UNIVARTE LEFT FRACTIONAL POLYNOMIAL HIGH
ORDER MONOTONE APPROXIMATION

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Abstract. Let \( f \in C^r ([−1, 1]) \), \( r \geq 0 \) and let \( L^* \) be a linear left frac-
tional differential operator such that \( L^* (f) \geq 0 \) throughout \([0, 1]\). We can
find a sequence of polynomials \( Q_n \) of degree \( \leq n \) such that \( L^* (Q_n) \geq 0 \)
over \([0, 1]\), furthermore \( f \) is approximated left fractionally and simulta-
neously by \( Q_n \) on \([-1, 1]\). The degree of these restricted approximations
is given via inequalities using a higher order modulus of smoothness for
\( f^{(r)} \).

1. Introduction

The topic of monotone approximation started in [6] has become a major
trend in approximation theory. A typical problem in this subject is: given a
positive integer \( k \), approximate a given function whose \( k \)th derivative is \( \geq 0 \) by
polynomials having this property.

In [3] the authors replaced the \( k \)th derivative with a linear differential oper-
ator of order \( k \). We mention this motivating result.

Theorem 1. Let \( h, k, p \) be integers, \( 0 \leq h \leq k \leq p \) and let \( f \) be a real function,
\( f^{(p)} \) continuous in \([-1, 1]\) with modulus of continuity \( \omega_1 (f^{(p)}, x) \) there. Let
\( a_j (x) \), \( j = h, h + 1, \ldots, k \) be real functions, defined and bounded on
\([-1, 1]\) and assume \( a_h (x) \) is either \( \geq \) some number \( \alpha > 0 \) or \( \leq \) some number \( \beta < 0 \)
throughout \([-1, 1]\). Consider the operator

\[
L = \sum_{j=h}^{k} a_j (x) \left[ \frac{d^j}{dx^j} \right]
\]

and suppose, throughout \([-1, 1]\),

\[
L (f) \geq 0.
\]

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Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that
\[ L(Q_n) \geq 0 \text{ throughout } [-1,1] \]
and
\[ \max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq Cn^{k-p}x^1 \left( f^{(p')} \frac{1}{n} \right), \]
where $C$ is independent of $n$ or $f$.

We use also the notation $I = [-1,1]$.

We would like to mention:

**Theorem 2** (Gonska and Hinemmamn [5]). Let $r \geq 0$ and $s \geq 1$. Then there exists a sequence $Q_n = Q_n^{(r,s)}$ of linear polynomial operators mapping $C^r(I)$ into $P_n$ (space of polynomials of degree $\leq n$), such that for all $f \in C^r(I)$, all $|x| \leq 1$ and all $n \geq \max(4(r+1), r+s)$ we have
\[ |f(x) - (Q_n f)(x)| \leq M_{r,s} (\Delta_n(x))^r \omega_s \left( f^{(r)}, \Delta_n(x) \right), \quad 0 \leq k \leq r, \]
where $\Delta_n(x) = \frac{x^n}{n!} + \frac{1}{n}$, and $M_{r,s}$ is a constant independent of $f$, $x$, and $n$. Above $\omega_s$ is the usual modulus of smoothness of order $s$ with respect to the supremum norm.

Theorem 2 implies the useful:

**Corollary 3** ([2]). Let $r \geq 0$ and $s \geq 1$. Then there exists a sequence $Q_n = Q_n^{(r,s)}$ of linear polynomial operators mapping $C^r(I)$ into $P_n$, such that for all $f \in C^r(I)$ and all $n \geq \max(4(r+1), r+s)$ we have
\[ \left\| f^{(k)} - (Q_n f)^{(k)} \right\|_\infty \leq \frac{C_{r,s}}{n^{k-r}} \omega_s \left( f^{(r)}, \frac{1}{n} \right), \quad k = 0, 1, \ldots, r, \]
where $C_{r,s}$ is a constant independent of $f$ and $n$.

In [2] we proved the motivational:

**Theorem 4.** Let $h, v, r$ be integers, $0 \leq h \leq v \leq r$ and let $f \in C^r(I)$, with $f^{(r)}$ having modulus of smoothness $\omega_s(f^{(r)}, \delta)$ there, $s \geq 1$. Let $\alpha_j(x)$, $j = h, h+1, \ldots, v$ be real functions, defined and bounded on $I$ and suppose $\alpha_h$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout $I$. Take the operator
\[ L = \sum_{j=h}^v \alpha_j(x) \left( \frac{d^j}{dx^j} \right) \]
and assume, throughout $I$,
\[ L(f) \geq 0. \]

Then for every integer $n \geq \max(4(r+1), r+s)$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that
\[ L(Q_n) \geq 0 \text{ throughout } I, \]
and
\[ \| f^{(k)} - Q_n^{(k)} \|_\infty \leq \frac{C}{n^{r-v}} \omega_s \left( f^{(r)} \frac{1}{n} \right), \quad 0 \leq k \leq h. \]

Moreover, we get
\[ \| f^{(k)} - Q_n^{(k)} \|_\infty \leq \frac{C}{n^{r-k}} \omega_s \left( f^{(r)} \frac{1}{n} \right), \quad h+1 \leq k \leq r, \]
were \( C \) is a constant independent of \( f \) and \( n \).

In this article we extend Theorem 4 to the fractional level. Indeed here \( L \) is replaced by \( L^* \), a linear left Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval \([0, 1]\). Simultaneous and fractional convergence remains true on all of \( I \).

We are also inspired by [1].

We make:

**Definition 5** ([4], p. 50). Let \( \alpha > 0 \) and \( \lceil \alpha \rceil = m \), \( (\lceil \cdot \rceil \) ceiling of the number). Consider \( f \in C^m([-1, 1]) \). We define the left Caputo fractional derivative of \( f \) of order \( \alpha \) as follows:
\[ (D_{-1}^{\alpha} f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{-1}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) \, dt, \]

for any \( x \in [-1, 1] \), where \( \Gamma \) is the gamma function.

We set
\[ D_0^{\alpha} f (x) = f (x), \]
\[ D_{-1}^{m} f (x) = f^{(m)} (x), \quad \forall \ x \in [-1, 1]. \]

**2. Main result**

We present:

**Theorem 6.** Let \( h, v, r \) be integers, \( 1 \leq h \leq v \leq r \) and let \( f \in C^r([-1, 1]) \), with \( f^{(r)} \) having modulus of smoothness \( \omega_s \left( f^{(r)}, \delta \right) \) there, \( s \geq 1 \). Let \( \alpha_j(x) \), \( j = h, h+1, \ldots, v \) be real functions, defined and bounded on \([-1, 1]\) and suppose \( \alpha_h(x) \) is either \( \alpha > 0 \) or \( \leq \beta < 0 \) on \([0, 1]\). Let the real numbers \( \alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \cdots < \alpha_r \leq r \). Here \( D_{-1}^{\alpha} f \) stands for the left Caputo fractional derivative of \( f \) of order \( \alpha_j \) anchored at \(-1\). Consider the linear left fractional differential operator
\[ L^* := \sum_{j=h}^{k} \alpha_j(x) [D_{-1}^{\alpha_j}], \]

and suppose, throughout \([0, 1]\),
\[ L^* (f) \geq 0. \]
Then, for any \( n \in \mathbb{N} \) such that \( n \geq \max (4 (r + 1), r + s) \), there exists a real polynomial \( Q_n (x) \) of degree \( \leq n \) such that

\[
(13) \quad L^* (Q_n) \geq 0 \text{ throughout } [0, 1],
\]

and

\[
(14) \quad \sup_{-1 \leq x \leq 1} \left| (D_{x-}^{\alpha_1} f) (x) - (D_{x-}^{\alpha_j} Q_n) (x) \right| \leq \frac{2^j \omega_s}{\Gamma (j - \alpha_j + 1) n^r} \omega_s \left( f^{(r)} \frac{1}{n} \right),
\]

\( j = h + 1, \ldots, r; \ C_{r,s} \) is a constant independent of \( f \) and \( n \). Set

\[
(15) \quad l_j \equiv \sup_{x \in [-1, 1]} \left| \alpha_h^{-1} (x) \alpha_j (x) \right|, \ h \leq j \leq v.
\]

When \( j = 1, \ldots, h \) we derive

\[
(16) \quad \sup_{-1 \leq x \leq 1} \left| (D_{x-}^{\alpha_1} f) (x) - (D_{x-}^{\alpha_j} Q_n) (x) \right| \leq \frac{C_{r,s}}{n^r} \omega_s \left( f^{(r)} \frac{1}{n} \right) \left( \frac{1}{n!} \sum_{\tau = h}^{v} \frac{2^{\tau - \alpha_r}}{\Gamma (\tau - \alpha_r + 1)} \right)
\]

Finally it holds

\[
(17) \quad \sup_{-1 \leq x \leq 1} \left| f (x) - Q_n (x) \right| \leq \frac{C_{r,s}}{n^r} \omega_s \left( f^{(r)} \frac{1}{n} \right) \left[ \frac{1}{n!} \sum_{\tau = h}^{v} \frac{2^{\tau - \alpha_r}}{\Gamma (\tau - \alpha_r + 1)} + 1 \right].
\]

Proof. Here let \( Q_n \) as in Corollary 3. Let \( \alpha_j > 0, j = 1, \ldots, r \), such that \( 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < \cdots < \alpha_v \leq r \). That is \( \left[ \alpha_j \right] = j, j = 1, \ldots, r \).

We consider the left Caputo fractional derivatives

\[
(18) \quad (D_{x-}^{\alpha_1} f) (x) = \frac{1}{\Gamma (j - \alpha_j)} \int_{-1}^{x} (x - t)^{j - \alpha_j - 1} f^{(j)} (t) dt,
\]

and

\[
(19) \quad (D_{x-}^{\alpha_j} f) (x) = f^{(j)} (x),
\]

and

\[
(20) \quad (D_{x-}^{\alpha_j} Q_n) (x) = \frac{1}{\Gamma (j - \alpha_j)} \int_{-1}^{x} (x - t)^{j - \alpha_j - 1} Q_n^{(j)} (t) dt,
\]

\[
(21) \quad (D_{x-}^{\alpha_j} Q_n) (x) = Q_n^{(j)} (x); \ j = 1, \ldots, r.
\]

We notice that

\[
\left| (D_{x-}^{\alpha_1} f) (x) - (D_{x-}^{\alpha_j} Q_n) (x) \right|
\]

\[
\leq \frac{C_{r,s}}{n^r} \omega_s \left( f^{(r)} \frac{1}{n} \right) \left( \frac{1}{n!} \sum_{\tau = h}^{v} \frac{2^{\tau - \alpha_r}}{\Gamma (\tau - \alpha_r + 1)} \right)
\]

\[
\leq \frac{C_{r,s}}{n^r} \omega_s \left( f^{(r)} \frac{1}{n} \right) \left[ \frac{1}{n!} \sum_{\tau = h}^{v} \frac{2^{\tau - \alpha_r}}{\Gamma (\tau - \alpha_r + 1)} + 1 \right].
\]
We proved for any $x \in [-1, 1]$ that

$$
(23) \quad |(D_{\ast-1}^\alpha f)(x) - (D_{\ast-1}^\alpha Q_n)(x)| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n}\right) .
$$

Hence it holds

$$(24) \quad \sup_{-1 \leq x \leq 1} |(D_{\ast-1}^\alpha f)(x) - (D_{\ast-1}^\alpha Q_n)(x)| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n}\right) ,$$

$j = 0, 1, \ldots, r.$

Above we set $D_{\ast-1}^0 f(x) = f(x), \ D_{\ast-1}^0 Q_n(x) = Q_n(x), \forall x \in [-1, 1],$ and $\alpha_0 = 0$, i.e., $[\alpha_0] = 0$.

Set also

$$
(25) \quad \rho_n := C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n}\right) \left(\sum_{j=h}^n l_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} n^{j-r}\right) .
$$

I. Suppose, throughout $[0, 1]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x), x \in [-1, 1],$ be a real polynomial of degree $\leq n$ so that

$$
(26) \quad \max_{-1 \leq x \leq 1} \left|D_{\ast-1}^\alpha \left(f(x) + \rho_n \frac{x^h}{h!}\right) - (D_{\ast-1}^\alpha Q_n)(x)\right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n}\right) , \quad j = 0, 1, \ldots, r .
$$
When \( j = h + 1, \ldots, r \), then

\[
\max_{-1 \leq x \leq 1} \left| (D_{r+1}^a f) (x) - (D_{r+1}^a Q_n) (x) \right|
\leq \frac{2^{j-a_j}}{\Gamma (j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j} \omega_s} \left( f^{(r)} \frac{1}{n} \right),
\]

proving (14).

For \( j = 1, \ldots, h \) we get

\[
D_{r+1}^a \left( \frac{x^h}{h!} \right) = \frac{1}{\Gamma (j - \alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} (t+1)^{h-j-\lambda-1} dt
\]

(we see that \( t = t+1 - 1 \), and

\[
th^{-j} = (t+1-1)^{h-j} = \sum_{\lambda=0}^{h-j} \left( \begin{array}{c} h-j \\ \lambda \end{array} \right) (t+1)^{h-j-\lambda-1} (-1)^\lambda
\]

\[
= \frac{1}{(h-j)! \Gamma (j - \alpha_j)} \cdot \sum_{\lambda=0}^{h-j} (-1)^\lambda \left( \begin{array}{c} h-j \\ \lambda \end{array} \right) \frac{\Gamma (j - \alpha_j) \Gamma (h-j+\lambda+1)}{\Gamma (h-\alpha_j - \lambda + 1)} (x+1)^{h-\alpha_j-\lambda}
\]

\[
\sum_{\lambda=0}^{h-j} \lambda! \Gamma (h-\alpha_j - \lambda + 1) (x+1)^{h-\alpha_j-\lambda}.
\]

Hence for \( j = 1, \ldots, h \) we found that

\[
D_{r+1}^a \left( \frac{x^h}{h!} \right) = \sum_{\lambda=0}^{h-j} (-1)^\lambda \frac{(x+1)^{h-\alpha_j-\lambda}}{\lambda! \Gamma (h-\alpha_j - \lambda + 1)}
\]

Therefore we get from (26) that

\[
\max_{-1 \leq x \leq 1} \left| (D_{r+1}^a f) (x) + \rho_n \sum_{\lambda=0}^{h-j} (-1)^\lambda \frac{(x+1)^{h-\alpha_j-\lambda}}{\lambda! \Gamma (h-\alpha_j - \lambda + 1)} - (D_{r+1}^a Q_n) (x) \right|
\leq \frac{2^{j-a_j}}{\Gamma (j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j} \omega_s} \left( f^{(r)} \frac{1}{n} \right), \ j = 1, \ldots, h.
\]

Therefore we get for \( j = 1, \ldots, h \), that

\[
\max_{-1 \leq x \leq 1} \left| ([D_{r+1}^a f] (x) - (D_{r+1}^a Q_n) (x)]
\]
\[(32) \leq \rho_n \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma (h - \alpha_j - \lambda + 1)} + \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} n^{r-j} \omega_s \left(f^{(r)}, \frac{1}{n}\right) \]

\[= C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n}\right) \left( \sum_{j=0}^{h-j} \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} n^{r-j} \right) \]

\[\cdot \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma (h - \alpha_j - \lambda + 1)} + \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} n^{r-j} \omega_s \left(f^{(r)}, \frac{1}{n}\right) \right) \]

\[(33) = C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n}\right) \left( \sum_{j=h}^{k} \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} n^{r-j} \right) \]

\[\cdot \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma (h - \alpha_j - \lambda + 1)} + \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} n^{r-j} \right) \]

\[(34) \leq C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n}\right) \left( \sum_{j=h}^{k} \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} n^{r-j} \right) \]

\[\left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma (h - \alpha_j - \lambda + 1)} + \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} \right) \]

Hence for \(j = 1, \ldots, h\) we derived (16):

\[(35) \max_{-1 \leq x \leq 1} |(D_{x-1}^{\alpha_j} f) (x) - (D_{a-1}^{\alpha_j} Q_n) (x)| \leq \frac{C_{r,s}}{n^{r-\tau} \omega_s \left(f^{(r)}, \frac{1}{n}\right)}. \]

\[\left( \sum_{\tau=1}^{v} \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} \right) \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma (h - \alpha_j - \lambda + 1)} + \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} \right) \]

From (26) when \(j = 0\) we obtain

\[(36) \max_{-1 \leq x \leq 1} \left| f (x) + \rho_n \frac{x^h}{h!} - Q_n (x) \right| \leq \frac{C_{r,s}}{n^{r} \omega_s \left(f^{(r)}, \frac{1}{n}\right)} + \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} \]

And

\[(37) \max_{-1 \leq x \leq 1} |f (x) - Q_n (x)| \leq \frac{\rho_n}{h!} + \frac{C_{r,s}}{n^{r} \omega_s \left(f^{(r)}, \frac{1}{n}\right)} \]

\[= \frac{C_{r,s}}{h!} \omega_s \left(f^{(r)}, \frac{1}{n}\right) \left( \sum_{\tau=1}^{v} \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1)} n^{r-\tau} \right) + \frac{C_{r,s}}{n^{r} \omega_s \left(f^{(r)}, \frac{1}{n}\right)} \]

\[= \frac{C_{r,s}}{h!} \omega_s \left(f^{(r)}, \frac{1}{n}\right) \left[ \frac{1}{h!} \sum_{\tau=1}^{v} \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1) n^{r-\tau}} + \frac{1}{n^{r}} \right] \]
that is proving (17).
Also if $0 \leq x \leq 1$, then

$$
\alpha_{h}^{-1}(x)L^{*}(Q_{n}(x))
= \alpha_{h}^{-1}(x)L^{*}(f(x)) + \rho_{n}\frac{(x+1)^{h-\alpha_{h}}}{\Gamma(h-\alpha_{h}+1)}
\geq \rho_{n}\left(\sum_{j=h}^{v} \alpha_{h}^{-1}(x)\alpha_{j}(x)\left[D_{r+1}^{\alpha_{j}}Q_{n}(x) - D_{r+1}^{\alpha_{j}}f(x) - \frac{\rho_{n}}{h!}D_{r+1}^{\alpha_{j}}x^{h}\right]\right)
\geq \rho_{n}\frac{(x+1)^{h-\alpha_{h}}-\rho_{n}}{\Gamma(h-\alpha_{h}+1)}\Gamma(h-\alpha_{h}+1) - \alpha_{j}(x)\left[D_{r+1}^{\alpha_{j}}Q_{n}(x) - D_{r+1}^{\alpha_{j}}f(x) - \frac{\rho_{n}}{h!}D_{r+1}^{\alpha_{j}}x^{h}\right]
\leq \rho_{n}\left(\frac{1-\Gamma(h-\alpha_{h}+1)}{\Gamma(h-\alpha_{h}+1)}\right) \geq 0.
$$

Explanation: We know that $\Gamma(1) = 1$, $\Gamma(2) = 1$, and $\Gamma$ is convex and positive on $(0, \infty)$. Here $0 \leq h-\alpha_{h} < 1$ and $1 \leq h-\alpha_{h}+1 < 2$. Thus $\Gamma(h-\alpha_{h}+1) \leq 1$ and $1 - \Gamma(h-\alpha_{h}+1) \geq 0$. Hence $L^{*}(Q_{n}(x)) \geq 0$, $x \in [0, 1]$.

II. Suppose on $[0, 1]$ that $\alpha_{h}(x) \leq \beta < 0$. Let $Q_{n}(x), x \in [-1, 1]$, be a real polynomial of degree $\leq n$ so that

$$
\max_{-1 \leq x \leq 1} \left|D_{r+1}^{\alpha_{j}}\left(f(x) - \rho_{n}\frac{x^{h}}{h!}\right) - (D_{r+1}^{\alpha_{j}}Q_{n})(x)\right|\leq \frac{2^{j-\alpha_{j}}C_{r,s}}{\Gamma(j-\alpha_{j}+1)}\omega_{s}\left(f^{(r)}, \frac{1}{n}\right), j = 0, 1, \ldots, r.
$$

Similarly we obtain again inequalities of convergence, see (14), (16) and (17).
Also if $0 \leq x \leq 1$, then

$$
\alpha_{h}^{-1}(x)L^{*}(Q_{n}(x))
= \alpha_{h}^{-1}(x)L^{*}(f(x)) - \rho_{n}\frac{(x+1)^{h-\alpha_{h}}}{\Gamma(h-\alpha_{h}+1)}
\geq \rho_{n}\left(\sum_{j=h}^{v} \alpha_{h}^{-1}(x)\alpha_{j}(x)\left[D_{r+1}^{\alpha_{j}}Q_{n}(x) - D_{r+1}^{\alpha_{j}}f(x) + \frac{\rho_{n}}{h!}D_{r+1}^{\alpha_{j}}x^{h}\right]\right)
\leq \rho_{n}\frac{(x+1)^{h-\alpha_{h}}}{\Gamma(h-\alpha_{h}+1)} + \rho_{n}\left(\sum_{j=h}^{v} \alpha_{h}^{-1}(x)\alpha_{j}(x)\left[D_{r+1}^{\alpha_{j}}Q_{n}(x) - D_{r+1}^{\alpha_{j}}f(x) + \frac{\rho_{n}}{h!}D_{r+1}^{\alpha_{j}}x^{h}\right]\right).
$$
\[ \rho_n \left( 1 - \frac{(x+1)^{h-\alpha_h}}{\Gamma(h) \Gamma(h-\alpha_h+1)} \right) = \rho_n \frac{\Gamma(h - \alpha_h + 1) - (x+1)^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \]

\[ \leq \rho_n \frac{1 - (x+1)^{h-\alpha_h}}{\Gamma(h - \alpha_h + 1)} \leq 0, \]

and hence on \([0,1]\) again holds \(L^*(Q_n(x)) \geq 0\). \(\square\)

**Remark 7 (to Theorem 6).** Suppose that \(\alpha_j(x), j = h, h+1, \ldots, v\) are continuous functions on \([-1,1]\), and we have on \([0,1]\) only \(L^*(f) > 0\). Relax the condition \(\alpha_h(x)\) is either \(\geq \alpha > 0\) or \(\leq \beta < 0\) on \([0,1]\). Let \(Q_n\) be the polynomial of degree \(\leq n\) corresponding to \(f\) from (24).

Then \(D_{h-1}^{\alpha_j}Q_n\) converges uniformly to \(D_{h-1}^{\alpha_j}f\) at a higher rate given by inequality (24), in particular for \(0 \leq j \leq h\). Moreover, because \(L^*(Q_n)\) converges uniformly to \(L^*(f)\) on \([-1,1]\), \(L^*(Q_n) > 0\) on \([0,1]\) for sufficiently large \(n\).

**References**


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