MODULES SATISFYING CERTAIN CHAIN CONDITIONS AND THEIR ENDMORPHISMS

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Abstract. In this paper, we characterize $w$-Noetherian modules in terms of polynomial modules and $w$-Nagata modules. Then it is shown that for a finite type $w$-module $M$, every $w$-epimorphism of $M$ onto itself is an isomorphism. We also define and study the concepts of $w$-Artinian modules and $w$-simple modules. By using these concepts, it is shown that for a $w$-Artinian module $M$, every $w$-monomorphism of $M$ onto itself is an isomorphism and that for a $w$-simple module $M$, $\text{End}_R M$ is a division ring.

1. Introduction

The question of when injective or surjective endomorphisms of certain modules over commutative rings are isomorphisms had been addressed in the literature. In [1], Bourbaki pointed out that if $M$ is a Noetherian module, then every surjective endomorphism of $M$ is an isomorphism. For the general case, Vasconcelos [5, 6] and Strooker [4] proved independently that if $M$ is a finitely generated module, then every surjective endomorphism of $M$ is an isomorphism. In [7], Vasconcelos also considered cases where an injective endomorphism of a finitely generated module is, actually, an isomorphism. It is a simple exercise that Artinian modules are endowed with this property [1, p. 23]. It is well known that if a module is simple, then its endomorphism ring is a division ring (this is sometimes called Schur’s lemma).

Let $D$ be an integral domain with quotient field $q(D)$. Following [11], a nonzero finitely generated ideal $J$ of $D$ is called a GV-ideal, denoted by $J \in GV(D)$, if $J^{-1} = D$; and a torsion-free $D$-module $M$ is called a $w$-module if $Jx \subseteq M$ for $x \in q(D) \otimes_D M$ and $J \in GV(D)$ implies $x \in M$. A $w$-module $M$ is called a strong Mori module if $M$ satisfies the ACC on $w$-submodules of $M$. G. W. Chang characterized strong Mori modules in terms of polynomial modules and $t$-Nagata modules and also studied the above question in [2] as follows. It is shown that $M$ is a strong Mori module over $D$ if and only if the polynomial

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module $M[X]$ is a strong Mori module over $D[X]$; if and only if $M[X]_{N_w}$ is a Noetherian module over $D[X]_{N_w}$, where $N_w = \{ f \in D[X] \mid c(f)_w = D \}$. And it is proved that if $\varphi : M \to M$ is an epimorphism, where $M$ is a strong Mori module, then $\varphi$ is an isomorphism. Certainly, this is the $w$-theoretic version of the aforementioned Bourbaki’s theorem.

In this paper, we show that the two results above of G. W. Chang still hold for a commutative ring with zero divisors if we use a new extended definition of $w$-modules (see [9, 14]) under more weaker conditions ($w$-epimorphisms not epimorphisms). We also address the above questions on endomorphisms. To do this, we introduce and study the concepts of $w$-Artinian modules and $w$-simple modules.

Throughout this paper, $R$ is a commutative ring with identity element and all modules are unitary. Following [14] a finitely generated ideal $J$ of $R$ is called a $GV$-ideal, if the natural homomorphism $R \to \text{Hom}_R(J, R)$ is an isomorphism. Denote by $GV(R)$ the set of GV-ideals of $R$. An $R$-module $M$ is called GV-torsion if for any $x \in M$, there is a $J \in GV(R)$ such that $Jx = 0$, and $M$ is said to be GV-torsion-free if $Jx = 0$ for $J \in GV(R)$ and $x \in M$ implies $x = 0$. Denote by $E(M)$ the injective envelope of $M$. For a GV-torsion-free $R$-module $M$, define

$$M_w = \{ x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R) \},$$

which is called the $w$-envelope of $M$. A GV-torsion-free module $M$ is called a $w$-module if $M_w = M$, equivalently, $\text{Ext}^1_R(R/J, M) = 0$ for any $J \in GV(R)$. Then it is easy to see that the $w$-operation on $R$ distributes over finite intersections since $GV(R)$ is a multiplicative system of $R$. A $w$-ideal $m$ of $R$ is called a maximal $w$-ideal if $m$ is maximal among proper integral $w$-ideals of $R$. It is shown that every maximal $w$-ideal of $R$ is prime [14, Proposition 3.8].

Let $M$ and $N$ be $R$-modules. Following [9], a homomorphism $f : M \to N$ is called a $w$-monomorphism (resp., $w$-epimorphism, $w$-isomorphism) if $f_m : M_m \to N_m$ is a monomorphism (resp., an epimorphism, an isomorphism) over $R_m$ for any maximal $w$-ideal $m$ of $R$. A sequence $A \to B \to C$ is said to be $w$-exact if the induced sequence $A_m \to B_m \to C_m$ is exact for any maximal $w$-ideal $m$ of $R$. An $R$-module $M$ is said to be of finite type if there is a $w$-exact sequence $F \to M \to 0$, where $F$ is finitely generated free. Thus, if $M$ is of finite type, then $M_m$ is finitely generated over $R_m$ for any maximal $w$-ideal $m$ of $R$. A module $M$ is said to be $w$-Noetherian if every submodule of $M$ is of finite type. Certainly, when $R$ is an integral domain, a torsion-free $w$-module $M$ is a strong Mori module if and only if $M$ is $w$-Noetherian.

2. Main results

Under the renewed notions we can not only generalize G. W. Chang’s results to a $w$-Noetherian module but also give a proof with different approach. To do this, we need a couple of lemmas.
Lemma 2.1. An $R$-module $M$ is of finite type if and only if there is a finitely generated submodule $N$ of $M$ such that $M/N$ is GV-torsion.

Proof. See [9, Proposition 1.2].

Let $X$ be an indeterminate over $R$. The content of a polynomial $f \in R[X]$, denoted by $c(f)$, is the ideal of $R$ generated by the coefficients of $f$. Set $S_w = \{ f \in R[X] \mid c(f)_w = R \}$ and $R\{X\} = R[X]_{S_w}$, which is called the $w$-Nagata ring of $R$. Let $M$ be an $R$-module and $M[X] = M \otimes_R R[X]$. Then $M[X]_{S_w}$ is an $R[X]_{S_w}$-module and is called the $w$-Nagata module of $M$ and set $M\{X\} = M[X]_{S_w}$. Note that if $R$ is a domain, then $S_w = N_v$ and $R\{X\} = R[X]_{N_v}$.

Lemma 2.2. (1) An $R$-module $M$ is GV-torsion if and only if $M\{X\} = 0$.

(2) An $R$-sequence $A \rightarrow B \rightarrow C$ is $w$-exact if and only if the $R\{X\}$-sequence $A\{X\} \rightarrow B\{X\} \rightarrow C\{X\}$ is exact.

(3) Let $\alpha : M \rightarrow N$ be an $R$-homomorphism. Then $\alpha$ is a $w$-monomorphism (resp., $w$-epimorphism, $w$-isomorphism) if and only if the canonical extension $\overline{\alpha} : M\{X\} \rightarrow N\{X\}$ is a monomorphism (resp., an epimorphism, an isomorphism).

(4) An $R$-module $M$ is of finite type if and only if $M\{X\}$ is finitely generated over $R\{X\}$.

Proof. See [10].

Lemma 2.3. If $J$ is a GV-ideal of $R[X]$, then there is $g \in J$ such that $c(g)_w = R$.

Proof. See [13, Corollary 2.5].

Theorem 2.4. The following statements are equivalent for an $R$-module $M$.

(1) $M$ is a $w$-Noetherian module over $R$.

(2) $M[X]$ is a $w$-Noetherian module over $R[X]$.

(3) $M\{X\}$ is a Noetherian module over $R\{X\}$.

Proof. (1) $\Rightarrow$ (2). Similar to the proof of [14, Theorem 4.9].

(2) $\Rightarrow$ (3). Let $A$ be a submodule of $M\{X\}$. Then there is a submodule $B$ of $M[X]$ such that $A = B_{S_w}$. Since $M[X]$ is $w$-Noetherian, $B$ is of finite type over $R[X]$. Thus by Lemma 2.1, there is a finitely generated submodule $C$ of $B$ such that $B/C$ is GV-torsion over $R[X]$. Let $u \in B$. Then there is a GV-ideal $J$ of $R[X]$ such that $Ju \subseteq C$. By Lemma 2.3 there is $g \in J$ such that $c(g)_w = R$. Hence $c(g) \in GV(R)$. From $gu \in C$ we have $\frac{u}{1} = \frac{gu}{g} \in C_{S_w}$. Therefore, $A = B_{S_w} = C_{S_w}$ is finitely generated over $R\{X\}$. So $M\{X\}$ is Noetherian.

(3) $\Rightarrow$ (1). Let $N$ be a submodule of $M$. Then $N\{X\}$ is a submodule of $M\{X\}$. Hence $N\{X\}$ is finitely generated by hypothesis. So $N$ is of finite type by Lemma 2.2(4). Consequently, $M$ is $w$-Noetherian.
As a corollary, we can recover [13, Proposition 4.3] in the following.

**Corollary 2.5.** The following statements are equivalent for a ring $R$.

1. $R$ is a $w$-Noetherian ring.
2. $R[X]$ is a $w$-Noetherian ring.
3. $R\{X\}$ is a Noetherian ring.

**Lemma 2.6.** Let $M$ and $N$ be $w$-modules and let $f : M \to N$ be a homomorphism. If $f$ is a $w$-isomorphism, then $f$ is an isomorphism.

*Proof.* This is a simple corollary of [9, Theorem 1.2].

The following is the $w$-theoretic version of Vasconcelos-Strooker’s theorem.

**Theorem 2.7.** Let $M$ be a finite type $w$-module and let $f : M \to M$ be a $w$-epimorphism. Then $f$ is an isomorphism.

*Proof.* Let $\mathfrak{m}$ be a maximal $w$-ideal of $R$. Then the induced map $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to M_{\mathfrak{m}}$ is an epimorphism over $R_{\mathfrak{m}}$. By Vasconcelos-Strooker’s theorem, $f_{\mathfrak{m}}$ is an isomorphism, that is, $f$ is a $w$-isomorphism. By Lemma 2.6, $f$ is an isomorphism.

In [3], Orzech proved that if $f : N \to M$ is an epimorphism, where $M$ is finitely generated and $N$ is a submodule of $M$, then $f$ is an isomorphism. This theorem is certainly a generalization of Vasconcelos’ theorem. The following is a $w$-version of this theorem.

**Theorem 2.8.** Let $M$ be a finite type $w$-module and let $N$ be a $w$-submodule of $M$. Suppose $f : N \to M$ is a $w$-epimorphism. Then $f$ is an isomorphism.

*Proof.* Similar to the proof of Theorem 2.7.

Recall from [15] that a nonzero $w$-module $M$ is said to be $w$-simple if $M$ has no nontrivial $w$-submodules. It was shown in [15, Example 3.7] that simple modules and $w$-simple modules are two mutually exclusive concepts.

In [1], Bourbaki pointed out that any injective endomorphism of an Artinian module is always an isomorphism. Now we can give a $w$-version of this theorem by defining $w$-Artinian modules.

**Definition 2.9.** Let $M$ be a $w$-module. If $M$ has the DCC on $w$-submodules, then we say that $M$ is a $w$-Artinian module.

It is natural that a $w$-simple module is certainly $w$-Artinian. Therefore, a $w$-Artinian module is not necessarily an Artinian module. Now we provide an explicit example of a module which is $w$-Artinian but not Artinian.

**Example 2.10.** Let $K$ be a field and $R = K[X,Y]$. Then $M = (R/(X))_w$ is a $w$-simple, and therefore, is $w$-Artinian. Write $y = Y$. Then $Ry \supsete R_y \supsete \cdots \supsete R_{y^n} \supsete \cdots$ is a descending chain of submodules of $M$ but not stationary. Therefore, $M$ is not Artinian.
Proposition 2.11. The following statements are equivalent for a \( w \)-module \( M \).

1. \( M \) is a \( w \)-Artinian module.
2. Any nonempty subset of \( w \)-submodules of \( M \) has a minimum element.

Proof. This is similar to the case of Artinian modules. \( \square \)

Theorem 2.12. Let \( A, B \) and \( C \) be \( w \)-modules and let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) be \( w \)-exact. Then \( B \) is a \( w \)-Artinian module if and only if \( A \) and \( C \) are \( w \)-Artinian.

Proof. Since \( A \) is GV-torsion-free and \( f \) is a \( w \)-monomorphism, \( f \) is a monomorphism. So we regard that \( A \) is a \( w \)-submodule of \( B \). Suppose \( B \) is \( w \)-Artinian. Clearly \( A \) is \( w \)-Artinian. Let \( C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots \) be a descending chain of \( w \)-submodules of \( C \). Set \( B_n = g^{-1}(C_n) \) for all \( n \). It is routine to verify that \( B_n \) is a \( w \)-submodule of \( B \) and \( B_n \supseteq B_{n+1} \). Therefore there is an integer \( m \) such that \( B_n = B_m \) for all \( n \geq m \). Note that \( C = g(B)_w \) since \( g \) is a \( w \)-epimorphism. Hence \( C_n = g(B_n)_w \). Consequently, \( C_n = C_m \) for all \( n \geq m \). It follows that \( C \) is \( w \)-Artinian.

Conversely, suppose \( A \) and \( C \) are \( w \)-Artinian. Let \( B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots \) be a descending chain of \( w \)-submodules of \( B \). Set \( A_n = A \cap B_n \) and \( C_n = g(B_n)_w \). Then \( A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \) and \( C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots \) are descending chains of \( w \)-submodules of \( A \) and \( C \), respectively. Thus there is an integer \( m \) such that \( A_n = A_m \) and \( C_n = C_m \) for all \( n \geq m \). Let \( b \in B_n \). Then \( g(b) \in C_n = C_m \). Therefore there is a GV-ideal \( J \) of \( R \) such that \( g(Jb) \subseteq g(B_m) \). For \( u \in J \), write \( g(ub) = g(x) \), \( x \in B_m \). Then \( ub - x \in A_n = A_m \). Hence \( Jb \subseteq B_m \). Since \( B_m \) is a \( w \)-module, we have \( b \in B_m \). Thus we get that \( B_n = B_m \) for all \( n \geq m \). Consequently, \( B \) is \( w \)-Artinian. \( \square \)

Corollary 2.13. A direct sum \( M_1 \oplus M_2 \oplus \cdots \oplus M_n \) is a \( w \)-Artinian module if and only if each \( M_i \) is a \( w \)-Artinian module.

Proposition 2.14. Let \( M \) be a \( w \)-Artinian module. Then \( M_m \) is Artinian for each maximal \( w \)-ideal \( m \) of \( R \).

Proof. Let \( A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \) be a descending chain of submodules of \( M_m \). Let \( \vartheta : M \to M_m \) be the natural map and set \( B_n = \vartheta^{-1}(A_n) \). Then \( (B_n)_m = A_n \) and \( B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots \) is a descending chain of \( w \)-submodules of \( M \). Thus there is an integer \( m \) such that \( B_n = B_m \) for \( n \geq m \). Therefore, \( A_n = A_m \), whence \( M_m \) is Artinian. \( \square \)

Recall that a ring \( R \) is called a \( DW \) ring if every ideal of \( R \) is a \( w \)-ideal; equivalently, \( GV(R) = \{ R \} \). By a slight modification of [8, Example 1.3(b)] we give a counterexample that the converse of Proposition 2.14 does not hold.

Example 2.15. Let \( E \) be a countable direct sum of copies of \( \mathbb{Z}_2 \) with addition and multiplication defined component-wise. Let \( R = \mathbb{Z}_4 \times E \), and define
addition and multiplication as follows:

\[(m, x) + (n, y) = (m + n, x + y)\]

and

\[(m, x)(n, y) = (mn, my + nx + xy),\]

where \(m, n \in \mathbb{Z}_4\) and \(x, y \in E\). Then \(R\) is a ring with identity \((1, 0)\). For \(\alpha = (2, 0) \in R\), we have that \(\text{ann}(\alpha) = 2\mathbb{Z}_4 \times E\) is not finitely generated. Hence \(R\) is not a coherent ring. Therefore, \(R\) is not an Artinian ring. For any maximal \(w\)-ideal \(m\) of \(R\), it follows easily that \(R_m = \mathbb{Z}_2\) or \(R_m = \mathbb{Z}_4\). Thus \(\dim(R) = 0\), and hence \(R\) is a DW ring. Therefore \(R\) is not a \(w\)-Artinian \(R\)-module, but for any maximal \(w\)-ideal \(m\), \(R_m\) is an Artinian module over \(R_m\).

Now we give a \(w\)-theoretic version of the other Bourbaki’s Theorem aforementioned.

**Theorem 2.16.** Let \(M\) be a \(w\)-Artinian module and let \(f : M \to M\) be a \(w\)-monomorphism. Then \(f\) is an isomorphism.

**Proof.** Since \(M\) is GV-torsion-free, \(f\) is actually a monomorphism. Consequently, \(f^n\) is also a monomorphism for all \(n\). Thus \(\text{Im}(f) \supseteq \text{Im}(f^2) \supseteq \cdots\) is a descending chain of \(w\)-submodules of \(M\). Hence there is an integer \(n\) such that \(\text{Im}(f^n) = \text{Im}(f^{n+1})\). Therefore, for each \(x \in M\), there is an element \(y \in M\) such that \(f^n(x) = f^{n+1}(y)\). It follows \(x = f(y)\). Consequently, \(\text{Im}(f) = M\). So \(f\) is an isomorphism. \(\square\)

The following is a \(w\)-theoretic version of Schur’s Lemma.

**Corollary 2.17.** Let \(M\) be a \(w\)-simple module. Then \(\text{End}_R M\) is a division ring.

**Proof.** Let \(f\) be a nonzero endomorphism of \(M\). Thus \(\ker(f) \neq M\). By [14, Theorem 2.7], \(\ker(f)\) is a \(w\)-submodule of \(M\). Hence \(\ker(f) = 0\). So \(f\) is a monomorphism. By Theorem 2.16, \(f\) is an isomorphism. Hence \(\text{End}_R M\) is a division ring. \(\square\)

In order to give a new characterization of Artinian rings, we need a couple of lemmas.

**Lemma 2.18.** Suppose that \(R\) satisfies the DCC on \(w\)-ideals. Then we have:

1. Non-zero-divisors of \(R\) are units.
2. \(R\) has only finitely many maximal \(w\)-ideals.

**Proof.** (1) Let \(a \in R\) be a non-zero-divisor. Then \((a) \supseteq (a^2) \supseteq \cdots \supseteq (a^n) \supseteq \cdots\) is a descending chain of \(w\)-ideals of \(R\). By hypothesis there is an integer \(n\) such that \((a^n) = (a^{n+1})\). It follows directly that \(a\) is a unit.

(2) If \(m_1, m_2, \ldots, m_n, \ldots\) are maximal \(w\)-ideals of \(R\), then

\[m_1 \supseteq (m_1m_2)_w \supseteq \cdots \supseteq (m_1m_2 \cdots m_n)_w \supseteq \cdots\]
is a descending chain of $w$-ideals of $R$. Hence there is an integer $n$ such that $(m_1m_2\cdots m_n)_w = (m_1m_2\cdots m_nm_{n+1})_w$. Hence $m_1m_2\cdots m_n \subseteq m_{n+1}$. It follows that $m_{n+1} = m_i$ for some $i$, $1 \leq i \leq n$. Hence $R$ has only finitely many maximal $w$-ideals.

Lemma 2.19 ([12, Corollary 3.22]). Let $R$ be a $w$-Noetherian ring. If $I$ is an ideal of $R$ with $\text{ann}(I) = 0$, then $I$ contains a non-zero-divisor of $R$. In particular, if $J \in \text{GV}(R)$, then $J$ contains a non-zero-divisor of $R$.

Theorem 2.20. A ring $R$ is Artinian if and only if $R$ satisfies the DCC on $w$-ideals.

Proof. It is enough to show “if” part. To show that $R$ is Artinian, we must prove that $R$ is a DW ring. Let $A$ be a $w$-ideal of $R$. From Lemma 2.18(2) we may assume that $m_1, \ldots, m_n$ are all maximal $w$-ideals of $R$. By Proposition 2.14, $Rm_i$ is Artinian, and hence $A_{m_i}$ is finitely generated. Take $\{a_{ij}\} \subseteq A$, for $j = 1, \ldots, m$, such that $\{m_{m_i}\}$ is a generating set of $A_{m_i}$ over $R_{m_i}$, $i = 1, \ldots, n$. It is routine to verify that $A = \{(a_{ij})\}_w$. Therefore, $A$ is of finite type, whence $R$ is $w$-Noetherian. Let $J \in \text{GV}(R)$. By Lemma 2.19, $J$ has a non-zero-divisor. By Lemma 2.18(1), $J = R$. Hence $R$ is a DW ring. \hfill $\square$

From Theorem 2.20, it is no use to define $w$-Artinian rings to satisfy the DCC on $w$-ideals.

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