CAYLEY-SYMMETRIC SEMIGROUPS

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Abstract. The concept of Cayley-symmetric semigroups is introduced, and several equivalent conditions of a Cayley-symmetric semigroup are given so that an open problem proposed by Zhu [19] is resolved generally. Furthermore, it is proved that a strong semilattice of self-decomposable semigroups $S_\alpha$ is Cayley-symmetric if and only if each $S_\alpha$ is Cayley-symmetric. This enables us to present more Cayley-symmetric semigroups, which would be non-regular. This result extends the main result of Wang [14], which stated that a regular semigroup is Cayley-symmetric if and only if it is a Clifford semigroup. In addition, we discuss Cayley-symmetry of Rees matrix semigroups over a semigroup or over a 0-semigroup.

1. Introduction

Many research papers were devoted to Cayley graphs of semigroups (see, for example, [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17]). Based on these works, the author first introduced the concept of generalized Cayley graphs of semigroups in [18], where some fundamental properties of generalized Cayley graphs of semigroups were studied. This work was extended in [19], where various combinatorial issues relating to generalized Cayley graphs were addressed. Especially, in Remark 3.8 of [19], the author proposed the following question: It may be interesting to characterize semigroups $S$ such that $Cay(S, S_l) = Cay(S, S_r)$, where $S_l$ and $S_r$ denote the left and right universal relations on $S$ respectively. This problem was partially solved by Wang in [14], where it was proved that for any regular semigroup $S$, $Cay(S, S_l) = Cay(S, S_r)$ if and only if $S$ is a Clifford semigroup.

Following [14, 18, 19], we continue to study the generalized Cayley graphs of semigroups in the present paper. Based on the problem mentioned as above, we introduce naturally the concept of Cayley-symmetric semigroups, in view of which the above problem may be restated as follows: When is a semigroup
Cayley-symmetric? Consequently, the main result of Wang [14] would be restated as follows: A regular semigroup is Cayley-symmetric if and only if it is a Clifford semigroup. As the most general answers to the above problem, several equivalent conditions of a Cayley-symmetric semigroup are given in this paper. We also generalize the notion of Clifford semigroups and establish a necessary and sufficient condition for a semilattices of semigroups to be Cayley-symmetric. Consequently, we can display some Cayley-symmetric semigroups which are not regular. In this sense, our results also extend the main result of [14]. In addition, the Cayley-symmetry of Rees matrix semigroups over a semigroup or over a 0-semigroup is also discussed so that the corresponding equivalent conditions are given.

2. Preliminaries

Recall that if $S$ is an ideal of a semigroup $T$, then we call $T$ an ideal extension of $S$. Let $T^1$ be the semigroup $T$ with an identity adjoined if necessary.

Definition 2.1 ([18, 19]). Let $T$ be an ideal extension of a semigroup $S$ and $\rho \subseteq T^1 \times T^1$. The Cayley graph $Cay(S, \rho)$ of $S$ relative to $\rho$ is defined as the graph with vertex set $S$ and edge set $E(Cay(S, \rho))$ consisting of those ordered pairs $(a, b)$, where $ax = y$ for some $(x, y) \in \rho$. We also call the Cayley graphs defined in this way the generalized Cayley graphs, in order to distinguish them from the usual ones.

Assume that $S$ is a semigroup and $a \in S$. Then $J(a) = S^1 a S^1$ ($L(a) = S^1 a$, $R(a) = a S^1$) is the principal (left, right) ideal generated by $a$ (cf. [2, 20]). Let $S_l = S^1 \times \{1\}$ (the left universal relation on $S^1$), $S_r = \{1\} \times S^1$ (the right universal relation on $S^1$) and $\omega_S = S^1 \times S^1$ (the universal relation on $S^1$). Then the generalized Cayley graphs $Cay(S, S_l)$, $Cay(S, S_r)$ and $Cay(S, \omega_S)$ are called the left universal, right universal and universal Cayley graphs of $S$, respectively.

As mentioned in [19, Remark 3.8], it would be interesting to characterize semigroups $S$ such that $Cay(S, S_l) = Cay(S, S_r)$. We shall answer this question in general in this paper. It would be convenient to give the following definition.

Definition 2.2. A semigroup $S$ is called Cayley-symmetric if $Cay(S, S_l) = Cay(S, S_r)$.

Assume that $T$ is an ideal extension of a semigroup $S$. Then the generalized Cayley graphs $Cay(S, T_l)$, $Cay(S, T_r)$ and $Cay(S, T_\omega)$ are called the left $T$-universal, right $T$-universal and $T$-universal Cayley graphs of $S$, respectively.

We also consider a more general symmetry problem in this paper. As a natural generalization of Definition 2.2, we have the following definition.

Definition 2.3. Let $T$ be an ideal extension of a semigroup $S$. If $Cay(S, T_l) = Cay(S, T_r)$, then we say that $S$ is Cayley-symmetric in $T$.

Throughout the paper, graphs are directed graphs without multiple edges, but possibly with loops, or digraphs in terms of [1, 11]. As in [18, 19], we always
equate two graphs isomorphic to each other if no confusion occurs. For a graph $\Gamma$, denote by $V(\Gamma)$ and $E(\Gamma)$ its vertex set and edge set, respectively. A graph $\Gamma_0$ is called a subgraph of a graph $\Gamma$ if $V(\Gamma_0) \subseteq V(\Gamma)$ and $E(\Gamma_0) \subseteq E(\Gamma)$. A graph $\Gamma_0$ is called an induced subgraph of a graph $\Gamma$ if $\Gamma_0$ is a subgraph of $\Gamma$ and the following condition is satisfied: for any $a, b \in V(\Gamma_0)$, $(a, b) \in E(\Gamma_0)$ if and only if $(a, b) \in E(\Gamma)$. Any book on graph theory, e.g., [4, 15], will provide terminology which may be used in this paper without definition.

For notions and notations of semigroup theory, we refer the reader to [2].

3. Cayley-symmetric semigroups

In this section, we answer the following question in general: When is a semigroup Cayley-symmetric? More generally, we answer the following question: When is a semigroup Cayley-symmetric in its a given ideal extension?

Assume that $T$ is an ideal extension of a semigroup $S$ and that $A$ is a subset of $S$. Then the ideal, left ideal and right ideal generated by $A$ in $T$ are denoted by $J_T(A)$, $L_T(A)$ and $R_T(A)$, respectively. If $A = \{a\}$, then instead of writing $J_T(A)$, $L_T(A)$ and $R_T(A)$, we write $J_T(a)$, $L_T(a)$ and $R_T(a)$, which are called the ideal, left ideal and right ideal generated by $a$ in $T$, respectively. In the case that $S = T$, we write $J(A)$, $L(A)$, $R(A)$, $J(a)$, $L(a)$ and $R(a)$ instead of writing $J_T(A)$, $L_T(A)$, $R_T(A)$, $J_T(a)$, $L_T(a)$ and $R_T(a)$, respectively. It is clear that $J_T(a) = T^1aT^1$, $L_T(a) = T^a$ and $R_T(a) = aT^1$. Thus $J(a) = S^1aS^1$, $L(a) = S^1a$ and $R(a) = aS^1$ (cf. [2]). It is easy to show the following lemma.

Lemma 3.1. Let $T$ be an ideal extension of a semigroup $S$ and $A$ a subset of $S$. Then $J_T(A) = \bigcup_{a \in A} J_T(a)$, $L_T(A) = \bigcup_{a \in A} L_T(a)$, and $R_T(A) = \bigcup_{a \in A} R_T(a)$.

Furthermore, we can prove the next lemma.

Lemma 3.2. If $T$ is an ideal extension of a semigroup $S$, then the following statements are equivalent:

1. $L_T(a) = R_T(a)$ for every $a \in S$;
2. $L_T(A) = R_T(A)$ for every $A \subseteq S$;
3. $L_T(a)$ is a right ideal of $T$ and $R_T(a)$ is a left ideal of $T$ for every $a \in S$;
4. $L_T(A)$ is a right ideal of $T$ and $R_T(A)$ is a left ideal of $T$ for every $A \subseteq S$.

Proof. $(1) \Rightarrow (2)$: In view of Lemma 3.1, we have $L_T(A) = \bigcup_{a \in A} L_T(a)$ and $R_T(A) = \bigcup_{a \in A} R_T(a)$, which together with the condition (1) implies that

$$L_T(A) = \bigcup_{a \in A} L_T(a) = \bigcup_{a \in A} R_T(a) = R_T(A).$$

$(2) \Rightarrow (1)$: This is trivial.

$(1) \Rightarrow (3)$: This is obvious since $L_T(a)$ is a left ideal and $R_T(a)$ is a right ideal of $T$.  

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Corollary 3.4. which answers the open question raised by the author in [19].

(3)⇒(1): Note that $L_T(a) \subseteq J_T(a)$ by definitions. Since $L_T(a)$ is a right ideal of $T$, $J_T(a) = T^1aT^1 = L_T(a)T^1 \subseteq L_T(a)$ and only if $(a, b) \in E(\text{Cay}(S, T_1))$ if and only if $(a, b) \in E(\text{Cay}(S, T_r))$. Thus for all $a, b \in S$ we have that $b \in L_T(a)$ if and only if $b \in R_T(a)$. Since $S$ is an ideal of $T$, $L_T(a) = T^1a \subseteq S$ and $R_T(a) = aT^1 \subseteq S$ for every $a \in S$. So $L_T(a) = R_T(a)$ for every $a \in S$.

Conversely, suppose that $L_T(a) = R_T(a)$ for every $a \in S$. Then for all $a, b \in S$ we have that $b \in L_T(a)$ if and only if $b \in R_T(a)$. Hence $(a, b) \in E(\text{Cay}(S, T_1))$ if and only if $(a, b) \in E(\text{Cay}(S, T_r))$. That is to say that $\text{Cay}(S, T_1) = \text{Cay}(S, T_r)$.

□

As the main result of this section, the following theorem characterizes the semigroups that are Cayley-symmetric in their ideal extensions.

**Theorem 3.3.** If $T$ is an ideal extension of a semigroup $S$, then the following statements are equivalent:

1. $S$ is Cayley-symmetric in $T$;
2. $L_T(a) = R_T(a)$ for every $a \in S$;
3. $L_T(A) = R_T(A)$ for every $A \subseteq S$;
4. $L_T(a)$ is a right ideal of $T$ and $R_T(a)$ is a left ideal of $T$ for every $a \in S$;
5. $L_T(A)$ is a right ideal of $T$ and $R_T(A)$ is a left ideal of $T$ for every $A \subseteq S$.

**Proof.** In light of Lemma 3.2, we only need to prove the equivalence of (1) and (2). According to the definition of generalized Cayley graphs, we know that

$$(a, b) \in E(\text{Cay}(S, T_1))$$

$\iff b = xa$ for some $x \in T^1$

$\iff b \in T^1a = L_T(a)$$

and that

$$(a, b) \in E(\text{Cay}(S, T_r))$$

$\iff b = ay$ for some $y \in T^1$

$\iff b \in aT^1 = R_T(a)$.

If $\text{Cay}(S, T_1) = \text{Cay}(S, T_r)$, then for any $a, b \in S$, $(a, b) \in E(\text{Cay}(S, T_1))$ if and only if $(a, b) \in E(\text{Cay}(S, T_r))$. Thus for all $a, b \in S$ we have that $b \in L_T(a)$ if and only if $b \in R_T(a)$. Since $S$ is an ideal of $T$, $L_T(a) = T^1a \subseteq S$ and $R_T(a) = aT^1 \subseteq S$ for every $a \in S$. So $L_T(a) = R_T(a)$ for every $a \in S$.

Conversely, suppose that $L_T(a) = R_T(a)$ for every $a \in S$. Then for all $a, b \in S$ we have that $b \in L_T(a)$ if and only if $b \in R_T(a)$. Hence $(a, b) \in E(\text{Cay}(S, T_1))$ if and only if $(a, b) \in E(\text{Cay}(S, T_r))$. That is to say that $\text{Cay}(S, T_1) = \text{Cay}(S, T_r)$.

□

Using Theorem 3.3 with $T = S$, we immediately have the following corollary, which answers the open question raised by the author in [19].

**Corollary 3.4.** For any semigroup $S$, the following statements are equivalent:

1. $S$ is Cayley-symmetric;
2. $L(a) = R(a)$ for every $a \in S$;
(3) \( L(A) = R(A) \) for every \( A \subseteq S \);
(4) \( L(a) \) is a right ideal and \( R(a) \) is a left ideal for every \( a \in S \);
(5) \( L(A) \) is a right ideal and \( R(A) \) is a left ideal for every \( A \subseteq S \);
(6) Every left ideal is a right ideal and every right ideal is a left ideal.

In the remainder of this section, we present some Cayley-symmetric semigroups.

Suppose that \( S \) is a semigroup with zero 0 and that \( S = \bigcup_{\alpha \in I} S_\alpha \), where each \( S_\alpha \) is a subsemigroup of \( S \), and where \( S_i S_j = S_i \cap S_j = \{0\} \).

Then we call \( S \) is a 0-direct union of \( S_\alpha \)'s (\[2\]). If we further suppose that every \( S_\alpha \) is Cayley-symmetric, then \( S_i^0 a = a S_i^0 \) for all \( a \in S_\alpha \) by Corollary 3.4. It follows that \( S^0 a = a S^0 \) for all \( a \in S \). Hence \( S \) is Cayley-symmetric by Corollary 3.4 again. Consequently, we obtain the following corollary.

**Corollary 3.5.** Assume that \( S \) is a 0-direct union of semigroups \( S_\alpha \)'s. If every \( S_\alpha \) is Cayley-symmetric, then so is \( S \).

If \( G \) is a group, then we have \( aG = G = Ga \) for every \( a \in G \), which means that \( G \) is Cayley-symmetric by Corollary 3.4. Let \( G^0 = G \cup \{0\} \), where 0 is an adjoined zero. Then we call \( G^0 \) a 0-group. Also we have \( aG^0 = G^0 a \) for every \( a \in G^0 \), which means that \( G^0 \) is also Cayley-symmetric by Corollary 3.4 again. As a direct consequence of Corollary 3.5, we have the following corollary.

**Corollary 3.6.** If \( S \) is a 0-direct union of some 0-groups, then \( S \) is Cayley-symmetric.

### 4. Cayley-symmetry of Rees matrix semigroups

This section is devoted to the Cayley-symmetry of Rees matrix semigroups so that a necessary and sufficient condition is given for a Rees matrix semigroup to be Cayley-symmetric. The main results of this section are Theorems 4.4 and 4.7.

Let \( S \) be a semigroup, let \( I, \Lambda \) be nonempty sets and let \( P = (p_{\lambda i}) \) be a \( \Lambda \times I \) matrix with entries in \( S^1 \). (Note that here is \( S^1 \) rather than \( S \).) Let \( T = I \times S \times \Lambda \), and define a multiplication on \( T \) by

\[
(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).
\]

Then \( T \) is a semigroup, which is called the \( I \times \Lambda \) Rees matrix semigroup over the semigroup \( S \) with the sandwich matrix \( P \) and denoted by \( M[S; I, \Lambda; P] \). Recall that a semigroup is called completely simple if it is simple and if it contains a primitive idempotent. By \[2, \text{Theorem 3.3.1}\], a semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup \( M[G; I, \Lambda; P] \) over a group \( G \).

To study the Cayley-symmetry of a Rees matrix semigroup over a semigroup, we need some new terminologies. As a generalization of the identity of a semigroup, a mid-identity of a semigroup is defined as follows.
Definition 4.1. An element \( u \) of a semigroup \( S \) is called a mid-identity, if for all \( x, y \in S \), \( xuy = xy \).

The next terms generalize the concept of invertible elements of a group.

Definition 4.2. Let \( S \) be a semigroup and \( p \in S \). If there exists \( q \in S \) such that \( pq \) (\( qp \)) is a mid-identity, then \( p \) is called left (right) factor of a mid-identity. If \( p \) is not only a left factor of a mid-identity but also a right factor of a mid-identity, then \( p \) is called mid-invertible.

It is clear that the identity of a semigroup is a mid-identity and it is also mid-invertible. Note that if \( S \) is a semigroup with an identity \( 1 \) and with a mid-identity \( u \), then \( 1u1 = 11 \) which means that \( u = 1 \). A non-trivial example is as follows.

Example 4.3. Let \( S = \langle p, q \rangle \) be the matrix semigroup (under the usual multiplication of matrices) generated by two matrices \( p = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \) and \( q = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). Then \( u = pq = qp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) is a mid-identity rather than an identity of \( S \). Furthermore, both \( p \) and \( q \) are mid-invertible.

As one of two main results of this section, the next theorem gives an equivalent condition for a Rees matrix semigroup over a semigroup to be Cayley-symmetric.

Theorem 4.4. Let \( T = M[S; I, \Lambda; P] \) be an \( I \times \Lambda \) Rees matrix semigroup over a semigroup \( S \), where the sandwich matrix \( P = (p_{\lambda i}) \) is a \( \Lambda \times I \) matrix with entries in \( S^1 \). Assume that \( p_{11} = p \) is a mid-invertible element of \( S^1 \). Then \( T \) is Cayley-symmetric if and only if \( |I| = |\Lambda| = 1 \) and \( S \) is Cayley-symmetric.

Proof. Necessity. Suppose that \( T \) is Cayley-symmetric. By (4.1), one can deduce that for any \( i \in I, \lambda \in \Lambda, a \in S \),
\[
(i, a, \lambda) T^1 = \{(i, ap_{\lambda j} b, \mu) \mid j \in I, \mu \in \Lambda, b \in S\},
\]
\[
T^1(i, a, \lambda) = \{(j, bp_{\mu a}, \lambda) \mid j \in I, \mu \in \Lambda, b \in S\}.
\]
Since \( T \) is Cayley-symmetric, we have
\[
(i, a, \lambda) T^1 = T^1(i, a, \lambda)
\]
by Corollary 3.4. From (4.2), (4.3) and (4.4), it follows that \( |I| = |\Lambda| = 1 \).
Thus we may set \( I = \{1\} \) and \( \Lambda = \{1\} \). Then we have that
\[
\{(1, a, 1), (1, ap b, 1) \mid b \in S\} = \{(1, a, 1), (1, cpa, 1) \mid c \in S\}
\]
holds for all \( a \in S \). It follows that for every \( a \in S \),
\[
\{a, ap b \mid b \in S\} = \{a, cpa \mid c \in S\}.
\]
By the assumption of the theorem, we can suppose that there exist \( u, v, q, r \in S \) such that \( u, v \) are mid-identity, and that \( pq = u \) and \( rp = v \). Thus for any \( a, b \in S \), according to (4.6), we have that
\[
ab = aub = apqb = \begin{cases} a \\ cpa. \end{cases}
\]
This shows that \( aS^1 \subseteq S^1a \). A symmetric argument shows that \( S^1a \subseteq aS^1 \). Hence \( aS^1 = S^1a \). So we have that \( S \) is Cayley-symmetric by Corollary 3.4.

**Sufficiency.** Assume that \( I = \{1\} \) and \( \Lambda = \{1\} \) and that \( S \) is Cayley-symmetric. To show that \( T \) is Cayley-symmetric, it only remains to prove that (4.6) holds. For this, let \( a, b \in S \). By Corollary 3.4 again, we obtain that
\[
\{a, ab \mid b \in S\} = \{a, ca \mid c \in S\}.
\]
Then \( apb = a \) or \( apb = ca \) (for some \( c \in S \)). If the case is the former, then it is easy to see that
\[
\{a, apb \mid b \in S\} \subseteq \{a, cpa \mid c \in S\}.
\]
If it is the latter case, i.e., if there exists \( c \in S \) such that \( apb = ca \), then we have that
\[
apb = ca = cva = arpa,
\]
which shows that (4.8) remains true. By a symmetric argument, we can prove the inverse conclusion of (4.8). Therefore, (4.6) holds. This completes the proof. \( \square \)

Note that each element of a group is mid-invertible. So we have the following corollary immediately.

**Corollary 4.5.** A completely simple semigroup is Cayley-symmetric if and only if it is isomorphic to a group.

Observe that if \( p_{\lambda i} = 1 \) for all \( i \in I \) and \( \lambda \in \Lambda \), then \( T = M[S; I, \Lambda; P] \) is isomorphic to the direct product of \( S \) and the rectangular band \( B = I \times \Lambda \). Consequently, we obtain the following corollary.

**Corollary 4.6.** A direct product of a semigroup \( S \) and a rectangular band \( B \) is Cayley-symmetric if and only if \( |B| = 1 \) and \( S \) is Cayley-symmetric.

Let us turn our attention to the Cayley-symmetry of a Rees matrix semigroup over a 0-semigroup \( S^0 \) (\( S^0 = S \cup \{0\} \) where 0 is an adjoined zero).

Assume that \( S \) is a semigroup. Let \( I, \Lambda \) be nonempty sets and let \( P = (p_{\lambda i}) \) be a \( \Lambda \times I \) matrix with entries in \( S^{1,0} \) (\( S^{1,0} = S^1 \cup \{0\} \)). Suppose that \( P \) is regular, in the sense that no row or column of \( P \) consists entirely of zeros. Formally,
\[
(\forall i \in I)(\exists \lambda \in \Lambda)p_{\lambda i} \neq 0,
(\forall \lambda \in \Lambda)(\exists i \in I)p_{\lambda i} \neq 0.
\]

(4.9)
Let $T^0 = (I \times S \times \Lambda) \cup \{0\}$, and define a composition on $T^0$ by

$$
(i, a, \lambda)(j, b, \mu) = \begin{cases} 
(i, a p_{\lambda j} b, \mu), & \text{if } p_{\lambda j} \neq 0 \\
0, & \text{if } p_{\lambda j} = 0.
\end{cases}
$$

Then $T^0$ is a semigroup, which is called the $I \times \Lambda$ Rees matrix semigroup over the 0-semigroup $S^0$ with the regular sandwich matrix $P$ and denoted by $M^0[S; I, \Lambda; P]$. Recall that a semigroup is called completely 0-simple if it is 0-simple and if it contains a primitive idempotent. By Rees Theorem (see [2, Theorem 3.2.3]), a semigroup is completely 0-simple if and only if it is isomorphic to a Rees matrix semigroup $M^0[G; I, \Lambda; P]$ over a 0-group $G^0$.

As the other main result of this section, the next theorem gives an equivalent condition of a Cayley-symmetric Rees matrix semigroup over a 0-semigroup.

**Theorem 4.7.** Let $T^0 = M^0[S; I, \Lambda; P]$ be an $I \times \Lambda$ Rees matrix semigroup over a 0-semigroup $S^0$, where the sandwich matrix $P = (p_{\lambda i})$ is a regular $\Lambda \times I$ matrix with entries in $S^1$. Assume that $p_{11} = p$ is a mid-invertible element of $S^{1,0}$. Then $T^0$ is Cayley-symmetric if and only if $|I| = |\Lambda| = 1$ and $S$ is Cayley-symmetric.

**Proof.** It is analogous to the proof of Theorem 4.4. \qed

As a direct consequence of Theorem 4.7, we have the following corollary.

**Corollary 4.8.** A completely 0-simple semigroup is Cayley-symmetric if and only it is isomorphic to a 0-group.

### 5. Cayley-symmetry of strong semilattices of semigroups

In this section, we shall introduce the concept of strong semilattices of semigroups, which is a natural generalization of the notion of Clifford semigroups. The main result of this section is Theorem 5.3, which gives a necessary and sufficient condition for a strong semilattice of semigroups to be Cayley-symmetric. Consequently, we can display more Cayley-symmetric semigroups which would not be regular. This means that our construction will provide a more universal class of Cayley-symmetric semigroups than the Clifford semigroups. In this sense, our result generalizes that of [14], which stated that a regular semigroup is Cayley-symmetric if and only if it is a Clifford semigroup.

Suppose that we have a semilattice $Y$ and a set of semigroups $S_\alpha$ indexed by $Y$, and suppose that, for all $\alpha \geq \beta$ in $Y$ there exists a morphism $\phi_{\alpha, \beta} : S_\alpha \to S_\beta$ such that: (1) for each $\alpha \in Y$, $\phi_{\alpha, \alpha} = 1_{S_\alpha}$, the identity mapping on $S_\alpha$; (2) $\phi_{\alpha, \beta} \phi_{\beta, \gamma} = \phi_{\alpha, \gamma}$ for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$. Let $S = \bigcup_{\alpha \in Y} S_\alpha$, the disjoint union of $S_\alpha$’s. Define a multiplication on $S$ by the rule that, for each $a \in S_\alpha$ and $b \in S_\beta$,

$$
ab = (a)\phi_{\alpha, \alpha \beta}(b)\phi_{\beta, \alpha \beta}.
$$

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$$
ab = (a)\phi_{\alpha, \alpha \beta}(b)\phi_{\beta, \alpha \beta}.
$$
Then $S$ is a semigroup, called the strong semilattice of semigroups $S_\alpha$ (cf. [18, Definition 3.11]). We write

$$S = S[Y; S_\alpha; \phi_{\alpha, \beta}].$$

To study Cayley-symmetry of a strong semilattice of semigroups, we need a new term, so called self-decomposable semigroups, which is also a generalization of the notion of monoids or that of regular semigroups.

**Definition 5.1.** Let $S$ be a semigroup. If for every $a \in S$, $a \in Sa \cap aS$, then we call $S$ self-decomposable.

Note that if $S$ is a self-decomposable semigroup, then $L(a) = S^1a = Sa$ and $R(a) = aS = aS$. Thus according to Corollary 3.4, we immediately obtain the following lemma, which provides a general answer to that problem in [19] for the self-decomposable semigroup class and will be used repeatedly later.

**Lemma 5.2.** Let $S$ be a self-decomposable semigroup. Then $S$ is Cayley-symmetric if and only if $Sa = aS$ for all $a \in S$.

As our main result of this section, the next theorem gives a necessary and sufficient condition for a strong semilattice of self-decomposable semigroups to be Cayley-symmetric.

**Theorem 5.3.** Suppose that $S = S[Y; S_\alpha; \phi_{\alpha, \beta}]$, where each $S_\alpha$ is self-decomposable. Then $S$ is Cayley-symmetric if and only if for every $\alpha \in Y$, $S_\alpha$ is Cayley-symmetric.

**Proof.** Observe that $S = S[Y; S_\alpha; \phi_{\alpha, \beta}]$ is self-decomposable since for every $\alpha$, $S_\alpha$ is self-decomposable.

**Necessity.** Assume that $S$ is Cayley-symmetric. Let $a, b \in S_\alpha$ with $\alpha \in Y$. According to Lemma 5.2, there exist $\beta \in Y$ and $c \in S_\beta$ such that $ab = ca$. It is clear that $\beta \geq \alpha$ since $ab \in S_\alpha$. Since $S_\alpha$ is self-decomposable, there exists $u \in S_\alpha$ such that $a = ua$. It follows that $ab = ca = c(ua) = (cu)a$, where $cu \in S_\alpha$. We have proved that $aS_\alpha \subseteq S_\alpha a$. Similarly, we have that $S_\alpha a \subseteq aS_\alpha$, which means that $S_\alpha a = aS_\alpha$. Thus by Lemma 5.2 again, $S_\alpha$ is Cayley-symmetric.

**Sufficiency.** Suppose that for all $\alpha$, $S_\alpha$ is Cayley-symmetric. Let $a \in S_\alpha$ and $b \in S_\beta$ with $\alpha, \beta \in Y$. Set $\alpha \beta = \gamma \in Y$, then $\alpha, \beta \geq \gamma$. Since $ab \in S_\gamma$ and $S_\gamma$ is self-decomposable, there exist $x, y \in S_\gamma$ such that $ab = x(ab) = (ab)y$. It follows that $ab = x(ab)y = (xa)(by)$, where $xa, by \in S_\gamma$. Since $S_\gamma$ is Cayley-symmetric, from Lemma 5.2 we deduce that there exists $z \in S_\gamma$ such that $(xa)(by) = z(xa) = (zx)a$, which implies that $ab = (zx)a \in Sa$. We have proved that $aS \subseteq Sa$. The inverse conclusion may be proved in a similar way. Hence $aS = Sa$. Therefore, by Lemma 5.2 again, $S$ is Cayley-symmetric. This completes the proof. □

Now suppose that $S$ is a Clifford semigroup. According to [2, Theorem 4.1], $S = S[Y; G_\alpha; \phi_{\alpha, \beta}]$, a strong semilattice of groups $G_\alpha$’s. Of course, every group
$G_\alpha$ is Cayley-symmetric. Thus using Theorem 5.3, we immediately obtain the following corollary, which is a half of the main theorem of Wang [14].

**Corollary 5.4** ([14]). Every Clifford semigroup is Cayley-symmetric.

Let us conclude our discussion by the following example, which shows that in light of Theorem 5.3, we can construct non-regular, non-commutative but Cayley-symmetric semigroups.

**Example 5.5.** Let $Y = \{\alpha, \beta, \gamma\}$ be a semilattice where the partial order $\geq$ is defined by

$\geq = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\alpha, \gamma), (\beta, \gamma)\}.$

Let $S_\alpha, S_\beta$ be two dihedral groups. Let $S_\gamma = \{1\} \cup \{2n \mid n \in \mathbb{Z}\}$. It is seen that $S_\gamma$ becomes a monoid with respect to the usual multiplication of integers. Define homomorphisms as follows:

$\phi_{\alpha, \gamma} : S_\alpha \rightarrow S_\gamma$, $a \mapsto 1$ for all $a \in S_\alpha$,

$\phi_{\beta, \gamma} : S_\beta \rightarrow S_\gamma$, $b \mapsto 1$ for all $b \in S_\beta$,

$\phi_{\delta, \delta} = 1_{S_\delta}$ for all $\delta \in Y$.

Let $S$ be the strong semilattice of semigroups $\{S_\delta\}_{\delta \in Y}$, that is,

$S = S[Y; S_\delta; \phi_{\delta, \lambda}]$.

It is clear that for each $\delta \in Y$, $S_\delta$ is not only self-decomposable but also Cayley-symmetric. By Theorem 5.3, $S$ is Cayley-symmetric. But note that the semigroup $S_\gamma$ is not regular. Furthermore, $S$ is also not regular. At last, $S$ is not commutative since nor is the dihedral group $S_\alpha$.

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**References**


