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# QUADRATIC RESIDUE CODES OVER $p$-ADIC INTEGERS AND THEIR PROJECTIONS TO INTEGERS MODULO $p^e$

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Abstract. We give idempotent generators for quadratic residue codes over $p$-adic integers and over the rings $\mathbb{Z}_{p^e}$.

## 1. Introduction

Let $R$ be a ring. A *code* of length $n$ over $R$ is a $R$-submodule of $R^n$. For generality on codes over fields, we refer [5] and [8]. For codes over $\mathbb{Z}_m$, see [3, 12], and for self dual codes, see [11]. See [1, 4] for codes over $p$-adic numbers.

Quadratic residue codes are cyclic codes of prime length $n$ defined over a finite field $\mathbb{F}_{p^e}$, where $p^e$ is a quadratic residue mod $n$. They comprise a very important family of codes. Examples of quadratic residue codes include the binary [7,4,3] Hamming code, the binary [23,12,7] Golay code, the ternary [11,6,5] Golay code and the quaternary Hexacode. Quadratic residue codes have rate close to 1/2 and tend to have high minimum distance. Extended quadratic residue codes are self-dual.

Denote by $\mathbb{Z}_{p^e}$ the ring of integers modulo $p^e$, and $\mathbb{Z}_{p^\infty}$ the ring of $p$-adic integers. In next section we are going to generalize these quadratic

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residue codes over the field $\mathbb{F}_p$ to rings $\mathbb{Z}_{p^e}$ and to the $p$-adic integers $\mathbb{Z}_{p^\infty}$.

In early papers [2,5,6,10,13], authors tried to generalize the quadratic residue codes to the rings $\mathbb{Z}_4, \mathbb{Z}_8, \mathbb{Z}_{16}, \mathbb{Z}_9$ by giving idempotent generators. In [7], author defined quadratic residue codes over the rings $\mathbb{Z}_{p^e}$ and $p$-adic integer ring $\mathbb{Z}_{p^\infty}$ in general and gave generating polynomials. In this article, we give their idempotent generators.

## 2. Quadratic residue codes over $\mathbb{Z}_{p^e}$

In the earlier works by several authors, quadratic residue codes over $\mathbb{Z}_{p^e}$ are usually defined by giving idempotent generators. See [2, 10] for quadratic residue codes over $\mathbb{Z}_8$, $\mathbb{Z}_{16}$ and [13] for codes over $\mathbb{Z}_9$ for example. However it is generally difficult to give a formula for such generators and hard to understand. We will define quadratic residue codes over $\mathbb{Z}_{p^e}$ in a similar way as in the field case. The $p$-adic case ($e = \infty$) is also included here. For codes over $p$-adic integers, we refer [1, 3, 4].

Let $p$ be a prime and let $n$ be a prime such that $p$ is a quadratic residue modulo $n$. Let $Q$ be the set of quadratic residues modulo $n$, and $N$ the set of quadratic nonresidues modulo $n$.

Let $\mathbb{Q}_p$ denote the field of $p$-adic numbers. Let $K$ be the splitting field of $x^n - 1$ over $\mathbb{Q}_p$. Since the roots of $x^n - 1$ in $K$ form a multiplicative group of order $n$, it is clear that there exists an element $\zeta$ such that $K = \mathbb{Q}_p[\zeta]$. By considering the map $\Psi_e : \mathbb{Z}_{p^\infty} \to \mathbb{Z}_{p^e}$ defined by $\Psi_e(a) = a \pmod{p^e}$ and extending it to $\mathbb{Z}_{p^\infty}[\zeta]$, we can easily see that

$$\mathbb{Z}_{p^e}[\zeta] \simeq \mathbb{Z}_{p^\infty}[\zeta]/(p^e).$$

$\mathbb{Z}_{p^e}[\zeta]$ is a Galois ring defined over $\mathbb{Z}_{p^e}$. Elements in $\mathbb{Z}_{p^e}[\zeta]$ can be written uniquely as a $\zeta$-adic expansion $u = \sum_{i=0}^{p-1} v_i\zeta^i$, $v_i \in \mathbb{Z}_{p^e}$ or in a $p$-adic expansion

$$u = u_0 + pu_1 + p^2u_2 + \cdots + p^{e-1}u_{e-1}$$

where $u_i \in \{0, 1, \zeta, \cdots, \zeta^{p-1}\} \simeq \mathbb{F}_p$, the finite field of $p$ elements. In $p$-adic integer case, this sum is infinite. The automorphism group of $\mathbb{Z}_{p^e}[\zeta]$ over $\mathbb{Z}_{p^e}$ is the cyclic group generated by the Frobenius automorphism

$$\mathcal{F}(\sum_{i=0}^{e-1} p^iu_i) = \sum_{i=0}^{e-1} p^iu_i^p.$$

We refer [1] or [9] for details. As in the field case, we let

$$Q_e(x) = \prod_{i \in Q} (x - \zeta^i), \quad N_e(x) = \prod_{i \in N} (x - \zeta^i).$$

Since $p \in Q$ we have

$$\mathcal{F}(Q_e(x)) = \prod_{i \in Q} (x - \zeta^{pi}) = \prod_{i \in Q} (x - \zeta^i) = Q_e(x)$$

and similarly $\mathcal{F}(N_e(x)) = N_e(x)$. Thus $Q_e(x)$ and $N_e(x)$ are polynomials in $\mathbb{Z}_{p^e}[x]$. We certainly have that

$$x^n - 1 = (x - 1)Q_e(x)N_e(x)$$

and for all $e' \geq e$,

$$Q_{e'}(x) \equiv Q_e(x) \pmod{p^e}, \quad N_{e'}(x) \equiv N_e(x) \pmod{p^e}.$$

$Q_\infty(x)$ and $N_\infty(x)$ may be defined as $p$-adic limits of $Q_e(x)$ and $N_e(x)$.

Definition 2.1. Cyclic codes $\mathcal{Q}^e, \mathcal{Q}_1^e, \mathcal{N}^e, \mathcal{N}_1^e$ of length $n$ with generator polynomials

$$Q_e(x), \quad (x-1)Q_e(x), \quad N_e(x), \quad (x-1)N_e(x),$$

respectively, are called **quadratic residue codes** over $\mathbb{Z}_{p^e}$.

## 3. Main Theorem

Let

$$f_Q(x) = \sum_{i \in Q} x^i, \quad f_N(x) = \sum_{i \in N} x^i.$$

the polynomials in $\mathbb{Z}_{p^e}[x]/(x^n - 1)$, where $e = 1, 2, \ldots, \infty$.

Theorem 3.1. *1. Suppose $n = 4k - 1$.*

$$f_Q^2 = \frac{n-3}{4} f_Q + \frac{n+1}{4} f_N$$
$$f_N^2 = \frac{n+1}{4} f_Q + \frac{n-3}{4} f_N$$
$$f_Q f_N = \frac{n-1}{2} + \frac{n-3}{4} f_Q + \frac{n-3}{4} f_N$$

*2. Suppose $n = 4k + 1$.*

$$f_Q^2 = \frac{n-5}{4} f_Q + \frac{n-1}{4} f_N + \frac{n-1}{2}$$
$$f_N^2 = \frac{n-1}{4} f_Q + \frac{n-5}{4} f_N + \frac{n-1}{2}$$
$$f_Q f_N = \frac{n-1}{4} f_Q + \frac{n-1}{4} f_N$$

*Proof.* It follows from the Perron's theorem. □

Let

$$\lambda = f_Q(\zeta) = \sum_{i \in Q} \zeta^i, \quad \mu = f_N(\zeta) = \sum_{i \in N} \zeta^i$$

Diffenrent choice of the root $\zeta$ may interchange $\lambda$ and $\mu$. Let

$$\theta = \lambda - \mu.$$

Then

$$\theta^2 = \pm n$$

for $n = 4k \pm 1$, where double signs are in the same order.

THEOREM 3.2. *1. If $n = 4k - 1$, then $\lambda$ and $\mu$ are roots of $x^2 + x + k = 0$.*
*2. If $n = 4k + 1$, then $\lambda$ and $\mu$ are roots of $x^2 + x - k = 0$.*

Note that $\mu + \lambda = -1$. For details, we refer [7].

THEOREM 3.3. *Let $p > 2$ be a prime and and $n = 4k \pm 1$ be a prime such that $p$ is a quadratic residue modulo $n$. Let $\theta^2 \equiv \pm 1 \pmod{p}$, where double signs are in the same order as in $n = 4k \pm 1$. The idempotent generators of the $p$-adic quadratic residue codes $\langle Q_\infty(x) \rangle, \langle (x-1)Q_\infty(x) \rangle, \langle N_\infty(x) \rangle, \langle (x-1)N_\infty(x) \rangle$ of length $n$ are given as follows, respectively:*

$$E_q(x) = a + b f_Q(x) + c f_N(x)$$
$$F_q(x) = a' - c f_Q(x) - b f_N(x)$$
$$E_n(x) = a + c f_Q(x) + b f_N(x)$$
$$F_n(x) = a' - b f_Q(x) - c f_N(x)$$

*where*

$$a = \frac{n+1}{2n}, \quad a' = \frac{n-1}{2n}, \quad b = \frac{1 \mp \theta}{2n}, \quad c = \frac{1 \pm \theta}{2n}.$$

*The idempotent generators of quadratic residue codes over $\mathbb{Z}_{p^e}$ can be obtained by projecting these generators modular $p^e$.*

*Proof.* We prove the formula for $E_q(x)$ in the case that $n = 4k - 1$. Let

$$E = 1 + f_Q(x) + f_N(x) + n + \theta(f_Q(x) - f_N(x)).$$

It is a lengthy but straightforward computation to show that $E^2 = 2nE$ using Theorem 3.1 and $\theta^2 = -n$. Therefore $(\frac{E}{2n})^2 = \frac{E}{2n}$. But $\frac{E}{2n} = E_q(x)$. Thus $E_q(x)$ is idempotent. Next, note that $1 + f_Q(x) + f_N(x) = Q_\infty(x)N_\infty(x)$. Thus for all $i \in Q$, we have $E(\zeta^i) = 0 + n + \theta(\lambda - \mu) = n + \theta^2 = 0$. For all $i \in N$, we have $E(\zeta^i) = 0 + n + \theta(\mu - \lambda) = n - \theta^2 = 2n$. Thus $E_q(\zeta^i) = 0$ if $i \in Q$ and $E_q(\zeta^i) = 1$ if $i \in N$. We also have that $E_q(1) = 1$. Thus $E_q(x) = V(x)Q_\infty(x)$ for some $V(x)$ and $E_q(x)$ is relatively prime to $N_\infty(x)(x-1)$. Therefore there exist $A(x), B(x)$ such that $A(x)E_q(x) + B(x)N_\infty(x)(x-1) = 1$. From this we get $A(x)E_q(x)Q(x) = Q(x)$. Hence $\langle E_q(x)\rangle = \langle Q_\infty(x)\rangle$.

All remaining cases can be proved in a similar way. □

Note that an idempotent generator for the binary case is given in [1].

## 4. An example

In this section, we use our Theorem 3.3 to find idempotent generators of the quadratic residue codes over $\mathbb{Z}_9$ as in [13].

First we note that $\left(\dfrac{n}{3}\right) = 1$ iff $n = 12r \pm 1$ for some $r$. In order to solve $\theta^2 \equiv \pm n \pmod 9$, we need to separate cases further according to $r$ modulo 3. We compute everything modulo 9.

Case I. $n = 12r - 1$.

1. $r = 3j$: $(n = 36j - 1)$.

   In this case $n = 36j - 1 = -1$. Inverse of $2n = -2$ is 4. Thus $a = 4(n+1) = 0$, $a' = 4(n-1) = 1$. Solving $\theta^2 = -n = 1$, we obtain $\theta = \pm 1$. Thus $b, c = 4(1 \pm \theta) = 8, 0$. Hence the idempotent generators of quadratic residue codes are

$$8f_Q, \quad 8f_N, \quad 1 - 8f_Q, \quad 1 - 8f_N.$$

2. $r = 3j + 1$: $(n = 36j + 11)$.

   In this case $n = 2$, and the inverse of $2n$ is 7. Thus $a = 3$ and $a' = 7$. From $\theta^2 = -n = 7$, we get $\theta = \pm 4$. Thus $b, c = 7(1 \pm 4) =$

8, 6. Thus the idempotent generators of quadratic residue codes are

$$3 + 8f_Q + 6f_N, \quad 3 + 6f_Q + 8f_N, \quad 7 + f_Q + 3f_N, \quad 7 + 3f_Q + f_N.$$

3. $r = 3j + 2$: $(n = 36j + 23)$.

Similarly, we find that the idempotent generators of quadratic residue codes for this case are

$$6 + 3f_Q + 8f_N, \quad 6 + 8f_Q + 3f_N, \quad 4 + 6f_Q + 1f_N, \quad 4 + 1f_Q + 6f_N.$$

Case II. $n = 12r + 1$.

1. $r = 3j$: $(n = 36j + 1)$.

In this case $n = 1$. Inverse of $2n = 2$ is 5. Thus $a = 1$, $a' = 0$. Solving $\theta^2 = n$, we obtain $\theta = \pm 1$. Thus $b, c = 5(1 \pm \theta) = 0, 1$. Hence the idempotent generators of quadratic residue codes are

$$1 + f_N, \quad 1 + f_Q, \quad 8f_N, \quad 8f_Q.$$

2. $r = 3j + 1$: $((n = 36j + 13)$.

The idempotent generators of quadratic residue codes are

$$4 + 6f_Q + f_N, \quad 4 + f_Q + 6f_N, \quad 6 + 3f_Q + 8f_N, \quad 6 + 8f_Q + 3f_N.$$

3. $r = 3j + 2$: $(n = 36j + 25)$.

The idempotent generators of quadratic residue codes for this case are

$$7 + f_Q + 3f_N, \quad 7 + 3f_Q + f_N, \quad 3 + 8_Q + 6f_N, \quad 3 + 6f_Q + 8f_N.$$

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