Journal of Korean Institute of Intelligent Systems
Vol. 25, No. 2, April 2015, pp. 197-202
http://dx.doi.org/10.5391/JKIIS.2015.25.2.197

ISSN(Print) 1976-9172
ISSN(Online) 2288-2324

JKIIS

Zadeh's extension principle for 2-dimensional triangular fuzzy numbers

2-차원 삼각퍼지수에 대한 Zadeh의 확장원리

Changil Kim* and Yong Sik Yun**†
김창일* · 윤용식**†

*Department of Mathematics Education, Dankook University
**Department of Mathematics, Jeju National University
*단국대학교 수학교육과, **제주대학교 수학과

Abstract
A triangular fuzzy number is one of the most popular fuzzy numbers. Many results for the extended algebraic operations between two triangular fuzzy numbers are well-known. We generalize the triangular fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$. By defining parametric operations between two regions valued α-cuts, we get the parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$.

Key Words : Extension Principle, Parametric Operation

요 약
삼각퍼지수는 가장 유명한 퍼지수 중의 하나이다. 두 삼각퍼지수 사이의 확장된 대수적 작용소에 대한 많은 결과들이 알려져 있다. 우리는 $\mathbb{R}$ 위에 정의된 삼각퍼지수를 $\mathbb{R}^2$ 위로 일반화하였다. 영역을 값으로 갖는 두 α-절단 사이에 매개변수 작용소를 정의함으로서 $\mathbb{R}^2$ 위에서 정의된 두 삼각퍼지수에 대한 매개변수 작용소를 얻을 수 있었다.

키워드 : 확장원리, 매개변수 작용소

Received: Dec. 22, 2014
Revised : Apr. 6, 2015
Accepted: Apr. 9, 2015
†Corresponding author(yunys@jejunu.ac.kr)

1. Introduction

A fuzzy set is characterized by its membership function. The membership function of triangular fuzzy number is very simple and consisting of monotonic increasing and decreasing functions. Thus a triangular fuzzy number defined on $\mathbb{R}$ is one of the most famous fuzzy number and many results for the extended algebraic operations between two triangular fuzzy numbers are well-known. The main idea of calculation of operations is to use the α-cuts.

In this paper, we generate the triangular fuzzy numbers on $\mathbb{R}$ to $\mathbb{R}^2$. By defining parametric operations between two regions valued α-cuts, we get the parametric operations for two triangular fuzzy numbers defined on $\mathbb{R}^2$.

2. Preliminaries

We define α-cut and α-set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$.

Definition 2.1. An *α-cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_0 = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_A(x) > 0\}$. For $\alpha \in (0,1)$, the set $A^\alpha = \{x \in \mathbb{R} \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 and A^1 are the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > 0\}$ and $\{x \in \mathbb{R} \mid \mu_A(x) = 1\}$, respectively.

In the calculations between two fuzzy numbers, the concept of α-cut is very important. Furthermore, some operations between α-cuts are very useful and α-set plays a very important role in a 2-dimensional case. Let X be a set.

Definition 2.2.([6]) A fuzzy set A is convex if

$$\mu_A(\lambda x_1+(1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)),$$

for all $x_1, x_2 \in X$ and $\lambda \in [0,1]$.

Zadeh had defined the extension principle([5]). Zimmermann introduced the same basic concepts in [6] as follows:

Definition 2.3.([6]) Let $X = X_1 \times \cdots \times X_n$ be a cartesian product and μ_i be a fuzzy set in X_i, respectively, and $f: X \to Y$ be a mapping. Then the extension principle allows us to define a fuzzy set ν in Y by

$$\nu(y) = \sup_{(x_1, \cdots, x_n) \in f^{-1}(y)} \min\{\mu_1(x_1), \cdots, \mu_n(x_n)\}$$

if $f^{-1}(y) \neq \varnothing$ and $\nu(y)=0$ if $f^{-1}(y)=\varnothing$.

For $n=1$, the extension principle reduces to a fuzzy set $\nu = f(\mu)$ defined by

$$\nu(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \varnothing, \\ 0, & f^{-1}(y) = \varnothing. \end{cases}$$

Definition 2.4.([6]) The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$ and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For $x \in A, y \in B$,

(1) $\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}$
(2) $\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}$
(3) $\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\}$
(4) $\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}$

Remark 2.5.([2]) Let A and B be fuzzy sets and $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ be the α-cuts of A and B, respectively. Then the α-cuts of $A(+)B$, $A(-)B$, $A(\cdot)B$ and $A(/)B$ can be calculated as follows.

(1) $(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)}+b_1^{(\alpha)}, a_2^{(\alpha)}+b_2^{(\alpha)}]$
(2) $(A(-)B)_\alpha = A_\alpha(-)B_\alpha = [a_1^{(\alpha)}-b_2^{(\alpha)}, a_2^{(\alpha)}-b_1^{(\alpha)}]$
(3) $(A(\cdot)B)_\alpha = A_\alpha(\cdot)B_\alpha = [a_1^{(\alpha)}b_1^{(\alpha)}, a_2^{(\alpha)}b_2^{(\alpha)}]$
(4) $(A(/)B)_\alpha = A_\alpha(/)B_\alpha = [a_1^{(\alpha)}/b_2^{(\alpha)}, a_2^{(\alpha)}/b_1^{(\alpha)}]$

Let X be a real line $\mathbb{R}$.

Definition 2.6.([6]) A fuzzy number A is a convex fuzzy set on $\mathbb{R}$ such that

(1) there exists unique $x \in \mathbb{R}$ with $\mu_A(x)=1$,
(2) $\mu_A(x)$ is piecewise continuous.

We call the fuzzy number A is *continuous* if the membership function $\mu_A(x)$ of A is continuous. If A is a continuous fuzzy number, then the α-cut A_α of A is a closed interval in $\mathbb{R}$.

One of the most famous fuzzy numbers is the triangular fuzzy number. And many results on a triangular fuzzy number have been suggested in many studies.

Definition 2.7. A triangular fuzzy number on $\mathbb{R}$ is a fuzzy number A which has a membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1,\ a_3 \leq x, \\ \dfrac{x-a_1}{a_2-a_1}, & a_1 \leq x < a_2, \\ \dfrac{a_3-x}{a_3-a_2}, & a_2 \leq x < a_3. \end{cases}$$

where $a_i \in \mathbb{R}$, $i=1,2,3$. It is denoted by $A=(a_1, a_2, a_3)$.

We defined the parametric operations for two fuzzy numbers defined on $\mathbb{R}$ and showed that the results for parametric operations are the same as those for the extended operations([1]). For this, we proved that for all fuzzy numbers A and all $\alpha \in [0,1]$, there exists a piecewise continuous function $f_\alpha(t)$ defined on $[0,1]$ such that $A_\alpha = \{f_\alpha(t) \mid t \in [0,1]\}$. If A is continuous, then the corresponding function $f_\alpha(t)$ is also continuous. The corresponding function $f_\alpha(t)$ is said to be the *parametric α-function* of A. The parametric α-function of A is denoted by $f_\alpha(t)$ or $f_A(t)$.

Definition 2.8. Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$ and $f_A(t), f_B(t)$ be the parametric α-functions of A and B, respectively. The parametric addition, parametric subtraction, parametric multiplication and parametric division are fuzzy numbers that have their α-cuts as follows.

(1) parametric addition $A(+)_pB$:
$(A(+)_pB)_\alpha = \{f_A(t)+f_B(t) \mid t \in [0,1]\}$
(2) parametric subtraction $A(-)_pB$:
$(A(-)_pB)_\alpha = \{f_A(t)-f_B(1-t) \mid t \in [0,1]\}$
(3) parametric multiplication $A(\cdot)_pB$:
$(A(\cdot)_pB)_\alpha = \{f_A(t)\cdot f_B(t) \mid t \in [0,1]\}$
(4) parametric division $A(/)_pB$:
$(A(/)_pB)_\alpha = \{f_A(t)/f_B(1-t) \mid t \in [0,1]\}$

Theorem 2.9. ([1]) Let A and B be two continuous fuzzy numbers defined on $\mathbb{R}$. Then we have the followings.

(1) $A(+)_p B = A(+)B$
(2) $A(-)_p B = A(-)B$
(3) $A(\cdot)_p B = A(\cdot)B$
(4) $A(/)_p B = A(/)B$

Corollary 2.10. ([1]) Let A and B be two triangular fuzzy numbers defined on $\mathbb{R}$. Then we have $A(+)_p B = A(+)B$, $A(-)_p B = A(-)B$, $A(\cdot)_p B = A(\cdot)B$ and $A(/)_p B = A(/)B$.

3. 2-dimensional triangular fuzzy numbers

In this section, we define the 2-dimensional triangular fuzzy numbers on $\mathbb{R}^2$ as a generalization of triangular fuzzy numbers on $\mathbb{R}$. Then we want to define the parametric operations between two 2-dimensional triangular fuzzy numbers. For that, we have to calculate operations between α-cuts in $\mathbb{R}^2$. The α-cuts are intervals in $\mathbb{R}$ but regions in $\mathbb{R}^2$, which makes the existing method of calculations between α-cuts unusable. We interpret the existing method from a different perspective and apply the method to the region valued α-cuts on $\mathbb{R}^2$.

Definition 3.1. A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} 1 - \sqrt{\dfrac{(x-x_1)^2}{a^2} + \dfrac{(y-y_1)^2}{b^2}}, & \\ \qquad b^2(x-x_1)^2 + a^2(y-y_1)^2 \le a^2 b^2, & \\ 0, \qquad \text{otherwise}, & \end{cases}$$

where $a, b > 0$ is called the *2-dimensional triangular fuzzy number* and denoted by $(a, x_1, b, y_1)^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha$ $(0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x,y)$ and the vertical planes $y - y_1 = k(x - x_1)$ $(k \in \mathbb{R})$ are symmetric triangular fuzzy numbers in those planes. If $a = b$, ellipses become circles. The α-cut A_α of a 2-dimensional triangular fuzzy number $A = (a, x_1, b, y_1)^2$ is an interior of ellipse in an xy-plane including the boundary

$$\begin{aligned} A_\alpha &= \{(x,y) \in \mathbb{R}^2 \mid b^2(x-x_1)^2 + a^2(y-y_1)^2 \\ &\qquad \le a^2 b^2 (1-\alpha)^2\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \left(\frac{x-x_1}{a(1-\alpha)}\right)^2 + \left(\frac{y-y_1}{b(1-\alpha)}\right)^2 \le 1\right\}. \end{aligned}$$

In Remark 2.5, if $A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$ is the α-cut of $A = (a_1, a_2, a_3)$ and $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$ is the α-cut of $B = (b_1, b_2, b_3)$, then

$$(A(+)B)_\alpha = A_\alpha(+)B_\alpha = [a_1^{(\alpha)} + b_1^{(\alpha)},\ a_2^{(\alpha)} + b_2^{(\alpha)}].$$

However, in a 2-dimensional case, $A_\alpha(+)B_\alpha$ can not be calculated by the same way since α-cuts are not intervals but subsets of $\mathbb{R}^2$. For the calculation in a 2-dimensional case, we consider the operations of α-cuts on $\mathbb{R}$ by using a parameter as in Definition 2.8.

Definition 3.2. A 2-dimensional fuzzy number A defined on $\mathbb{R}^2$ is called *convex* fuzzy number if for all $\alpha \in (0,1)$, the α-cuts

$$A_\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) \ge \alpha\}$$

are convex subsets in $\mathbb{R}^2$.

Theorem 3.3. Let A be a convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x,y) \in \mathbb{R}^2 \mid \mu_A(x,y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0,1)$, there exist piecewise continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

Proof. Let $\alpha \in (0,1)$ be fixed. Since A is a convex fuzzy number defined on $\mathbb{R}^2$, the α-cut A_α is convex subset in $\mathbb{R}^2$. Let

$$l = \inf\{x \mid \mu_A(x,y) = \alpha\} \text{ and } m = \sup\{x \mid \mu_A(x,y) = \alpha\}$$

The upper boundary of A_α is the graph of a piecewise continuous concave function $h_1(x)$ and the lower boundary of A_α is also the graph of a piecewise continuous convex function $h_2(x)$ defined on $[l, m]$.

Since $h_1(x)$ is piecewise continuous, $h_1(x)$ is continuous on $[l, m]$ except finitely many points $l < x_n < x_{n-1} < \cdots < x_1 < m$. Note that x_1 and x_n may equal to the end points m and l, respectively. Similarly, since $h_2(x)$ is also piecewise continuous, $h_2(x)$ is continuous on $[l, m]$ except finitely many points $l < x_{n+1} < x_{n+2} < \cdots < x_{n+m} < m$. Note that x_{n+1} and x_{n+m} may equal to the end points l and m, respectively. If the end points l and m (or one of them) equal to some x_i, we can prove the above facts similarly. Define

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m, \text{ if } t \in [0, \pi],$$

except the points

$$t_i = \cos^{-1}\left(\frac{2(x_i - m)}{m-l} + 1\right),\ i = 1, 2, \cdots, n.$$

Then $f_1^\alpha(t)$ is piecewise continuous on $[0, \pi]$ and

$$\{l \le x \le m \mid x \ne x_i,\ i = 1, 2, \cdots, n\}$$
$$= \{f_1^\alpha(t) \mid t \in [0, \pi],\ t \ne t_i,\ i = 1, 2, \cdots, n\}.$$

Define

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m, \text{ if } t \in [\pi, 2\pi],$$

except the points

$$t_j = \cos^{-1}\left(\frac{2(x_{n+j} - m)}{m-l} + 1\right),\ j = 1, 2, \cdots, m.$$

Then $f_1^\alpha(t)$ is piecewise continuous on $[\pi, 2\pi]$ and

$$\{l \le x \le m \mid x \ne x_{n+j},\ j = 1, 2, \cdots, m\}$$
$$= \{f_1^\alpha(t) \mid t \in [\pi, 2\pi],\ t \ne t_{n+j},\ j = 1, 2, \cdots, m\}.$$

The explicit proof for piecewise continuity can be proved by the same way in the proof of Theorem 3.2([1]). Focussing the construction of functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$, we outline our proof. Define $f_1^\alpha(t)$ and $f_2^\alpha(t)$ by

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m,\ t \in [0, 2\pi],$$

and

$$f_2^\alpha(t) = \begin{cases} h_1(f_1^\alpha(t)), & 0 \le t \le \pi, \\ h_2(f_1^\alpha(t)), & \pi \le t \le 2\pi. \end{cases}$$

Then we have $A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$.
The proof is complete.

If A is a continuous convex fuzzy number defined on $\mathbb{R}^2$, then the α-set A^α is a closed circular convex subset in $\mathbb{R}^2$.

Corollary 3.4. Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x, y) \in \mathbb{R}^2 \mid \mu_A(x, y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0, 1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}.$$

Definition 3.5. Let A and B be convex fuzzy numbers defined on $\mathbb{R}^2$ and

$$A^\alpha = \{(x, y) \in \mathbb{R}^2 \mid \mu_A(x, y) = \alpha\}$$
$$= \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$
$$B^\alpha = \{(x, y) \in \mathbb{R}^2 \mid \mu_B(x, y) = \alpha\}$$
$$= \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively. For $\alpha \in (0, 1)$, we define that the parametric addition, parametric subtraction, parametric multiplication and parametric division of two fuzzy numbers A and B are fuzzy numbers that have their α-sets as follows.

(1) parametric addition $A(+)_p B$:

$$(A(+)_p B)^\alpha = \{(f_1^\alpha(t) + g_1^\alpha(t), f_2^\alpha(t) + g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

(2) parametric subtraction $A(-)_p B$:

$$(A(-)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

where

$$x_\alpha(t) = \begin{cases} f_1^\alpha(t) - g_1^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_1^\alpha(t) - g_1^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

and

$$y_\alpha(t) = \begin{cases} f_2^\alpha(t) - g_2^\alpha(t+\pi), & \text{if } 0 \le t \le \pi \\ f_2^\alpha(t) - g_2^\alpha(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

(3) parametric multiplication $A(\cdot)_p B$:

$$(A(\cdot)_p B)^\alpha = \{(f_1^\alpha(t) \cdot g_1^\alpha(t), f_2^\alpha(t) \cdot g_2^\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

(4) parametric division $A(/)_p B$:

$$(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 \mid 0 \le t \le 2\pi\}$$

where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi),$$
$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi),$$
$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi),$$
$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi).$$

For $\alpha=0$, $(A(*)_p B)^0 = \lim_{\alpha\to 0^+}(A(*)_p B)^\alpha$ and if $\alpha=1$, $(A(*)_p B)^1 = \lim_{\alpha\to 1^-}(A(*)_p B)^\alpha$, where $*$ = $+, -, \cdot, /$.

Theorem 3.6. Let $A=(a_1, x_1, b_1, y_1)^2$ and $B=(a_2, x_2, b_2, y_2)^2$ be two 2-dimensional triangular fuzzy numbers. Then we have the following.

(1) $A(+)_p B = (a_1+a_2, x_1+x_2, b_1+b_2, y_1+y_2)^2$

(2) $A(-)_p B = (a_1+a_2, x_1-x_2, b_1+b_2, y_1-y_2)^2$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \,|\, 0\le t\le 2\pi\}$, where

$$\begin{aligned} x_\alpha(t) &= x_1x_2+(x_1a_2+x_2a_1)(1-\alpha)\cos t \\ &\quad +a_1a_2(1-\alpha)^2\cos^2 t, \\ y_\alpha(t) &= y_1y_2+(y_1b_2+y_2b_1)(1-\alpha)\sin t \\ &\quad +b_1b_2(1-\alpha)^2\sin^2 t. \end{aligned}$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \,|\, 0\le t\le 2\pi\}$, where

$$x_\alpha(t) = \frac{x_1+a_1(1-\alpha)\cos t}{x_2-a_2(1-\alpha)\cos t},$$
$$y_\alpha(t) = \frac{y_1+b_1(1-\alpha)\sin t}{y_2-b_2(1-\alpha)\sin t}.$$

Thus $A(+)_p B$ and $A(-)_p B$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_p B$ and $A(/)_p B$ need not to be 2-dimensional triangular fuzzy numbers.

Proof. Since A and B are convex fuzzy numbers defined on $\mathbb{R}^2$, by Theorem 3.3, there exists $f_i^\alpha(t)$, $g_i^\alpha(t)$ $(i=1,2)$ such that

$$\begin{aligned} A^\alpha &= \{(x,y)\in\mathbb{R}^2 \,|\, \mu_A(x,y)=\alpha\} \\ &= \{(f_1^\alpha(t), f_2^\alpha(t))\in\mathbb{R}^2 \,|\, 0\le t\le 2\pi\} \\ B^\alpha &= \{(x,y)\in\mathbb{R}^2 \,|\, \mu_B(x,y)=\alpha\} \\ &= \{(g_1^\alpha(t), g_2^\alpha(t))\in\mathbb{R}^2 \,|\, 0\le t\le 2\pi\} \end{aligned}$$

Since $A=(a_1, x_1, b_1, y_1)^2$ and $B=(a_2, x_2, b_2, y_2)^2$, we have

$$f_1^\alpha(t) = x_1+a_1(1-\alpha)\cos t,\ f_2^\alpha(t)=y_1+b_1(1-\alpha)\sin t,$$
$$g_1^\alpha(t) = x_2+a_2(1-\alpha)\cos t,\ g_2^\alpha(t)=y_2+b_2(1-\alpha)\sin t.$$

(1) Since

$$f_1^\alpha(t)+g_1^\alpha(t) = x_1+x_2+(a_1+a_2)(1-\alpha)\cos t,$$
$$f_2^\alpha(t)+g_2^\alpha(t) = y_1+y_2+(b_1+b_2)(1-\alpha)\sin t,$$

we have

$$(A(+)_p B)^\alpha = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \left(\frac{x-x_1-x_2}{(a_1+a_2)(1-\alpha)}\right)^2 + \left(\frac{y-y_1-y_2}{(b_1+b_2)(1-\alpha)}\right)^2 = 1\right\}.$$

Thus $A(+)_p B = (a_1+a_2, x_1+x_2, b_1+b_2, y_1+y_2)^2$.

(2) If $0\le t\le\pi$,

$$f_1^\alpha(t)-g_1^\alpha(t+\pi) = x_1-x_2+(a_1+a_2)(1-\alpha)\cos t,$$
$$f_2^\alpha(t)-g_2^\alpha(t+\pi) = y_1-y_2+(b_1+b_2)(1-\alpha)\sin t.$$

In the case of $\pi\le t\le 2\pi$, we have

$$f_1^\alpha(t)-g_1^\alpha(t-\pi) = f_1^\alpha(t)-g_1^\alpha(t+\pi),$$
$$f_2^\alpha(t)-g_2^\alpha(t-\pi) = f_2^\alpha(t)-g_2^\alpha(t+\pi).$$

Thus

$$(A(-)_p B)^\alpha = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \left(\frac{x-x_1+x_2}{(a_1+a_2)(1-\alpha)}\right)^2 + \left(\frac{y-y_1+y_2}{(b_1+b_2)(1-\alpha)}\right)^2 = 1\right\},$$

i.e., $A(-)_p B = (a_1+a_2, x_1-x_2, b_1+b_2, y_1-y_2)^2$.

(3) Let $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \,|\, 0\le t\le 2\pi\}$. Since

$$f_1^\alpha(t)+g_1^\alpha(t) = x_1+x_2+(a_1+a_2)(1-\alpha)\cos t,$$
$$f_2^\alpha(t)+g_2^\alpha(t) = y_1+y_2+(b_1+b_2)(1-\alpha)\sin t,$$

we have

$$\begin{aligned} x_\alpha(t) &= f_1^\alpha(t)\cdot g_1^\alpha(t) = x_1x_2+(x_1a_2+x_2a_1) \\ &\quad \times(1-\alpha)\cos t + a_1a_2(1-\alpha)^2\cos^2 t, \\ y_\alpha(t) &= f_2^\alpha(t)\cdot g_2^\alpha(t) = y_1y_2+(y_1b_2+y_2b_1) \\ &\quad \times(1-\alpha)\sin t + b_1b_2(1-\alpha)^2\sin^2 t. \end{aligned}$$

(4) Let $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \,|\, 0\le t\le 2\pi\}$. Similarly, we have

$$x_\alpha(t) = \frac{x_1+a_1(1-\alpha)\cos t}{x_2-a_2(1-\alpha)\cos t},\ y_\alpha(t) = \frac{y_1+b_1(1-\alpha)\sin t}{y_2-b_2(1-\alpha)\sin t}.$$

The proof is complete.

Example 3.7. Let $A=(6,3,8,5)^2$ and $B=(4,2,5,3)^2$. Then by Theorem 3.6, we have the following.

(1) $A(+)_p B = (10, 5, 13, 8)^2$

(2) $A(-)_p B = (10, 1, 13, 2)^2$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) | 0 \leq t \leq 2\pi\}$, where

$$x_\alpha(t) = 6 + 24(1-\alpha)\cos t + 24(1-\alpha)^2 \cos^2 t,$$
$$y_\alpha(t) = 15 + 49(1-\alpha)\sin t + 40(1-\alpha)^2 \sin^2 t.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) | 0 \leq t \leq 2\pi\}$, where

$$x_\alpha(t) = \frac{3+6(1-\alpha)\cos t}{2-4(1-\alpha)\cos t},\ y_\alpha(t) = \frac{5+8(1-\alpha)\sin t}{3-5(1-\alpha)\sin t}.$$

Thus $A(+)_p B$ and $A(-)_p B$ become 2-dimensional triangular fuzzy numbers, but $A(\cdot)_p B$ and $A(/)_p B$ need not to be 2-dimensional triangular fuzzy numbers.

References

[1] J. Byun and Y. S. Yun, Parametric operations for two fuzzy number, Communications of Korean Mathematical Society, vol. 28, no. 3, pp. 635-642, 2013.

[2] A. Kaufmann and M. M. Gupta, Introduction To Fuzzy Arithmetic : Theory and Applications, Van Nostrand Reinhold Co., New York, 1985.

[3] Y. S. Yun, S. U. Ryu and J. W. Park, The generalized triangular fuzzy sets, Journal of the Chungcheong Mathematical Society, vol. 22, no. 2, pp. 161-170, 2009.

[4] Y. S. Yun, J. C. Song and J. W. Park, Normal fuzzy probability for quadratic fuzzy number, Journal of fuzzy logic and intelligent systems, vol. 15, no. 3, pp. 277-281, 2005.

[5] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning – I, Information Sciences, vol. 8, pp. 199-249, 1975.

[6] H. J. Zimmermann, Fuzzy set Theory - and Its Applications, Kluwer-Nijhoff Publishing, Boston-Dordrecht-Lancaster, 1985.

저 자 소 개

김창일(Chang-Il Kim)
1986년 : 단국대학교 수학교육학과 이학사
1988년 : 서강대학교 수학과 이학석사
1992년 : 서강대학교 수학과 이학박사
1993년~현재 : 단국대학교 수학교육학과 교수

관심분야 : Fuzzy, Topology, Functional Analysis
Phone : +82-31-8005-3832
E-mail : kci206@hanmail.net

윤용식(Yong Sik Yun)
1988년 : 부산대학교 수학과 이학사
1990년 : 부산대학교 수학과 이학석사
1995년 : 교토대학 수학과 이학박사
1995년~현재 : 제주대학교 수학과 교수

관심분야 : Fuzzy Probability, Probability Theory
Phone : +82-64-754-3565
E-mail : yunys@jejunu.ac.kr