Korean J. Math. **23** (2015), No. 1, pp. 153–161
http://dx.doi.org/10.11568/kjm.2015.23.1.153

NONTRIVIAL SOLUTIONS FOR AN ELLIPTIC SYSTEM

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Abstract. In this work, we consider an elliptic system

$$\begin{cases} -\triangle u = au + bv + \delta_1 u^+ - \delta_2 u^- + f_1(x,u,v) & \text{in } \Omega, \\ -\triangle v = bu + cv + \eta_1 v^+ - \eta_2 v^- + f_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^N$ be a bounded domain with smooth boundary. We prove that the system has at least two nontrivial solutions by applying linking theorem.

1. Introduction and Background

Presently there are many significant results with respect to the elliptic system

$$\begin{cases} -\triangle u = \lambda u + \delta v + h_1(x,u,v), \\ -\triangle v = \theta u + \nu v + h_2(x,u,v), \end{cases}$$

in Ω, where $\Omega \subset R^n$ is bounded smooth domain, subject to Dirichlet boundary conditions $u = v = 0$ on $\partial\Omega$, h_i, $i = 1, 2$ are real valued functions and λ, δ, ν and θ are real numbers. [2, 6–8]

Received December 2, 2014. Revised March 16, 2015. Accepted March 16, 2015.
2010 Mathematics Subject Classification: 35J55; 49J35.
Key words and phrases: Elliptic system, Linking inequality.
This work was supported by Namseoul University.
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Many authors also investigated the problem

$$\begin{cases} -\triangle u = au + bv + (u^+)^p + f_1 & \text{in } \Omega, \\ -\triangle v = bu + av + (v^+)^q + f_2 & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

where $u^+ = \max\{0, u(x)\}$. Here Ω is a bounded smooth domain in R^n with $n \geq 2$. [4, 5]

In this paper we prove the existence of two nontrivial solutions for a general elliptic system. We use a variational approach and look for critical points of a suitable functional I on a Hilbert space H. Since the functional is strongly indefinite, it is convenient to use the notion of linking theorem. In Section 2, we find a suitable functional I on a Hilbert space H. In Section 3, we prove the suitable version of the Palais-Smale condition for the topological method. In Section 4, we apply the two critical points theorem.

We recall some basic theorem and set up some terminology. Let H be a Hilbert space and V a C^2 complete connected Finsler manifold. Suppose $H = H_1 \oplus H_2$ and let $H_n = H_{1n} \oplus H_{2n}$ be a sequence of closed subspaces of H such that

$$H_{in} \subset H_i, \quad 1 \leq \dim H_{in} < +\infty \quad \text{for each} \quad i = 1, 2 \quad \text{and} \quad n \in N$$

Moreover suppose that there exist $e_1 \in \cap_{n=1}^{\infty} H_{1n}$, and $e_2 \in \cap_{n=1}^{\infty} H_{2n}$, with $\|e_1\| = \|e_2\| = 1$.

For any Y subspace of H, consider $B_\rho(Y) := \{u \in Y | \|u\| \leq \rho\}$ and denote by $\partial B_\rho(Y)$ the boundary of $B_\rho(Y)$ relative to Y. Furthermore define, for any $e \in H$,

$$Q_R(Y, e) := \{u + ae \in Y \oplus [e] | u \in Y, a \geq 0, \|u + ad\| \leq R\}$$

and denote by $\partial Q_R(Y, e)$ its boundary relative to $Y \oplus [e]$, and denote by $X = H \times V$.

We recall the two critical points theorem in [3].

Theorem 1.1. *Suppose that f satisfies the $(PS)^*$ condition with respect to H_n. In addition assume that there exist ρ, R, such that $0 < \rho < R$ and*

$$\sup_{\partial Q_R(H_2, e_1) \times V} f \;<\; \inf_{\partial B_\rho(H_1) \times V} f,$$

$$\sup_{Q_R(H_2, e_1) \times V} f < +\infty, \qquad \inf_{B_\rho(H_1) \times V} f < -\infty,$$

Then there exist at least 2 critical levels of f. Moreover the critical levels satisfy the following inequalities

$$\inf_{B_\rho(H_1)\times V} f \le c_1 \le \sup_{\partial Q_R(H_2,e_1)\times V} f < \inf_{\partial B_\rho(H_1)\times V} f \le c_2 \le \sup_{Q_R(H_2,e_1)\times V} f,$$

and there exist at least $2+2$ cuplength(V) critical points of f.

2. Notations and main result

Let $\Omega \subset R^N$ be a bounded domain with smooth boundary and $H = W_0^{1,p}(\Omega)$, the usual Sobolev space with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$.

In this paper, we consider the existence of nontrivial solutions to the elliptic system

$$(1) \quad \begin{cases} -\triangle u = au + bv + \delta_1 u^+ - \delta_2 u^- + f_1(x,u,v) & \text{in } \Omega, \\ -\triangle v = bu + cv + \eta_1 v^+ - \eta_2 v^- + f_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

And there exists a function $F : \bar{\Omega} \times R^2 \to R$ such that $\frac{\partial F}{\partial u} = f_1$ and $\frac{\partial F}{\partial v} = f_2$ without loss of generality, we set

$$F(x,u,v) = \int_{(0,0)}^{(u,v)} f_1(x,u,v)du + f_2(x,u,v)dv.$$

Then $F \in C^1(\bar{\Omega} \times R^2, R)$.

We consider the following assumptions.

(F1) There exist $M > 0$ and $\alpha > 2$ such that

$$0 < \alpha F(x,u,v) \le uF_u(x,u,v) + vF_v(x,u,v)$$

for all $(x,u,v) \in \bar{\Omega} \times R^2$ with $u^2 + v^2 > M^2$.

(F2) There exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$|F_u(x,u,v)| + |F_v(x,u,v)| \le a_1 + a_2(|u|^r + |v|^r)$$

where $1 \le r < (N+2)/(N-2)$ if $N > 2$, $1 \le r < \infty$ otherwise.

(F3) For $(0,v) \to (0,0)$,

$$\frac{F(x,0,v)}{v^2} \to 0.$$

Remark 2.1. *The condition (F1) shows that there exist constants $b_1 > 0$ and b_2 such that(cf. [1])*

$$F(x,u,v) \ge b_1(|u|^\alpha + |v|^\alpha) - b_2.$$

Let λ_k denote the eigenvalues and e_k the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, with Dirichlet boundary condition, where each eigenvalue λ_k is respected as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \lambda_i \to +\infty$ and that $e_1 > 0$ for all $x \in \Omega$. Then $H = \text{span}\{e_i | i \in N\}$.

Let $e_i^1 = (e_i, 0)$ and $e_i^2 = (0, e_i)$. We define $H_j = \text{span}\{e_i^j | i \in N\}$, for $j = 1, 2$ and $E = H_1 \oplus H_2$ with the norm $\|(u,v)\|_E^2 = \|u\|^2 + \|v\|^2$.

We define the energy functional associated to (1) as

$$
\begin{aligned}
I(u,v) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx - \frac{1}{2}\int_\Omega (au^2 + 2buv + cv^2)dx \\
&\quad -\frac{1}{2}\int_\Omega (\delta_1(u^+)^2 - \delta_2(u^-)^2 + \eta_1(v^+)^2 - \eta_2(v^-)^2)dx \\
&\quad -\int_\Omega F(x,u,v,w)dx
\end{aligned} \tag{2}
$$

It is easy to see that $I \in C^1(E, R)$ and thus it makes sense to lock for solutions to (1) in weak sense as critical points for I *i.e.*$(u,v) \in E$ such that $I'(u,v) = 0$, where

$$
\begin{aligned}
I'(u,v)\cdot(\phi,\psi) &= \int_\Omega (\nabla u \nabla \phi + \nabla v \nabla \psi)dx \\
&\quad - \int_\Omega (au\phi + bv\phi + bu\psi + cv\psi)dx \\
&\quad - \int_\Omega (\delta_1 u^+\phi - \delta_2 u^-\phi + \eta_1 v^+\psi - \eta_2 v^-\psi)dx \\
&\quad - \int_\Omega (f_1(x,u,v)\phi + f_2(x,u,v)\psi)dx.
\end{aligned}
$$

We will prove the following theorem.

THEOREM 2.1. *Assume F satisfies (F1), (F2) and (F3) with $\alpha = r+1$. If a, b, c, δ, and η are positive with $a + b + \delta_1 + \delta_2 < \lambda_1$ and $b + c + \eta_1 + \eta_2 < \lambda_1$ then system (1) has at least two nontrivial solutions.*

3. The Palais Smale star condition

In this section we will prove the $(PS)_c^*$ condition which was required for the application of Theorem 1.1. In the following, we consider the following sequence of subspaces of E :

$$E_n = \text{span}\{e_i^j | i = 1, \cdots, n \quad \text{and} \quad j = 1, 2\}, \qquad \text{for } n \geq 1.$$

LEMMA 3.1. *Assume F satisfies (F1) and (F2) with $\alpha = r + 1$. If $a + b + \delta_1 + \delta_2 < \lambda_1$ and $b + c + \eta_1 + \eta_2 < \lambda_1$, then any $(PS)_c^*$ sequence is bounded.*

Proof. Let $\{(u_n, v_n)\} \subset E$ be a sequence such that

$$(u_n, v_n) \in E_n, \quad I(u_n, v_n) \to c, \quad I_n'(u_n, v_n) \to 0 \quad \text{as} \quad n \to \infty$$

In the following we denote different constants by C_1, C_2 etc. (F1) and Remark imply that

$$\begin{aligned} C_1 + \frac{1}{2} o(1)(\|u_n\| + \|v_n\|) &\geq I(u_n, v_n) - \frac{1}{2} I_n'(u_n, v_n) \cdot (u_n, v_n) \\ &= \frac{1}{2} \int_\Omega (u_n f_1 + v_n f_2) dx - \int_\Omega F dx \\ &\geq (\frac{\alpha}{2} - 1) \int_\Omega F(x, u_n, v_n) dx \qquad (3) \\ &\geq (\frac{\alpha}{2} - 1) b_1 \int_\Omega (|u_n|^\alpha + |v_n|^\alpha) dx - C_2 \\ &\geq (\frac{\alpha}{2} - 1) b_1 (\|u_n\|_{L^\alpha}^\alpha + \|v_n\|_{L^\alpha}^\alpha) - C_2 \end{aligned}$$

On the other hand,

$$\begin{aligned} o(1)\|u_n\| &\geq I_n'(u_n, v_n) \cdot (u_n, 0) \\ &= \|u_n\|^2 - \int_\Omega (a u_n^2 + b u_n v_n) dx \\ &\quad - \int_\Omega (\delta_1 (u_n^+)^2 - \delta_2 (u_n^-)^2) dx - \int_\Omega f_1(x, u_n, v_n) u_n dx, \\ o(1)\|v_n\| &\geq I_n'(u_n, v_n) \cdot (0, v_n) \\ &= \|v_n\|^2 - \int_\Omega (b u_n v_n + c v_n^2) dx \\ &\quad - \int_\Omega (\eta_1 (v_n^+)^2 - \eta_2 (v_n^-)^2) dx - \int_\Omega f_2(x, u_n, v_n) v_n dx. \end{aligned}$$

We know that

$$\int_\Omega (u^+)^2 dx \leq \|u\|_{L^2}^2 \leq \frac{1}{\lambda_1} \|u\|^2$$

and

$$\int_\Omega (u^-)^2 dx \le \|u\|_{L^2}^2 \le \frac{1}{\lambda_1}\|u\|^2$$

for $u \in H$. Using (F2), we obtain

$$\begin{aligned}
\|u_n\|^2 + \|v_n\|^2 &\le o(1)(\|u_n\| + \|v_n\|) \\
&\quad + \int_\Omega (au_n^2 + 2bu_nv_n + cv_n^2)dx + \int_\Omega (\delta_1(u_n^+)^2 - \delta_2(u_n^-)^2)dx \\
&\quad + \int_\Omega (\eta_1(v_n^+)^2 - \eta_2(v_n^-)^2)dx + \int_\Omega (u_nf_1 + v_nf_2)dx \\
&\le o(1)(\|u_n\| + \|v_n\|) \\
&\quad + \frac{a+b+\delta_1+\delta_2}{\lambda_1}\|u_n\|^2 + \frac{a+b+\eta_1+\eta_2}{\lambda_1}\|v_n\|^2 \\
&\quad + C_3\int_\Omega (|u_n|^{r+1} + |v_n|^{r+1})dx + C_4.
\end{aligned} \tag{4}$$

(4) imply that if $a+b+\delta_1+\delta_2 < \lambda_1$ and $b+c+\eta_1+\eta_2 < \lambda_1$ then

$$\begin{aligned}
\|u_n\|^2 + \|v_n\|^2 &\le o(1)C_5(\|u_n\| + \|v_n\|) \\
&\quad + C_6\int_\Omega (|u_n|^{r+1} + |v_n|^{r+1})dx + C_7.
\end{aligned} \tag{5}$$

Combining (3), (5) and using $\alpha = r+1$, one infers that

$$\|u_n\|^2 + \|v_n\|^2 \le o(1)C_8(\|u_n\| + \|v_n\|) + C_9.$$

This yields $\{(u_n, v_n)\}$ is bounded. □

LEMMA 3.2. *Assume F satisfies (F1) and (F2) with $\alpha = r+1$. If $a+b+\delta_1+\delta_2 < \lambda_1$ and $b+c+\eta_1+\eta_2 < \lambda_1$, then the functional I satisfies the $(PS)_c^*$ condition with respect to E_n.*

Proof. By Lemma 3.1, any $(PS)_c^*$ sequence $\{(u_n, v_n)\}$ in E is bounded and hence $\{(u_n, v_n)\}$ has a weakly convergent subsequence. That is there exist a subsequence $\{(u_{n_j}, v_{n_j})\}$ and $(u, v) \in E$, with $u_{n_j} \rightharpoonup u$ and $v_{n_j} \rightharpoonup v$. Since $\{u_{n_j}\}$ and $\{v_{n_j}\}$ are bounded, by Remark of Rellich-Kondrachov compactness theorem [4], $u_{n_j} \to u$, $v_{n_j} \to v$ and thus I satisfies $(PS)_c^*$ condition. □

4. Proof of main theorem

LEMMA 4.1. *Assume F satisfies (F3). If $c < \lambda_1$, then there exists $\rho_1 > 0$ such that*

$$\inf_{\partial B_{\rho_1}(H_2)} I > 0.$$

Proof. By (F3), for any $\varepsilon > 0$, there exists $\rho > 0$ such that

$$0 < \|v\| < \rho \Rightarrow |F(x,0,v)| < \varepsilon |v|^2.$$

Then $|\int_\Omega F(x,0,v)dx| < \int_\Omega |F(x,0,v)|dx < \int_\Omega \varepsilon |v|^2 dx < \frac{\varepsilon}{\lambda_1}\|v\|^2$ and hence

$$\begin{aligned} I(0,v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{c}{2}\int_\Omega v^2 dx \\ &\quad -\frac{1}{2}\int_\Omega (\eta_1(v^+)^2 - \eta_2(v^-)^2)dx - \int_\Omega F(x,0,v)dx \\ &> \frac{1}{2}\|v\|^2 - \frac{c+\eta_1+\eta_2}{2\lambda_1}\|v\|^2 - \frac{\varepsilon}{\lambda_1}\|v\|^2 \\ &= \frac{1}{2}(1 - \frac{c+\eta_1+\eta_2+2\varepsilon}{\lambda_1})\|v\|^2 > 0 \end{aligned}$$

which gives the result for sufficiently small ε. Therefore we can choose $0 < \rho_1 < \rho$ such that $I(0,v) > 0$ for any $\|v\| = \rho_1$. □

LEMMA 4.2. *Assume F satisfies (F1). If a, b, c, δ_1, δ_2, η_1, and η_2 are positive, then there exists an $R > 0$ such that for any $R_1 > R$*

$$\sup_{\partial Q_{R_1}(H_1, e_1^2)} I < 0.$$

Proof. In the following we denote different constants by C_1, C_2 etc. Remark implies that

$$\begin{aligned}
I(u,\beta e_1) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda_1\beta^2}{2} - \frac{1}{2}\int_\Omega au^2 dx - b\lambda_1\beta - \frac{c\beta^2}{2}\\
&\quad -\frac{1}{2}\int_\Omega (\delta_1(u^+)^2 - \delta_2(u^-)^2)dx\\
&\quad -\frac{1}{2}\int_\Omega (\eta_1((\beta e_1)^+)^2 - \eta_2((\beta e_1)^-)^2)dx - \int_\Omega F(x,u,\beta e_1)dx\\
&\le \frac{1}{2}\|u\|^2 + \frac{\lambda_1\beta^2}{2} - b\lambda_1\beta + \frac{\delta_2}{2}\int_\Omega (u^-)^2 dx\\
&\quad +\frac{\eta_2}{2}\int_\Omega ((\beta e_1)^-)^2 dx - \int_\Omega F(x,u,\beta e_1)dx\\
&\le \frac{1}{2}\|u\|^2 + \frac{\lambda_1\beta^2}{2} - b\lambda_1\beta + \frac{\delta_2}{2\lambda_1}\|u\|^2 + \frac{\eta_2\beta^2}{2\lambda_1}\\
&\quad -b_1\int_\Omega (|u|^\alpha + |\beta e_1|^\alpha)dx + C_1\\
&\le \frac{\lambda_1+\delta_2}{2\lambda_1}\|u\|^2 + \frac{(\lambda_1^2+\eta_2)\beta^2}{2\lambda_1} - b\lambda_1\beta - C_2\|u\|^\alpha - C_3|\beta|^\alpha + C_4,
\end{aligned}$$

for any $(u,0) \in H_1$ and any constant β. Since $\alpha > 2$, $I(u,\beta e_1) \to -\infty$ for $\|u\| \to \infty$ or $|\beta| \to \infty$. Therefore we can choose $0 < R_1 < \infty$ such that $I(u,\beta e_1) < 0$ for any $\|(u,\beta e_1)\|_E = R_1$. □

Proof of Theorem 2.1.
By Lemma 4.1 and 4.2, there exists $0 < \rho_1 < R_1$ such that

$$\sup_{\partial Q_{R_1}(H_1,e_1^2)} I < 0 < \inf_{\partial B_{\rho_1}(H_2)} I.$$

By Theorem 1.1, $I(u,v)$ has at least two nonzero critical values c_1, c_2

$$\inf_{B_{\rho_1}(H_2)} I \le c_1 \le \sup_{\partial Q_{R_1}(H_1,e_1^2)} I < \inf_{\partial B_{\rho_1}(H_2)} I \le c_2 \le \sup_{Q_{R_1}(H_1,e_1^2)} I.$$

Therefore, (1) has at least two nontrivial solutions. □

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