HOW THE PARAMETER \( \epsilon \) INFLUENCE THE GROWTH RATES OF THE PARTIAL QUOTIENTS IN GCF\( \epsilon \) EXPANSIONS

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Abstract. For generalized continued fraction (GCF) with parameter \( \epsilon(k) \), we consider the size of the set whose partial quotients increase rapidly, namely the set

\[ E_\epsilon(\alpha) := \left\{ x \in (0, 1] : k_{n+1}(x) \geq k_n(x)^\alpha \text{ for all } n \geq 1 \right\}, \]

where \( \alpha > 1 \). We in [6] have obtained the Hausdorff dimension of \( E_\epsilon(\alpha) \) when \( \epsilon(k) \) is constant or \( \epsilon(k) \sim k^\beta \) for any \( \beta \geq 1 \). As its supplement, now we show that:

\[
\dim H E_\epsilon(\alpha) = \begin{cases} 
\frac{1}{\alpha}, & \text{when } -k \delta \leq \epsilon(k) \leq k \text{ with } 0 \leq \delta < 1; \\
\frac{1}{\alpha + 1}, & \text{when } -k - \rho < \epsilon(k) \leq -k \text{ with } 0 < \rho < 1; \\
\frac{1}{\alpha + 2}, & \text{when } \epsilon(k) = -k - 1 + \frac{1}{k}.
\end{cases}
\]

So the bigger the parameter function \( \epsilon(k_n) \) is, the larger the size of \( E_\epsilon(\alpha) \) becomes.

1. Introduction

In 2003, F. Schweiger [2] introduced a new class of continued fractions with parameters, called generalized continued fractions (GCF\( \epsilon \)), which are induced by the map \( T_\epsilon : (0, 1] \to (0, 1] \)

\[
T_\epsilon(x) := \frac{-1 + (k + 1)x}{1 + \epsilon - kex} \quad \text{for } x \in \left( \frac{1}{k + 1}, \frac{1}{k} \right],
\]

where the parameter \( \epsilon : \mathbb{N} \to \mathbb{R} \) satisfies

\[
\epsilon(k) + k + 1 > 0 \quad \text{for all } k \geq 1.
\]

For any \( x \in (0, 1] \), its partial quotients \( \{k_n\}_{n \geq 1} \) in the GCF\( \epsilon \) expansion are defined as

\[
k_1 = k_1(x) := \left\lfloor \frac{1}{x} \right\rfloor, \quad \text{and} \quad k_n = k_n(x) := k_1(T_\epsilon^{n-1}(x)).
\]
By the algorithm (1.1), it follows [2] that
\[ x = \frac{A_n + B_n T^n(x)}{C_n + D_n T^n(x)} \]
for all \( n \geq 1 \),
where the numbers \( A_n, B_n, C_n, D_n \) are given by the recursive relations
\[
\begin{pmatrix}
C_0 & D_0 \\
A_0 & B_0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]
\[
(n) \begin{pmatrix}
C_n & D_n \\
A_n & B_n
\end{pmatrix} = \begin{pmatrix}
C_{n-1} & D_{n-1} \\
A_{n-1} & B_{n-1}
\end{pmatrix}\begin{pmatrix}
k_n + 1 & k_n \epsilon(k_n) \\
1 & 1 + \epsilon(k_n)
\end{pmatrix},
\]
for all \( n \geq 1 \).

It is interesting to see how the parameter functions \( \epsilon \) influence the growth rates of the partial quotients in \( \text{GCF}_\epsilon \). Under the condition (1.2), it is easy to see that for any \( x \in [0, 1) \),
\[
\lim_{n \to \infty} \frac{\log k_n(x)}{n} = 1.
\]
As far as a general parameter \( \epsilon \) is concerned, there is no general result concerning the growth rate of \( k_n(x) \). However, it is believed that the bigger the parameter \( \epsilon \) is, the faster the growth rate of \( k_n(x) \) should be. In this paper, we consider this question from the viewpoint of Hausdorff dimension. Namely, we consider the size of the following set:
\[
E_\epsilon(\alpha) := \{ x \in [0, 1) : k_{n+1}(x) \geq k_n(x)^\alpha \text{ for all } n \geq 1 \}.
\]
In [6], Zhong and Tang showed that:

**Theorem 1.1.**
\[
\dim_H E_\epsilon(\alpha) = \begin{cases}
\frac{\alpha}{\alpha - \beta + 1}, & \text{when } \epsilon(k) \equiv \epsilon_0 \text{ (constant)}; \\
\frac{\alpha}{\alpha}, & \text{when } \epsilon(k) \sim k^\beta \text{ and } \alpha \geq \beta \geq 1; \\
1, & \text{when } \epsilon(k) \sim k^\beta \text{ and } \alpha \leq \beta,
\end{cases}
\]
where \( \dim_H \) denotes the Hausdorff dimension.

In this paper, we will prove that:

**Theorem 1.2.**
\[
\dim_H E_\epsilon(\alpha) = \begin{cases}
\frac{\alpha}{\alpha - \beta + 1}, & \text{when } -k^\delta \leq \epsilon(k) \leq k \text{ with } 0 \leq \delta < 1; \\
\frac{\alpha}{\alpha - \beta + 2}, & \text{when } -k - \rho < \epsilon(k) \leq -k \text{ with } 0 < \rho < 1; \\
\frac{\alpha}{\alpha - 1}, & \text{when } \epsilon(k) = -k - 1 + \frac{1}{k}.
\end{cases}
\]
The above two theorems imply that
(1) The bigger \( \epsilon \) is, the larger the set in question. This gives some evidence that the bigger \( \epsilon \) is, the faster the partial quotients \( k_n \) grows.
(2) If and only if \( -k^\delta \leq \epsilon(k) \leq c k \) (where \( 0 \leq \delta < 1 \) and \( c \) is constant), the set \( E_\epsilon(\alpha) \) is of Hausdorff dimension \( \frac{1}{\alpha} \). This is the same with the Engel series expansion (see [4]).
(3) If \( \epsilon = \epsilon(k, t) = -k^t \), then \( t = 1 \) is a jump discontinuity of \( \dim_H E_\epsilon(\alpha) \).

In fact, it follows from Theorem 1.2 that

\[
\dim_H E_\epsilon(\alpha) = \begin{cases} 
\frac{1}{n}, & \text{when } 0 \leq t < 1; \\
\frac{1}{n+1}, & \text{when } t = 1.
\end{cases}
\]

2. Preliminary

In this section, we present some simple facts about GCF\( \epsilon \) expansion for later use. The first lemma concerns the relationships between \( A_n, B_n, C_n, D_n \) which are recursively defined by (1.3).

**Lemma 2.1** ([2, 3, 5]). For all \( n \geq 1 \) we have

(i) \( C_n = (k_n + 1)C_{n-1} + D_{n-1} > 0, \ C_0 = 1 \).

(ii) \( D_n = k_n \epsilon(k_n)C_{n-1} + (1 + \epsilon(k_n))D_{n-1}, \ D_0 = 0, \) and \( D_n \geq 0 \) when \( \epsilon \geq 0; D_n < 0 \) when \( \epsilon < 0 \).

(iii) \( k_n C_n + D_n = (k_n C_{n-1} + D_{n-1})(k_n + 1 + \epsilon(k_n)) > 0 \).

(iv) \( B_n C_n - A_n D_n = (B_N C_N - A_N D_N) \prod_{i=N+1}^{n}(k_i + 1 + \epsilon(k_i)) > 0, \forall 0 \leq N < n \).

Now we define the cylinder set as follows. For any non-decreasing integer vector \( (k_1, \ldots, k_n) \), define the \( n \)-th order cylinders as follows

\[
B(k_1, \ldots, k_n) = \{ x \in (0, 1]: k_j(x) = k_j, \forall 1 \leq j \leq n \}.
\]

Then it is just the interval with the endpoints \( L_n = \frac{A_n}{C_n} \) and \( R_n = \frac{A_n B_n}{k_n C_n + D_n} . \)

As a consequence, the length of \( B(k_1, \ldots, k_n) \) is

\[
|B(k_1, k_2, \ldots, k_n)| = \frac{B_n C_n - A_n D_n}{C_n(k_n C_n + D_n)}.
\]

A further calculation shows \( (k_{n+1} = k) \)

\[
|B(k_1, k_2, \ldots, k_n, k)| = \frac{B_n C_n - A_n D_n}{(k C_n + D_n)((k + 1)C_n + D_n)}.
\]

From now on until the end of this paper, we fix a point \( x \in E_\epsilon(\alpha) \), and let \( k_n = k_n(x) \) be the \( n \)-th partial quotient of \( x \). The numbers \( A_n, B_n, C_n, D_n \) be recursively defined by (1.3) for \( x \).

The following simple inequalities will be used frequently. For easy reference, we list them as a lemma.

**Lemma 2.2.** When \( n \) is large enough (say \( n \geq N \)) we have,

(a) \( k_n - k_{n-1} \geq \frac{k_n}{2} \).
(b) \( k_n - k_n^{\delta} \geq \frac{k_n}{2} \) for \( \delta < 1 \).
(c) \( k_n^{\alpha} - k_n \geq \frac{k_n}{2} \).
Proof. Since $x \in E_\varepsilon(x)$ with $\alpha > 0$, and $k_n$ is integer, it’s obvious that
$$k_n \geq k_n^\alpha \Rightarrow k_n \geq k_{n-1} + 1 \Rightarrow k_n \geq n.$$ So when $n \geq \max \{2^\alpha, 2^{1-\delta}\}$, all of the inequalities (a), (b) and (c) hold. □

The next result concerns the growth of $C_n = C_n(x)$.

Lemma 2.3. For all $n \geq N_1$ we have
$$C_n \geq \frac{k_n}{2} \frac{k_n-1}{2} \cdots \frac{k_{N_1}}{2} C_{N_1-1}.$$ Proof. Since $k_i \geq k_{i-1}$ for all $i > 1$, so by using Lemma 2.1(iii) $n$ times we get
$$k_n C_n + D_n \geq (k_1 C_0 + D_0) \prod_{i=1}^n (k_i + 1 + \varepsilon(k_i)) > 0.$$ This gives
$$D_n \geq -k_n C_n$$ for all $n \geq 1.$

Then by Lemma 2.1(i) and Lemma 2.2(a), we get
$$C_n \geq (k_n + 1 - k_{n-1}) C_{n-1} \geq (k_n + 1 - k_{n-1}) C_{n-1} \geq \frac{k_n}{2} C_{n-1} \quad \text{when } n \geq N_1.$$ Iterating this process enables us to conclude the result. □

The second one concerns the growth of $k_n^\alpha C_n + D_n$.

Lemma 2.4. For any $n \geq N_1$ we have
$$k_n^\alpha C_n + D_n \geq \frac{k_n^\alpha}{2} \frac{k_n^\alpha-1}{2} \cdots \frac{k_{N_1}^\alpha}{2} C_{N_1}.$$ Proof. By (2.3) and Lemma 2.2(c), we get
$$k_n^\alpha C_n + D_n \geq (k_n^\alpha - k_n) C_n \geq \frac{k_n^\alpha}{2} C_n, \quad \text{when } n \geq N_1.$$ Using $k_n \geq k_{n-1}$, the result (2.4) also gives that
$$C_n \geq \frac{k_n}{2} C_{n-1} \geq \frac{k_{n-1}}{2} C_{n-1} \geq \cdots \geq \frac{k_{N_1}}{2} C_{N_1}.$$ Substituting (2.6) into (2.5) to get the result. □

The following corollary will be used for getting the upper bound of $\dim_H E_\varepsilon(x)$. 


Corollary 2.5. Let
\[ L_1 = \frac{3^{N_1-1}k_1k_2 \cdots k_{N_1-1}}{C_{N_1}C_{N_1-1}}, \quad L_2 = \frac{1}{C_{N_1}C_{N_1-1}} \quad \text{and} \quad L_3 = \frac{2/(k_1k_2 \cdots k_{N_1-1})}{C_{N_1}C_{N_1-1}}, \]
where \( N_1 \) is given by Lemma 2.2. Then for any \( n \geq N_1 \) we have
1. If \( \epsilon(k_n) \leq k_n \) for all \( n \geq 1 \), then
   \[ \frac{B_nC_n - A_nD_n}{C_n(k_n^2C_n + D_n)} \leq L_1 \cdot \frac{12}{k_1^\alpha} \frac{12}{k_2^\alpha} \cdots \frac{12}{k_{N_1}^\alpha}; \]
2. If \( \epsilon(k_n) \leq -k_n \) for all \( n \geq 1 \), then
   \[ \frac{B_nC_n - A_nD_n}{C_n(k_n^2C_n + D_n)} \leq L_2 \cdot \frac{4}{k_1^{\alpha+\epsilon}} \frac{4}{k_2^{\alpha+\epsilon}} \cdots \frac{4}{k_{N_1}^{\alpha+\epsilon}}; \]
3. If \( \epsilon(k_n) = -k_n - 1 + \frac{1}{k} \) for all \( n \geq 1 \), then
   \[ \frac{B_nC_n - A_nD_n}{C_n(k_n^2C_n + D_n)} \leq L_3 \cdot \frac{4}{k_1^{\alpha+\epsilon}} \frac{4}{k_2^{\alpha+\epsilon}} \cdots \frac{4}{k_{N_1}^{\alpha+\epsilon}}. \]

Proof. From Lemma 2.1(iv) we can get immediate
\[ B_nC_n - A_nD_n \leq 3^n k_1 k_2 \cdots k_n \quad \text{when} \quad \epsilon(k_n) \leq k_n \quad \text{for all} \quad n \geq 1; \]
\[ B_nC_n - A_nD_n \leq 1 \quad \text{when} \quad \epsilon(k_n) \leq -k_n \quad \text{for all} \quad n \geq 1; \]
\[ B_nC_n - A_nD_n = \frac{1}{(k_1k_2 \cdots k_n)} \quad \text{when} \quad \epsilon(k_n) = -k_n - 1 + \frac{1}{k} \quad \text{for all} \quad n \geq 1. \]
Combining these with Lemma 2.3 and the Lemma 2.4, we get the three results. \( \square \)

The following results will be used for getting the lower bound of \( \dim_H E_c(\alpha) \)

Lemma 2.6. For any \( n \geq N_1 \) we have,
1. \( \frac{(k_n^\alpha - k_n)C_n}{k_n^\alpha C_n + D_n} \geq \frac{1}{k} \) if \( \epsilon(k_n) \leq k_n \),
2. \( C_n(k_nC_n + D_n) \leq 2^{n-2N_1+1} k_n (k_n k_{n-1} \cdots k_{N_1+1} C_{N_1})^2 \) if \( \epsilon(k_n) \leq k_n \),
3. \( C_n(k_nC_n + D_n) \leq 2^{n-2N_1} k_n (k_n k_{n-1} \cdots k_{N_1+1} C_{N_1})^2 \) if \( \epsilon(k_n) \leq -k_n \).

Proof. We first show \( D_n \leq k_nC_n \) for \( \epsilon(k_n) \leq k_n \). This is true for \( n = 1 \).
Furthermore, suppose \( D_n-1 \leq k_{n-1}C_{n-1} \). Then by Lemma 2.1 and \( \epsilon(k_{n-1}) \leq k_{n-1} \),
\[ D_n = k_n \epsilon(k_n) C_{n-1} + \epsilon(k_n) D_{n-1} + D_{n-1} \]
\[ \leq k_n^2 C_{n-1} + k_n D_{n-1} + k_{n-1} C_{n-1} \]
\[ \leq k_n^2 C_{n-1} + k_n D_{n-1} + k_n C_{n-1} = k_n C_n. \]
Thus when \( \epsilon(k_n) \leq k_n \) we have
\[ k_n^2 C_n + D_n \leq (k_n^2 + k_n) C_n \leq 2k_n^2 C_n, \]
and by Lemma 2.1(i),
\[ C_n = (k_n + 1) C_{n-1} + D_{n-1} \]
By induction one has that, for any $1 \leq N_1 < n$,
\begin{equation}
C_n \leq 2^{n-N_1}(k_n k_{n-1} \cdots k_{N_1+1})C_{N_1}.
\end{equation}

From Lemma 2.2(c) we have $k_n - k_{n-1} \geq \frac{k_n}{2}$ when $n \geq N_1$. Combine this with (2.7) to get that
\[
\frac{(k_n^2 - k_n)C_n}{k_n^2 C_n + D_n} \geq \frac{\frac{1}{4} k_n^2 C_n}{2 k_n^2 C_n} = \frac{1}{4} \quad \text{when } n \geq N_1.
\]

Combine (2.7) and (2.8), we get
\[
C_n(k_n C_n + D_n) \leq 2^{2n-2N_1+1} k_n (k_n k_{n-1} \cdots k_{N_1+1})^2 C_{N_1}^2
\]
when $\epsilon(k_n) \leq k_n$.

In case of $\epsilon(k_n) = -k_n$. By using Lemma 2.1(i),(ii), we get
\[
D_n \leq -k_n^2 C_{n-1} + (1 - k_n)D_{n-1}
= -k_n C_n + k_n C_{n-1} + D_{n-1}
= -k_n C_n + C_n - C_{n-1}.
\]

It gives that
\[
D_n + k_n C_n \leq C_n \quad \text{when } \epsilon(k_n) \leq -k_n.
\]

Thus by (2.8), we obtain
\[
C_n(k_n C_n + D_n) \leq C_n^2 \leq 2^{2n-2N_1+1} (k_n k_{n-1} \cdots k_{N_1+1})^2 C_{N_1}^2
\]
when $\epsilon(k_n) \leq -k_n$.

\[
\square
\]

**Corollary 2.7.** Let $L_4 = \frac{B_n C_n - A_n D_n}{2^{n-N_1+1} C_{N_1}}$, $L_5 = \frac{(1-\rho)^{N_1}}{2^{2n-N_1}(C_{N_1})^2}$ and $L_6 = \frac{1/(k_{n+1} \cdots k_{N_1+1})}{2^{2n-N_1}(C_{N_1})^2}$ be three constants, where $N_1$ is given by Lemma 2.2. Then for any $n \geq N_1$ we have

1. If $-k_n^2 \leq \epsilon(k_n) \leq k_n$ with $0 \leq \delta < 1$ for all $n \geq 1$, then
\[
\frac{B_n C_n - A_n D_n}{C_n (k_n C_n + D_n)} \cdot \frac{(k_n^2 - k_n) C_n}{k_n^2 C_n + D_n} \geq L_4 \cdot \frac{1}{2^{4n} k_n (k_n k_{n-1} \cdots k_{N_1+1})};
\]

2. If $-k_n - \rho < \epsilon(k_n) \leq -k_n$ for all $n \geq 1$, then
\[
\frac{B_n C_n - A_n D_n}{C_n (k_n C_n + D_n)} \cdot \frac{(k_n^2 - k_n) C_n}{k_n^2 C_n + D_n} \geq L_5 \cdot \frac{(1-\rho)^{N_1}}{2^{2n} (k_n k_{n-1} \cdots k_{N_1+1})};
\]

3. If $\epsilon(k_n) = -k_n - 1 + \frac{1}{k_n}$ for all $n \geq 1$, then
\[
\frac{B_n C_n - A_n D_n}{C_n (k_n C_n + D_n)} \cdot \frac{(k_n^2 - k_n) C_n}{k_n^2 C_n + D_n} \geq L_6 \cdot \frac{1}{2^{2n} (k_n k_{n-1} \cdots k_{N_1+1})};
\]
Proof. By Lemma 2.1(iv) and Lemma 2.2(b), when \( n \geq N_1 \) we have,

1. If \( \epsilon(k_n) \geq -k_n^\delta \) with \( 0 \leq \delta < 1 \), then
   \[ B_nC_n - A_nD_n \geq \frac{k_n}{2} \frac{k_{n+1} - 1}{2} (B_{N_1}C_{N_1} - A_{N_1}D_{N_1}) \]

2. If \( -k - \rho < \epsilon(k_n) \leq -k_n \) with \( 0 < \rho < 1 \), then
   \[ B_nC_n - A_nD_n \geq (1 - \rho)^n \]

3. If \( \epsilon(k_n) = -k_n - 1 + \frac{1}{k_n} \), then
   \[ B_nC_n - A_nD_n = \frac{1}{(k_1k_2 \cdots k_n)} \]

Combine these with Lemma 2.6 to get the above three results. \( \square \)

3. The Hausdorff dimension of \( E_\delta(\alpha) \)

The proof of Theorem 1.2 is divided into two parts: one for upper bound, the other for lower bound.

3.1. Upper bound

For any non-decreasing integer vector \((k_1, k_2, \ldots, k_n)\), define

\[ I(k_1, k_2, \ldots, k_n) = \bigcup_{k=\kappa_n}^{\infty} \{ x \in (0, 1) : k_i(x) = k_i, \forall 1 \leq i \leq n, k_{n+1}(x) = k \} \]

Then it is clear that

\[ E_\epsilon(\alpha) \subset \bigcup_{n=1}^{\infty} \bigcup_{\kappa_{i+1} \geq \kappa_i, 1 \leq i \leq n-1} I(k_1, k_2, \ldots, k_n) \]

So by (2.2), one has

\[ |I(k_1, k_2, \ldots, k_n)| = \sum_{k=\kappa_n}^{\infty} \frac{B_nC_n - A_nD_n}{((k+1)C_n + D_n)(kC_n + D_n)} \]
\[ = \frac{B_nC_n - A_nD_n}{C_n} \sum_{k=\kappa_n}^{\infty} \left( \frac{1}{kC_n + D_n} - \frac{1}{(k+1)C_n + D_n} \right) \]
\[ = \frac{B_nC_n - A_nD_n}{C_n(k_1^\alpha C_n + D_n)} \]

(i) Let’s first consider the case of \(-k^\delta \leq \epsilon(k) \leq k\) with \( 0 \leq \delta < 1 \).

Since the series \( \sum_{k=1}^{\infty} \left( \frac{1}{k^s} \right) \) is convergent for any \( s_1 > 1/\alpha \), there exists an integer \( n_1 \) large enough such that for any \( n \geq n_1 \),

\[ \sum_{k=n}^{\infty} \left( \frac{12}{k^s} \right) \leq 1. \]
By (3.1), (3.2), (3.3) and Corollary 2.5(1), we find that for any 
\( n \geq N = \max\{N_1, n_1\} \), the \( s_1 \)-dimensional Hausdorff measure of \( E_\epsilon(\alpha) \) can be estimated as

\[
\mathcal{H}^{s_1}(E_\epsilon(\alpha)) 
\leq \liminf_{n \to \infty} \sum_{k_{i+1} \geq k_i^N} |I(k_1, k_2, \ldots, k_n)|^{s_1}
\leq \liminf_{n \to \infty} \sum_{k_{i+1} \geq k_i^N} L_1(\alpha)^{s_1} \sum_{k_{N+1} \geq k_N} \left( \frac{12}{k_{n+1}} \right)^{s_1} \cdots \sum_{k_{N} \geq k_{n-1}} \left( \frac{12}{k_{n-1}} \right)^{s_1}
\leq \liminf_{n \to \infty} \sum_{k_{i+1} \geq k_i^N} L_1(\alpha)^{s_1} < \infty.
\]

which gives that \( \dim_H E_\epsilon(\alpha) \leq s_1 \). Since \( s_1 > \frac{1}{\alpha} \) is arbitrary, we get \( \dim_H E_\epsilon(\alpha) \leq \frac{1}{\alpha} \).

(ii) In case of \(-k - \rho < \epsilon(k) \leq -k\) with constant \(0 < \rho < 1\). By using Corollary 2.5(2), we can prove, in the same way as we prove (i) that \( \dim_H E_\epsilon(\alpha) \leq \frac{1}{\alpha^2} \).

(iii) In case of \( \epsilon(k) = -k - 1 + \frac{1}{k} \). By using Corollary 2.5(3), we can also prove, in the same way as we prove (i) that \( \dim_H E_\epsilon(\alpha) \leq \frac{1}{\alpha^2} \).

3.2. Lower bound

In order to estimate the lower bound, we recall the classical dimensional result concerning a specially defined Cantor set.

**Lemma 3.1** (Falconer [1]). Let \( I = E_0 \supset E_1 \supset E_2 \supset \cdots \) be a decreasing sequence of sets, with each \( E_n \), a union of a finite number of disjoint closed intervals. If each interval of \( E_{n-1} \) contains at least \( m_n \) intervals of \( E_n \) in \( n = 1, 2, \ldots \) which are separated by gaps of at least \( \eta_n \), where \( 0 < \eta_{n+1} < \eta_n \) for each \( n \). Then the lower bound of the Hausdorff dimension of \( E \) can be given by the following inequality:

\[
\dim_H \left( \cap_{n \geq 1} E_n \right) \geq \liminf_{n \to \infty} \frac{\log(m_1m_2 \cdots m_{n-1})}{-\log(m_n\eta_n)}.
\]

Let \( f_n(\alpha) = 2^{n+\alpha^2+\cdots+\alpha^n} = 2^{\frac{\alpha^{n+1} - \alpha}{\alpha - 1}} \). Define

\[
(3.4) \quad E(f) = \{x \in (0, 1) : f_n(\alpha) \leq k_n(x) < 2f_n(\alpha) \ \forall n \geq 1\}.
\]

It is easy to see that, when \( x \in E(f) \), we have

\[
k_n(x)^\alpha < (2f_n(\alpha))^\alpha = f_{n+1}(\alpha) \leq k_{n+1}(x).
\]

This implies that

\[
E(f) \subset E_\epsilon(\alpha).
\]
For each $n \geq 1$, let $E_n(f)$ be the collection of cylinders

$\bigcup_{j=k_n^a}^{k_n^{-1}} \{B(k_1, \ldots, k_n) : \; f_i(\alpha) \leq k_i(x) < 2f_i(\alpha) \; \forall \; 1 \leq i \leq n, \; k_{n+1}(x) = j\}$.

Then

$E(f) = \bigcap_{n=1}^{\infty} E_n(f)$

and $E(f)$ fulfills the construction of the Cantor set in Lemma 3.1. Now we specify the integers $\{m_n, n \geq 1\}$ and the real numbers $\{\eta_n, n \geq 1\}$.

Due to the definition of $E_n$, each interval of $E_{n-1}$ contains $m_n = f_n(\alpha) = 2^{n^{-1} \cdot \frac{1}{\alpha}}$ intervals of $E_n$, and

$$m_1m_2 \cdots m_{n-1} \geq 2^{\sum_{i=1}^{n-1} \frac{\alpha-\alpha_i-\alpha_i+1}{n-1}} = 2^{\frac{\alpha_{n+1}-\alpha_{n+2}+\cdots+\alpha_{n+1}}{n+1}}.$$  

In addition, any two of intervals in $E_n$ are separated by at least an interval $J_n(f)$ defined by

$\bigcup_{j=k_n}^{k_n^a} \{B(k_1, \ldots, k_n) : \; f_i(\alpha) \leq k_i(x) < 2f_i(\alpha), \; \forall \; 1 \leq i \leq n, \; k_{n+1}(x) = j\}$.

From (2.2) we get

$$|J_n(f)| = \sum_{j=k_n}^{k_n^a} \frac{B_nC_n - A_nD_n}{(j+1)C_n + D_n} = \frac{B_nC_n - A_nD_n}{C_n} \sum_{j=k_n}^{k_n^a} \frac{1}{jC_n + D_n} - \frac{1}{(j+1)C_n + D_n}$$

$$= \frac{B_nC_n - A_nD_n}{C_n} \left( \frac{1}{k_nC_n + D_n} - \frac{1}{k_nC_n + D_n} \right)$$

$$= \frac{B_nC_n - A_nD_n}{C_n(k_nC_n + D_n)} \cdot \frac{(k_n^a - k_n)C_n}{(k_n^aC_n + D_n)}.$$  

Thus and by Corollary 2.7(1), we get when $-k^\delta \leq \epsilon(k) \leq k$ with $0 < \delta < 1$,

$$|J_n(f)| \geq L_4 \cdot \frac{1}{2^{2n}k_n(k_nk_{n-1} \cdots k_{N+1})}.$$  

And when $-k - \rho < \epsilon(k) \leq -k$ with $0 < \rho < 1$, by Corollary 2.7(2), we get

$$|J_n(f)| \geq L_5 \cdot \frac{(1 - \rho)^{n-N_i}}{2^{2n}(k_nk_{n-1} \cdots k_{N+1})^2}.$$  

When $\epsilon(k) = -k - 1 + \frac{1}{k}$, by Corollary 2.7(3), we get

$$|J_n(f)| \geq L_6 \cdot \frac{1}{2^{2n}(k_nk_{n-1} \cdots k_{N+1})^3}.$$
In view of (3.4), the partial quotients $k_n$ satisfying that $2^{\frac{n+1}{\alpha-1}} \leq k_n \leq 2^{\frac{n+1}{\alpha-1}}$ for all $n \geq 1$. Therefore,

(i) when $-k \delta \leq \epsilon(k) \leq k$ with $0 < \delta < 1$,

$$|J_n(f)| \geq L_4 \left( 2^{3n} \cdot \frac{2^{\frac{n+1}{\alpha-1}}}{(\alpha-1)^2} \right) \left( 2^{\sum_{i=N+1}^{N+2-[(n-N)(\alpha-1)]}} \right)^{-1} \geq L_4 \left( 2^{3n} \cdot \frac{2^{\frac{n+2-\alpha N+2-[(n-N)(\alpha-1)]}{\alpha-1}}} {(\alpha-1)^2} \right) = \eta_n.$$ 

(ii) when $-k - \rho < \epsilon(k) \leq -k$ with $0 < \rho < 1$,

$$|J_n(f)| \geq L_5 \cdot (1 - \rho)^{n-N_1} \left( 2^{2n} \cdot \frac{2^{\frac{n+2-\alpha N+2-[(n-N)(\alpha-1)]}{\alpha-1}}} {(\alpha-1)^2} \right)^{-1} \geq L_5 \cdot (1 - \rho)^{n-N_1} \left( 2^{2n} \cdot \frac{2^{\frac{n+2-\alpha N+2-[(n-N)(\alpha-1)]}{\alpha-1}}} {(\alpha-1)^2} \right)^{-1} = \eta'_n.$$ 

(iii) when $\epsilon(k) = -k - 1 + \frac{1}{k}$,

$$|J_n(f)| \geq L_6 \left( 2^{2n} \cdot \frac{2^{\frac{n+1}{\alpha-1}}} {\alpha} \right)^{-1} \geq L_6 \left( 2^{2n} \cdot \frac{2^{\frac{n+2-\alpha N+2-[(n-N)(\alpha-1)]}{\alpha-1}}} {(\alpha-1)^2} \right)^{-1} = \eta''_n.$$ 

As a result of (3.5), in the case (i), we get

$$\lim_{n \to \infty} \frac{\log_2(m_1 \cdots m_{n-1})}{\alpha^{n+1}} \geq \frac{1}{(\alpha-1)^2},$$

$$\lim_{n \to \infty} \frac{-\log_2 m_n \eta_n}{\alpha^{n+1}} = \frac{\alpha}{(\alpha-1)^2}.$$

Combining this with Lemma 3.1, we get when $-k \delta \leq \epsilon(k) \leq k$ with $0 < \delta < 1$,

$$\dim H E(\alpha) \geq \dim H E(f) \geq \frac{1}{\alpha}.$$ 

Similarly, when $-k - \rho < \epsilon(k) \leq -k$ with $0 < \rho < 1$ (case ii),

$$\dim H E(\alpha) \geq \frac{1}{\alpha+1}.$$ 

And when $\epsilon(k) = -k - 1 + \frac{1}{k}$ (case iii),

$$\dim H E(\alpha) \geq \frac{1}{\alpha+2}.$$ 

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