains $\sharp$ entry it is convinient to expand the sign set as follows $\Gamma = \{1,\ 0,\ -1,\ \sharp\}$. A square matrix with its entries in the sign set $\Gamma = \{1,\ 0,\ -1,\ \sharp\}$ is said to be the *generalized sign pattern matrix*. A sign pattern matrix A is said to be *powerful* if each power of A contains no $\sharp$ entry. And A is *non-powerful* if it is not powerful. When we deal with the non-powerful sign pattern matrix we use the generalized one. Since we use non-powerful sign pattern matrix, throughout this paper we simply say the sign pattern matrix instead of the generalized sign pattern matrix.

Let $A = (a_{ij})$ be the adjacency matrix of the signed digraph S, that is $(i, j) \in A$ and $\text{sgn}(i, j) = \alpha$ if and only if $a_{ij} = \alpha$ where $\alpha = 1$, or -1. Hence A is the sign pattern matrix. A least positive integer l such that there is a positive integer p satisfying $A^l = A^{l+p}$ is said to be the *base* of A, and denoted by $l(A)$. Li et al. [3], showed that if a sign pattern matrix A is powerful, then $l(A) = l(|A|)$ where $|A|$ is the matrix by assigning each non-zero entries of A to 1. If A is non-powerful, then the $\sharp$ entry appears and we have different situations. It follows directly from the definitions $l(S) = l(A)$ where A is the adjacency matrix of S. Gao, Huang and Shao [2], Shao and Gao [6] and Li and Liu [4] studied the base and local base of the primitive non-powerful signed symmetric digraphs with loops. Song and Kim [7] computed the base of the non-powerful signed complete graphs.

In this paper we show that for $n \geq 5$ the upper bound of the base $l(T_n)$ of a primitive non-powerful signed tournament T_n of order n is $\max\{2n+2, n+11\}$ and this bound is extremal when $n \neq 5, 7$. When $n = 5$ or 7, we prove that $l(T_n) \leq n + 10$ by providing some examples.

2. Bases of Signed Tournament

A tournament T_n of order n is a digraph which can be obtained from the complete graph K_n by assigning a direction to each of its edges. It is well known that T_n is primitive if and only if T_n is strongly connected. Moon and Pullman [5] studied invariant structure of the primitive tournament. Throughout this paper we assume that T_n is a primitive non-powerful signed tournament of order n. The following on T_n is well known.

LEMMA 1. [1] *If T_n is strongly connected then each vertex of T_n is contained in a simple cycle of length l for each $3 \leq l \leq n$.*

The following characteristics of the non-powerful primitive signed digraph are useful to obtain the main results.

LEMMA 2. [8] *A signed digraph S is non-powerful primitive if and only if S contains a pair of cycles C_1 and C_2 of length p_1 and p_2 respectively satisfying one of the following holds.*
(1) p_1 is odd and p_2 is even with $\text{sgn}(C_2) = -1$
(2) p_1 and p_2 is odd and $\text{sgn}(C_1) = -\text{sgn}(C_2)$.

Let T_n be a primitive non-powerful signed tournament of order n and u, v be two vertices of T_n which are not necessarily distinct. By Lemma 1 there is a cycle C_0 of length n. If we assume that C_0 is the cycle $v_0v_1 \cdots v_{n-1}v_0$ where $v_0 = u$, then the vertex set is $\{v_0, v_1, \ldots, v_{n-1}\}$. For each cycle C of T_n we define $d(C) = \min\{k | v_k \text{ is a vertex of } C\}$ and $|C|$ to be the length of C. Since a primitive tournament T_n contains every cycle of length $3 \leq l \leq n$, thus Lemma 2 can be rewritten as follows:

LEMMA 3. *Let T_n be a primitive signed tournament. Then T_n is non-powerful if and only if T_n contains a cycle C satisfying one of the following holds.*
(1) $|C|$ is even with $\text{sgn}(C) = -1$
(2) $|C|$ is odd and $\text{sgn}(C) = -\text{sgn}(C')$ *for some odd cycle C' of T_n.*

THEOREM 1. *If $n \geq 5$, then for each pair of vertices u, v of T_n there is a pair of SSSD walks from u to v of length less than or equals to* $\max\{2n-1, n+8\}$.

Proof. Since T_n is non-powerful we let C be the the first cycle in T_n which causes the situation of (1) or (2) in Lemma 3. In other words C is a cycle of T_n satisfying one of the followings.

(A): If C is an even cycle, then $\text{sgn}(C) = -1$ and every even cycle C' in T_n such that $d(C') < d(C)$, or $d(C') = d(C)$ and $|C'| < |C|$ satisfies $\text{sgn}(C') = 1$. Moreover every odd cycle C' in T_n such that $d(C') < d(C)$, or $d(C') = d(C)$ and $|C'| < |C|$ have the same sign.

(B): If C is an odd cycle with $d(C) \geq 1$, or $d(C) = 0$ and $|C| > 3$, then every odd cycle C' in T_n such that $d(C') < d(C)$, or $d(C') = d(C)$ and $|C'| < |C|$ satisfies $\text{sgn}(C) = -\text{sgn}(C')$. Moreover every even cycle C' in T_n such that $d(C') < d(C)$, or $d(C') = d(C)$ and $|C'| < |C|$ satisfies $\text{sgn}(C') = 1$.

(C): If $|C| = 3$ and $d(C) = 0$, then there is an odd cycle C' in T_n such that $d(C') = 0$ and $|C'| = 3$ with $\text{sgn}(C) = -\text{sgn}(C')$.

If $d(C) = k \geq 1$, then since T_n is a tournament and by **(A)** or **(B)** there is C' such that $d(C') = 0$ and $|C'| = |C|$ such that $\text{sgn}(C) = -\text{sgn}(C')$. Since $d(C) = k$, $v_0, v_1, \ldots, v_{k-1}$ is not a vertex of C. We have $|C| = m \leq n - k$. If $v = v_j$ with $0 \leq j < k$ then the walk

$$W_1 = (v_0 v_1 \cdots v_k) + C + (v_k v_{k+1} \cdots v_{n-1} v_0 \cdots v_j)$$

is the walk of length $|W_1| = k + m + (n-1-k) + 1 + j = m + n + j$ from $u = v_0$ to $v = v_j$. And the walk

$$W_2 = C' + (v_0 v_1 \cdots v_{n-1} v_0 \cdots v_j)$$

is the walk of length $|W_2| = m + n + j$. Since $m \leq n - k$ and $j < k$, the common length $m + n + j$ of W_1 and W_2 is less than or equals to $2n - 1$. We have

$$\begin{aligned} \text{sgn}(W_1) &= \text{sgn}(C) \times \text{sgn}(v_0 \cdots v_{n-1} v_0 \cdots v_j) \\ &= -\text{sgn}(C') \times \text{sgn}(v_0 \cdots v_{n-1} v_0 \cdots v_j) \\ &= -\text{sgn}(W_2). \end{aligned}$$

So there is a pair of SSSD walks of length less than or equals to $2n - 1$ from u to v. If $v = v_j$ with $k \leq j \leq n-1$, then the walk

$$W_1 = (v_0 v_1 \cdots v_k) + C + (v_k \cdots v_j)$$

is the walk of length $|W_1| = k + m + j - k = m + j$ from u to v. And the walk

$$W_2 = C' + (v_0 v_1 \cdots v_j)$$

is the walk of length $|W_2| = m + j$. The common length $m + j$ of W_1 and W_2 is less than or equals to $2n - 1$. We also have

$$\begin{aligned} \mathrm{sgn}(W_1) &= \mathrm{sgn}(C) \times \mathrm{sgn}(v_0 \cdots v_j) \\ &= -\mathrm{sgn}(C') \times \mathrm{sgn}(v_0 \cdots v_j) \\ &= -\mathrm{sgn}(W_2). \end{aligned}$$

So there is a pair of SSSD walks of length less than or equals to $2n - 1$ from u to v.

If $d(C) = 0$ and $|C| = 3$, then by **(C)** there is a cycle C' in T_n where $d(C') = 0$ and $|C'| = 3$ such that $\mathrm{sgn}(C) = -\mathrm{sgn}(C')$. The walks

$$W_1 = C + (v_0 v_1 \cdots v_j)$$

and

$$W_2 = C' + (v_0 v_1 \cdots v_j)$$

are a pair of SSSD walks with common length $3 + j (\leq 2n - 1)$ from u to v.

Let $d(C) = 0$ and $|C| = m \geq 6$. If m is even, then there is a cycle $C_{\frac{m}{2}}$ in T_n where $d(C_{\frac{m}{2}}) = 0$ and $|C_{\frac{m}{2}}| = \frac{m}{2}$. By (A) $\mathrm{sgn}(C) = -1$ and since $\mathrm{sgn}(2C_{\frac{m}{2}}) = 1$ the walks

$$W_1 = C + (v_0 v_1 \cdots v_j)$$

and

$$W_2 = 2C_{\frac{m}{2}} + (v_0 v_1 \cdots v_j)$$

are a pair of SSSD walks with common length $m + j (\leq 2n - 1)$ from u to v. If m is odd, then since $m - 3 (\geq 4)$ is even and by **(B)** there is a cycle C_{m-3} in T_n where $d(C_{m-3}) = 0$, $|C_{m-3}| = m - 3$ and $\mathrm{sgn}(C_{m-3}) = 1$. Also by **(B)** there is a cycle C_3 in T_n where $d(C_3) = 0$, $|C_3| = 3$ and $\mathrm{sgn}(C_3) = -\mathrm{sgn}(C)$. Since $\mathrm{sgn}(C_3 + C_{m-3}) = \mathrm{sgn}(C_3)\mathrm{sgn}(C_{m-3}) = -\mathrm{sgn}(C)$ the walks

$$W_1 = C + (v_0 v_1 \cdots v_j)$$

and

$$W_2 = C_3 + C_{m-3} + (v_0 v_1 \cdots v_j)$$

are a pair of SSSD walks with common length $m + j (\leq 2n - 1)$ from u to v.

If $d(C) = 0$ and $|C| = 5$, then by **(B)** there are cycles C_3 and C_4 in T_n where $d(C_3) = d(C_4) = 0$, $|C_3| = 3$ and $|C_4| = 4$ such that $\mathrm{sgn}(C) = -\mathrm{sgn}(C_3)$. In this case we have $\mathrm{sgn}(C_3 + C) = -1$ and so the walks

$$W_1 = C + C_3 + (v_0 v_1 \cdots v_j)$$

and

$$W_2 = 2C_4 + (v_0 v_1 \cdots v_j)$$

are a pair of SSSD walks with common length $8 + j(\leq n + 7)$ from u to v.

Let $d(C) = 0$ and $|C| = 4$. Since $n \geq 5$, there are cycles C_3 and C_5 in T_n with $d(C_3) = d(C_5) = 0$ and $|C_3| = 3$ and $|C_5| = 5$.

If $\mathrm{sgn}(C_3) = \mathrm{sgn}(C_5)$, then $\mathrm{sgn}(3C_3) = -\mathrm{sgn}(C_5 + C)$. So the walks

$$W_1 = 3C_3 + (v_0 v_1 \cdots v_j)$$

and

$$W_2 = C_5 + C + (v_0 v_1 \cdots v_j)$$

are a pair of SSSD with common length $9 + j(\leq n + 8)$ from u to v. If $\mathrm{sgn}(C_3) = -\mathrm{sgn}(C_5)$, then $\mathrm{sgn}(C_3 + C_5) = -\mathrm{sgn}(2C)$. So the walks

$$W_1 = C_3 + C_5 + (v_0 v_1 \cdots v_j)$$

and

$$W_2 = 2C + (v_0 v_1 \cdots v_j)$$

are a pair of SSSD walks with common length $8 + j(\leq n + 7)$ from u to v. □

Since T_n is a primitive tournament and $n \geq 5$, there is a closed walk of length l passing through u for each vertex u of T_n and $l \geq 3$. We obtain the following corollary.

Corollary 1. *If $n \geq 5$, then the base $l(T_n)$ of the primitive non-powerful signed tournament T_n of order n satisfies*

$$l(T_n) \leq \max\{2n + 2, n + 11\}.$$

The following examples reveal that the upper bound of the base given in Corollary 1 is extremal when $n \geq 5$ and $n \neq 5,\ 7$.

Examples: Let $S_n = (V, A)$ be the signed tournament such that

$$V = \{0, 1, \ldots, n - 1\}$$

$$A = \{(i, i+1) | 0 \leq i \leq n-2\} \bigcup \{(i, j) | 0 \leq j \leq i-2 \leq n-2\}.$$

1: For $n \geq 9$ if we assign 1 to each arc of S_n except $(n-1, 0)$ to which we assign -1, then there is no walk of length $2n+1$ from 0 to $n-1$ with sign -1. So the upper bound $2n+2$ is extremal.

2: For $n = 8$ if we assign 1 to each arc of S_8 except the 7 arcs

$$(7, 4),\ (6, 3),\ (5, 2),\ (4, 1),\ (3, 0),\ (7, 1),\ (6, 0)$$

to which we assign -1, then there is no walk of length 18 from 0 to 7 with sign -1. So the upper bound 19 is extremal.

3: For $n = 7$ we assign 1 to each arc of S_7 except $(6, 2),\ (5, 1),\ (4, 0)$ to which we assign -1, then there is no walk of length 16 from 0 to 6 with sign -1. In this case there is a pair of SSSD walks of length 14 from 0 to 6, so $l(S_7) = 17 = n + 10$.

4: For $n = 6$ we assign 1 to each arc of S_6 except $(5, 2),\ (4, 1),\ (3, 0)$ to which we assign -1, then there is no walk of length 16 from 0 to 5 with sign -1. So the upper bound 17 is extremal. Figure 1 displays the signed tournament S_6, in which the sign of the arcs with no symbols is 1.

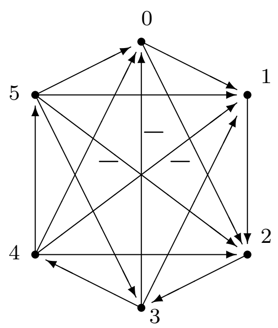

Figure 1. Signed tournament S_6

5: For $n = 5$ we assign 1 to each arc of S_5 except $(4, 0)$ to which we assign -1, then there is no walk of length 14 from 0 to 4 with sign -1. In this case there is a pair of SSSD walks of length 12 from 0 to 4, so $l(S_5) = 15 = n + 10$.

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