

The Pseudo-Covariational Reasoning Thought Processes in Constructing Graph Function of Reversible Event Dynamics Based on Assimilation and Accommodation Frameworks¹

SUBANJI, Rajiden

Department of Mathematics, State University of Malang, East Java,
Indonesia; Email: subanjimat@yahoo.co.id

SUPRATMAN, Ahman Maedi*

Mathematics Education Courses and Pedagogy, Faculty of Education,
University of Siliwangi Tasikmalaya, West Java, Indonesia;
Email: supratman_id@yahoo.com

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This study discussed about how pseudo-thinking process actually occurs in the mind of the students, used Piaget's frame work of the assimilation and accommodation process. The data collection is conducted using *Think-Out-Loud* (TOL) method. The study reveals that pseudo thinking process of covariational reasoning occurs originally from incomplete assimilation, incomplete accommodation process or both. Based on this, three models of incomplete thinking structure constructions are established:

- (1) Deviated thinking structure,
- (2) Incomplete thinking structure on assimilation process, and
- (3) Incomplete thinking structure on accommodation process.

Keywords: pseudo-thinking process, assimilation, accommodation covariational reasoning, graph, invert dynamics events

MESC Classification: C30

MSC2010 Classification: 97C30

1. INTRODUCTION

¹ A draft version of the article was presented at the 2013 Joint International Conference on Mathematics Education held at Seoul National University, Seoul 151-742, Korea; November 1–2, 2013 (*cf.* Supratman, 2013).

* Corresponding author

There have been a plenty of studies on pseudo thinking processes conducted. However, they were different only in using the terms to refer to the same contents and objects. Besides, they were focused on describing the existence of pseudo thinking process. Vinner (1997), for example, used the term of *Pseudo-Analytic versus Analytic*. According to Vinner (1997), the pseudo analytic thought process is in this way: when a problem is given to the students, the problem will evoke their mental schema. The students will recall the similarity between a given question and the types of questions they have in their mind. Next, the mental scheme will evoke the types of questions and how to solve them (procedures). The process of remembering how to solve the questions can take place fuzzy (fuzzy memory). How to solve the questions are then applied in order to get the solutions.

Besides, Lithner (2000) used *the Established Experience (EE) versus Plausible Reasoning (PR)* to refer to pseudo thinking process. In this point, the students' thinking in solving problems is described in four structures:

- (1) Understanding the problem,
- (2) Selecting a strategy,
- (3) The implementing the strategy, and
- (4) Drawing conclusions (results obtained).

The structures 1–4 is a process of thinking of PR when the components involved in reasoning contains mathematical properties. The structures 1–4 is the thought process of understanding EE when discovered and procedures based on the experiences.

Furthermore, Leron & Hazzn, O. (2009) studied *Dual Process Theory* from Kahneman (2002) (*the System 1 process versus the System 2 process*). In the dual process, the theory and behavioral processes in completing tasks can be grouped into two different model scalled the process of system1 (S1) and the system 2 (S2). The process S1 has several characteristics: fast, automatic, effort less, unconscious, and difficult to adjust. The S2 processes are slow, careful, hard effort, accurate, and relatively flexible. The two systems are different in terms of "catch". It is about how the speed and easiness come to the mind. In many situations, the S1 and S2 work together to generate the appropriate answers, but in some cases (such as non-routine problems), the S1 generates in appropriate answers (non-normative) quickly and automatically, particularly when the S2 does not control by supervising and criticizing the way the S1 answers or rejecting the truth.

Problems solving are essential of teaching and learning mathematics process (Lee, Brown & Orrill, 2011; Wu & Adam, 2006; Lee, 2005). But in teaching and learning process, there are many mathematical teachers who taught the procedure without explaining why the particular procedure was used. As a result, the students believed that in resolving the problem, it was enough to choose the resolution procedure in accordance with the problem given. In this case, the focus of learning is not why the certain procedure was

used to resolve the problem, but *which* procedure was chosen to solve the problem and *how* to solve the problem using the procedure. *Procedure-emphasized-learning process* resulted in the students' behavior only "to copy" the procedure that was carried out by the teacher, without understanding why they must use the procedure. Therefore, when the problem was slightly changed, students were incapable to resolve the modified problem. The emphasis on the procedure without giving the appropriate reasons was the beginning of the pseudo thinking process formation.

1.1. Research questions

How did the pseudo thinking occur? This study aims at answering such a question. In particular, the study is concerned with how pseudo-thinking process actually occurs in the mind of the students. For the purpose, Piaget's framework of the assimilation and accommodation process is accommodated

1.2. Review of the Literature

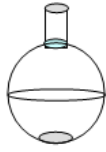
This section discusses two issues. The first deals with the pseudo thinking process in the covariational reasoning and the second: deals with Piaget's assimilation and accommodation framework.

1.3. Pseudo Thinking Process in the Covariational reasoning

As a matter of fact, the pseudo thinking process in the covariational reasoning is called as pseudo covariational reasoning. The covariational reasoning is the mental activity in coordinating two quantities (the independent and dependent variable) linked with the change methods from one quantity to another.

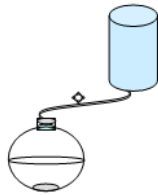
The results of the pseudo thinking process in the covariational reasoning are known as the pseudo covariational reasoning behavior. The pseudo covariational reasoning behavior could appear from the students' answers "true" but the students could not give any justification, or the answers "false" but in fact the students understood (or could resolve) the problem after the reflection. This study discusses only the pseudo covariational reasoning from the answer "false", and hereinafter will be mentioned as the pseudo covariational reasoning.

The problem of the covariation studied in this research is the development of the problem studied by Carlson, Jacobs, Coe, Larsen & Hsu (2002). The difference/development is presented in the following figures.



Imagine this bottle filling with water. Sketch a graph of the height. as a function of the amount of water that is in the bottle

Figure 1. The Carlson' Covariation Problem



To Notice the figure!

The tube-shaped upper bottle and the ball-shaped lower bottle are connected with water-pipe. The upper bottle is water-full and the lower-bottle is water-empty. The water-pipe has water-tap. If the water-tap is opened, then the water in upper bottle flows through the lower bottle. Sketch a graph of the height water in the lower bottle as a function of the height water in upper bottle. Given are reasoning of the solution!

Figure 2. This Research' Covariation Problem

There are 3 (three) differences between the Carlson's covariation problem and the co-variation problem in this article:

- (1) The graph characteristics,
- (2) The process of the co-variation, and
- (3) The level of the co-variation.

In his study, Carlson found the existence of pseudo-analytic behavior, but he did not explain how this pseudo-analytic occurred. This research studies the occurrence process of pseudo covariational reasoning by using the assimilation and accommodation framework from Piaget.

1.4. Piaget's Assimilation and Accommodation Framework

Piagetian theory (Tall, 2004) explained a tripartite theory of abstraction. The first, empirical abstraction focusing on how the child constructs meaning for the properties of objects. The second, pseudo-empirical abstraction, focusing on construction of meaning for the properties of actions on objects. The third, reflective abstraction focused on the idea of how 'actions and operations become thematized objects of thought or assimilation. Piaget argued that predisposition to adjust to environment, involves assimilation and accommodation. To Piaget, this concept of adaptation is the most important principle of human functioning. Included in this process is assimilation and accommodation.

The assimilation process is the process of new stimulus integration into the scheme that has been formed. The accommodation process is the process of new stimulus integration through the modification of the old scheme or through the formation of the new scheme

to adapt with the new acquired stimulus. In solving the problem, the process of assimilation and accommodation will continue to take place until the existence of balance (equilibrium).

To have more understanding, the occurrences of assimilation and accommodation process are visualized in Figure 3 as follows.

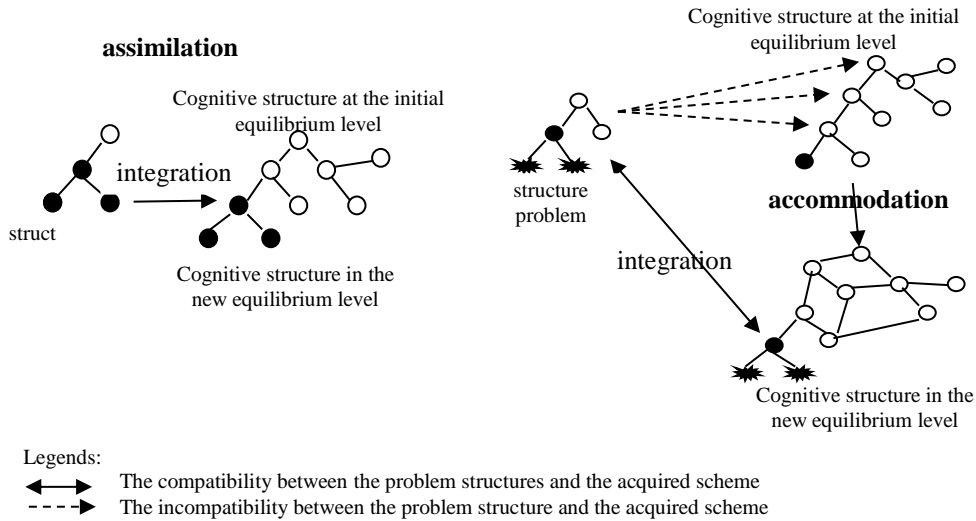


Figure 3. The occurrence of the Assimilation and Accommodation Process

For the problem having a complex structure, it will be difficult for the assimilation or accommodation to happen. Therefore, it is necessary to divide the problem structure into simpler parts so that the process of assimilation and accommodation could take place. The division of problem into its simpler parts is known as the analytical process.

The study of pseudo thinking in the mathematics problem solving by Vinner (1997), Lithner (2000), Leron & Hazzn (2009), Pape (2004), and Carlson, Jacobs, Coe, Larsen & Hsu (2002) have not reached the investigation of the pseudo thinking process, but they merely showed the existence of pseudo thinking.

2. METHODOLOGY

A qualitative design was chosen for the present study in order to investigate the intricate thinking process (Bogdan & Biklen, 1992). To see the thinking process, the data were gathered using the *Think Out Loud* method (Olson, Duffy & Mack, 1984). This method was conducted by asking the research subjects to solve the problems and at the same time to tell how their thinking process is. *Think Aloud* was developed by the cogni-

tive psychologists aiming to study how someone solves the problem. When someone solves a problem, then what he or she thought about could be recorded and analyzed to decide the cognitive process related to the problems.

The research subjects were undergraduate students the 7th semester of Mathematics Department State University of Malang-Indonesia, who had taken the Calculus 2. In this study, the students were asked to solve the problem of covariance (task sheet instruments), and express, what he was thinking strongly (Think Out louds). Following which, the students obtain a solution; researchers examine the truth of student answers. If the student produces the correct answer, it is not used as a research subject. Conversely, if a student produces incorrect answers, the students are given the opportunity to reflection. If after reflection, students are not able to answer correctly, then the student will not be used as a research subject. Next take a student again to resolve the problem of covariance. The process is repeated until obtaining a student who shows the wrong answer, but after reflection students be able to fix to be the correct answer. Students who demonstrate an answer wrong and after reflection be able to fix to be the correct answer chosen as the subject of research.

The students had finished Calculus 2. In this study, students were to solve a covariational problem and to reveal (out loud) their thinking. Three activities of data collection were used, namely video taping, audio taping, and field notes. Video tape, audio tape, dan fieldnote were used simultaneously. Video tape was to record the subjects' expressions, audio tape was to record the subjects' voices, and fieldnote was to record the unique things occurred when the subjects solved the problems. The students' solutions were evaluated in terms of true or false answers. The students who had a false solution and were able to make changes to the solution after the reflection only, were used as the subjects. The students' answers were then analyzed and grouped based on the three characteristics of pseudo thinking which include

- (1) Deviated thinking structure,
- (2) Incomplete thinking structure on assimilation process, and
- (3) Incomplete thinking structure on accommodation process.

This process was continuously carried out until the data were saturated. Therefore, *snowball sampling* was used in the selection of the participants.

In this study, ten (10) students were selected as the participants/subjects. The subject's solutions were grouped according to characteristics of pseudo thinking. The five students' solutions were in the first characteristic, the four students' solutions were in the second characteristic, and one student's solution was in the last characteristic. To analyze the data, two students were asked to elaborate the first characteristic, two students to elaborate the second characteristic, and one student to elaborate the last characteristic.

3. RESULTS

The results of this study are divided into three sub-topics, namely incompleteness of thinking sub-structure in the assimilation process, incompleteness of thinking sub-structure in the accommodation, and incompatibility of thinking sub-structure use in the process of assimilation or accommodation. They are presented in the following.

3.1. Incompleteness of thinking sub-structure in the Assimilation Process

The occurrence process of the first form of pseudo covariational reasoning started from the incompleteness of thinking sub-structure in the process of assimilation and was not immediately followed by the reflection process. The nonexistence of reflection process results in the absence of control for the truth of the answer. Therefore, “wrong answer” produced was the final answer before the reflection. After the reflection, the students realized that there was “incompatibility” between the problem and the answer, and finally could fix it out into the correct answer.

In this case, the occurrence process of pseudo covariational reasoning:

- (1) was started from the incompleteness the thinking sub-structure in the process of the assimilation so as to produce the wrong answer,
- (2) The absence of the reflection the process, and
- (3) The true answer produced after the reflection process.

Incompleteness of substructure in the process of assimilation is a process of direct interpretation of the problem with more complex structure using a simple thinking structure. This thinking process was preceded by the imperfect assimilation process. In the process of problem solving, the assimilation took place, but the complex problem was interpreted to the simple problem. Therefore, it produced an inappropriate answer.

The first finding of the occurrence process of pseudo covariational reasoning is illustrated in Picture 4 as follows.

In the process of problem solving (before the reflection), the students only conduct the assimilation process, but did not produce the appropriate structure to the structure of the problem. In this case, their thinking structure was still incomplete; nonetheless it had been used to interpret a complex problem structure. Yet, it produced an inappropriate answer (wrong). After receiving the answer, the students did not go through the reflection again.

Furthermore, when given the opportunity for reflection, the disequilibrium took place again in the students' thinking process, with the result that they continued to the assimilation and accommodation process.

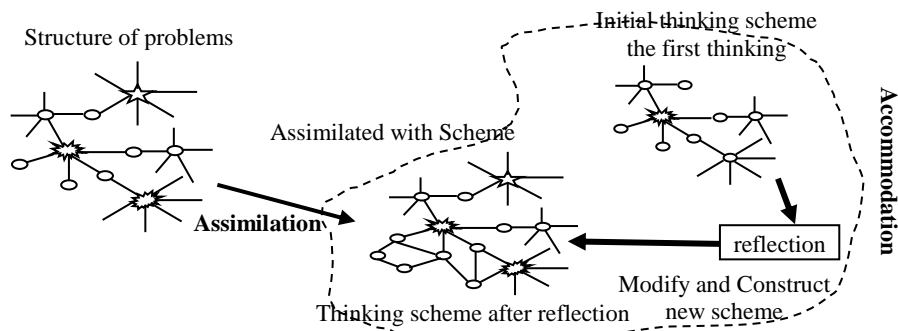


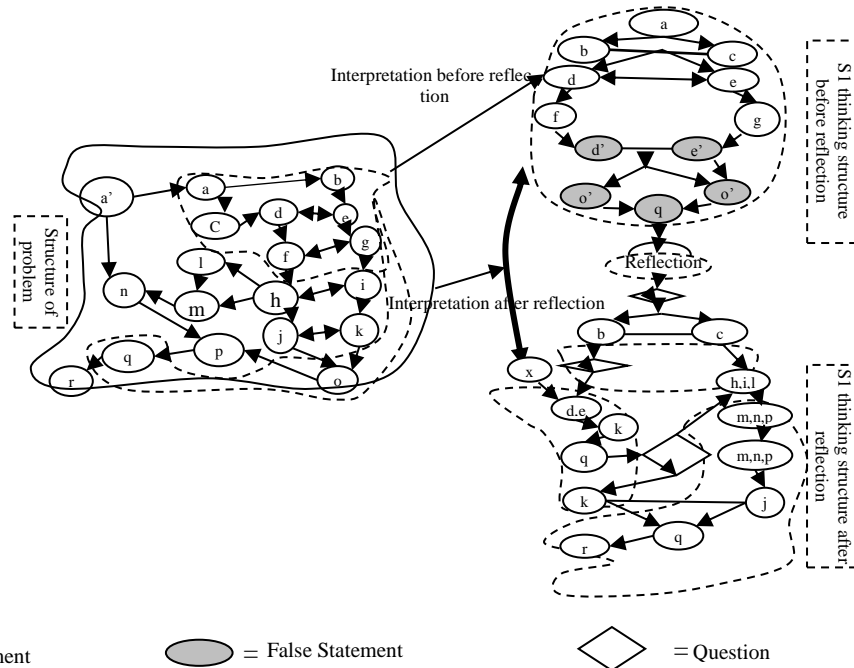
Figure 4. The sub-structure incompleteness in the process of assimilation

In the long run, they could get the correct answers. The students' pseudo thinking process (S1) that was started with thinking incompleteness substructure in assimilation process (in resolving the problem of the co-variation) is presented in Figure 5.

In the thinking process, it is obvious that S1 knew the problem, knew some of its relations, and knew some solution strategies. Therefore, the problem assimilation process and the relations assimilation processes happened, as well as the strategy assimilation. Before the reflection, the assimilation process dominated their thinking process.

S1 knew the relations between problems that if the tap was opened, water would flow from the upper bottle to the lower bottle. In this case, S1 had been able to integrate that the relations happened reversible event dynamics, "the water level in the upper bottle would decrease and caused the water level in the lower bottle to increase". Therefore, S1 thinking process was the relation assimilation.

From a simple relation, "the water level in the upper bottle decreased and the water level in the lower bottle water increased". Furthermore, S1 would focus on the final event that was when the water in upper bottle ran out, therefore the water level in the lower bottle was maximal. It was to acquired one point on the vertical axis ($0, h_{\max}$ (B)). Eventually, S1 also knew event when water in the lower bottle was empty and the upper bottle was full, therefore it was to acquire one other point (h_{\max} (A), 0). In this case, S1 chose the strategy to construct the graph by determining the point on the horizontal and the vertical axis.



Legends:

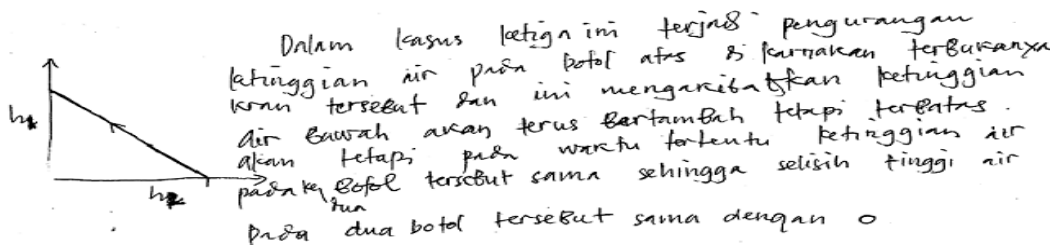
- = Statement
- = False Statement
- ◇ = Question

a	the problem presented: “construct the invert dynamic events function graph”	j	the increase in the dynamic height
a'	the problem encountered in the everyday life	k	the decrease in the constant height
d	the increase of the lower bottle volume	l	the narrow area of the bottom, the spacious middle area and the narrow upper part (symmetrical)
d'	water is full	m	the speed of the height change
e	the volume decrease in the upper bottle;	n	the narrow flat surface (the speed is high), the spacious flat surface (low speed)
e'	water runs out	o	pointing axis co-ordinate
f	the water height of the lower bottle water increases	o'	the point in the axis
g	the decrease in the water height upper bottle	p	the solution to the interval
h	the area of the flat surface changing	q	construct the graph: increased from fast to slow (from the bottom as far as the middle), from slow to fast (from the middle to the top)
I	the area of the fixed flat surface		
R	finished,		
S	other (pressure, pipe size)		

Figure 5. The Occurring of Pseudo Thinking Process of S1

By having one point in the horizontal axis and one point in the vertical axis, S1 immediately connected them by drawing a straight line. And the graph produced was the chosen conclusion, without having any further suspicion. S1 did not do the reflection again whether the answer had been appropriate or not to the problem. This means that the S1

thinking process had achieved the condition of equilibrium. They had felt the existence of compatibility between the thinking structure and the problem structure. The graph produced by S1 before the reflection is presented in Figure 6 as follows



English translation:

In this case, there is a reduction in the water level due to the opening of the bottle over the tap and this can make the lower water levels continue to grow but limited. However, at certain times the height of the water in the two bottles is the same, thus the difference in the two bottles of water zero (0).

Figure 6. The Graph produced by S1 before Reflection

Because the incomplete structure integration happened, the answer acquired by S1 was wrong. Nevertheless, it did not necessarily show that S1 could not solve this problem. Possibly, S1 had not used their thinking process optimally. Because S1 did not go through the reflection again, they were satisfied with the answer.

When the researcher asked S1: whether S1 were convinced about the answer? S1 fell into silence while holding the head, did the reflection “inquire (match) the graph with the given problem”. S1 began to doubt the answer. S1 muttered “it seems that something is wrong”. With this suspicion, S1 began to do the reflection again, by studying the problem from the very beginning.

In this process, S1 began to know the shape of the bottle: the upper bottle took the shape of a tube while the lower a ball. However, S1 still did not comprehend the influence of the shape of these bottles on the graph. On the other hand, S1 even questioned the speed of the water reduction in the upper bottle, when it was full and comparing it to the condition for the bottle with decreased water level, so as S1 looked for relations between the pressure and the speed of the water decrease in the upper bottle. This happened because the S1 thinking process S1 was a relation accommodation. S1 found that at first the decline was fast (because the pressure was high), and it became increasingly slow (because the pressure was increasingly low).

In this process, S1 studied only the upper bottle and did not consider the condition of the lower bottle, and it made the thinking structure formed was not yet enough (was not yet complete) to solve the given problem structure. The thinking structure formed from

the incomplete assimilation and accommodation processes had been used to interpret the problem S1 was dealing with. Therefore, the second graph made by S1 was wrong.

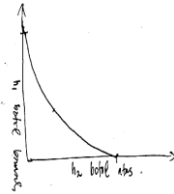


Figure 7. S1's second graph

In this problem solving process, S1 made a mistake on the analysis which was carried out only to the upper bottle, but it was used to construct the relations graph between the height of the upper bottle and the lower.

When acquiring the answer in Figure 7, S1 still doubted his answer (condition of disequilibria still took place), S1 questioned the influence of the ball-shaped lower bottle that had a different area in each of its heights. Therefore, the process of accommodation of the bottle shape happened (including the area of its layer), relations between the area of the layer and the height change speed. This could be seen from the statement S1 as follows.

S1: In this lower bottle, first the height of the water level grows fast, and increasingly slow, because it is affected by the shape of this bottle which is a ball, within which has narrow bottom, and it is getting increasingly bigger in the middle, this will influence the height of the water, the travel speed of the water.

From this statement, it is obvious that S1 is in the process of accommodation, that is the formation of the new thinking structure in relation to the speed of the water height change in the lower bottle (from the bottom through the middle). The change in height is from fast to slow, since it is affected by the area of the profile of the lower bottle. With this thinking structure, further development is by forming new structure for the case of the lower bottle (from the middle through the top), like this following statement.

S1: The water level in the lower bottle increases fast after the water level reaches the middle of the lower bottle, because the state of being affected by the shape of this bottle.

This S1's statement implicitly revealed the ball-shaped bottle, from the middle to the top, the area of its diameter from big to small and height changed speed from slow headed fast. With the formation of the more complete thinking structure through the process of the assimilation and accommodation especially for the lower bottle, S1 began to form the structure of relations between the decline in the water height in the upper bottle and the rise in the water height in the lower bottle. When the water reached the middle of the lower bottle, the water level in the upper bottle declined faster compared to the rise of the water height in the lower bottle (because of the *profile* of the upper bottle smaller com-

pared to the middle area of the lower bottle). With the development of this new structure, continued to the condition in the bottom and in the top of the lower bottle. There will be a complete structure that is used to integrate the structure of the given problem. The last graph produced by S1 is shown in the Figure 8 as follows.

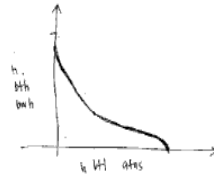


Figure 8. S1's Last Graph

After receiving this answer, the *equilibration* process took place. S1 had been absolutely certain of the answer. This showed that S1 was in the stage pseudo thinking (in this case pseudo covariational reasoning) from the wrong answer. When receiving the answer S1 did not do the reflection. In fact, S1 could resolve the problem correctly, but the habit “just to get the answer” dominated more of their thinking process, so as the received answer did not yet reflect the actual process of thinking (*pseudo*).

3.2. Incompleteness of Thinking Sub-Structure in the Accommodation

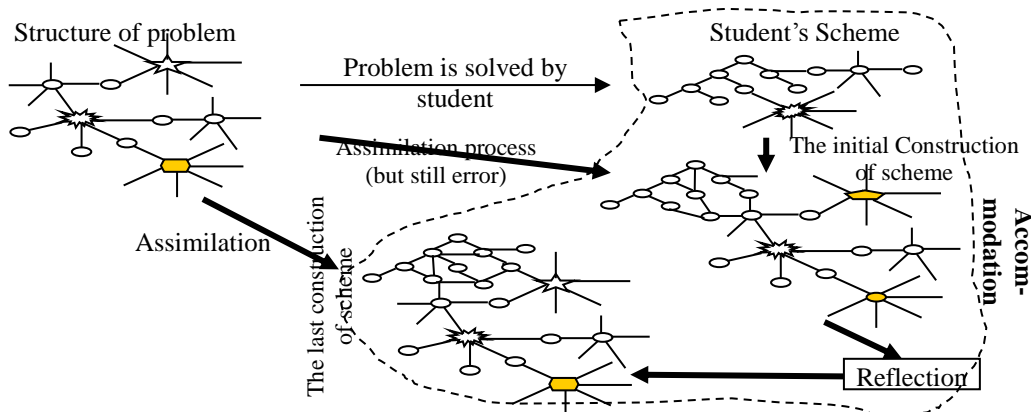


Figure 9. Incompleteness Thinking Substructure in the Accommodation Process

In problem solving process, the students' thinking process experiences incompleteness of the thinking sub-structure in the accommodation process, being marked with the occurrence of assimilation and accommodation process, but the thinking structures formed have not been appropriate with the structure of the problem (was incomplete), but it has been used to interpret the problem. Therefore, the answer produced was *inappropriate*

(wrong). After the reflection, there comes the awareness for the sub-structure incompleteness, it stimulates the process of accommodation to appear (the modification of the old scheme or the formation of the new scheme), eventually leads to the true answer.

The occurrence of the pseudo thinking process caused by incompleteness of sub-structure in the accommodation process is illustrated in Figure 9.

When resolving the problem of the covariation, the student (hereinafter mentioned S2) has done accommodation, however S2's accommodation did not produce a structure appropriate to the structure of the problem. With the incomplete structure, S2 used them to interpret the structure of a complex problem; therefore S2 was to receive an inappropriate answer (wrong). After receiving the answer, S2 did not do the reflection again. After being given an opportunity for reflection, disequilibrium took place again in S2's thinking, and it continued to the process of the assimilation, accommodation, and analytical. In the long run S2 could receive the true answer. The S2's pseudo thinking process in solving the problem of the covariation is illustrated in Figure 10.

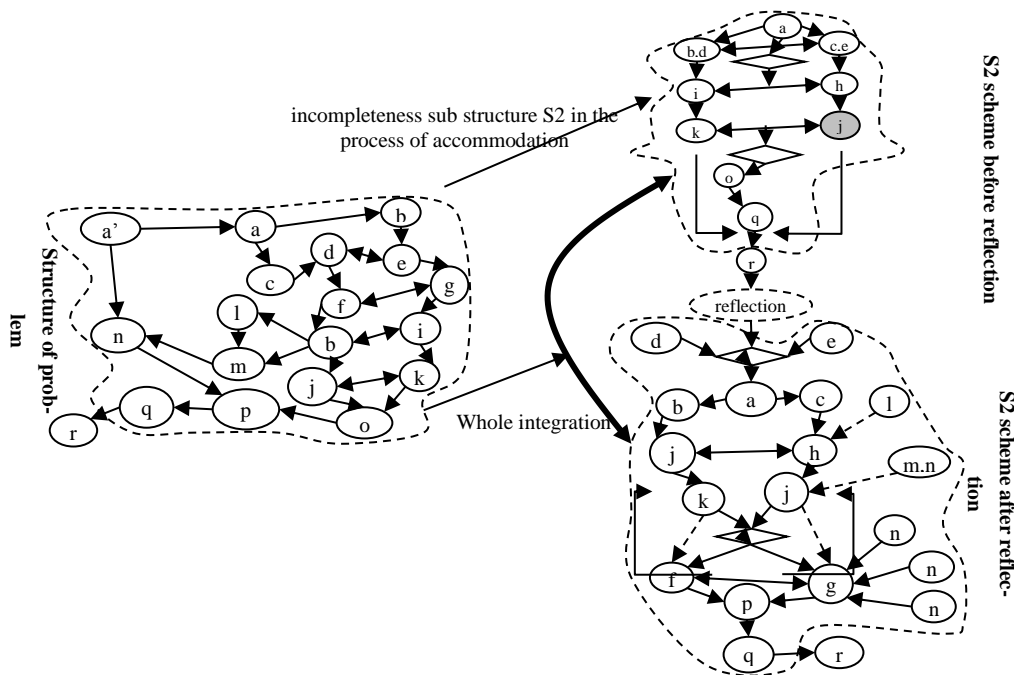


Figure 10. The Occurrence of the S2 Pseudo Thinking Process

In dealing with the problem of the covariation, S2 has understood some of the problems and some of their relations, which triggered the process of simple problem assimilation and relations assimilation to happen. S2 knew some problems and relations of the two bottles, because this problem had been encountered in the everyday life. Like the water reservoir problem, when the tap is as opened, water will flow from the reservoir to the

bath. Moreover, the problem of the shape and the condition of the bottle, and relations between the bottles (related to the tap) are the first parts S2 acknowledged (interpreted) directly. However, this assimilation process could not give the problem solution. Because in S2's thinking process disequilibrium happened, that is the imbalance between the structure of the problem and the thinking structure. In the disequilibrium condition, the S2's thinking process continued to the accommodation and the analytical process.

In the accommodation process, S2 still produced an incomplete thinking structure, but S2 has used them to interpret the problem, therefore S2 had the wrong answer. S2 expressed the thinking structure into the various function graphs (as the sketch) as follows

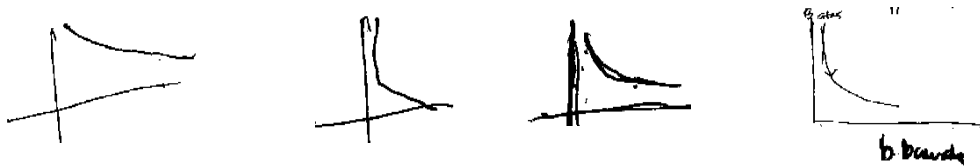
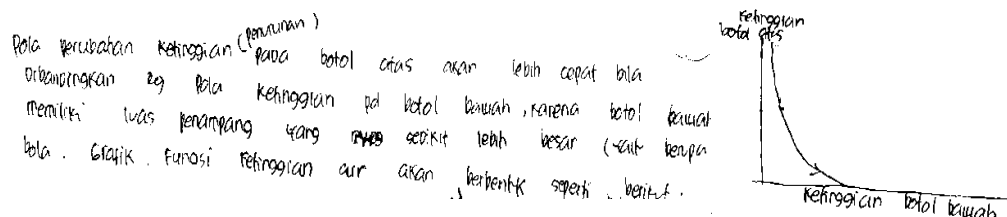


Figure 11. The Graph Sketches before Reflection

After making the graph sketches, S2 decides his answer graph (before the reflection) as follows



English translation:

The pattern of elevation changes on the bottle top will decline more rapidly when compared with the height of the pattern on the bottom of the bottle, because the bottom of the bottle has cross-sectional areas lightly larger (in the form of balls, graphs). The function of the water level will be shaped like this.

Figure 12. The Graph that was produced before the Reflection

Given the opportunity to conduct the reflection, S2 questioned (compared) the graph with the problem he was dealing with and S2 began to doubt the answer. This means that S2 entered the condition of disequilibrium again and continued to the accommodation process. Eventually S2 was able to form the thinking structure in accordance with the structure of the problem. The S2's answers after the reflection are as follows

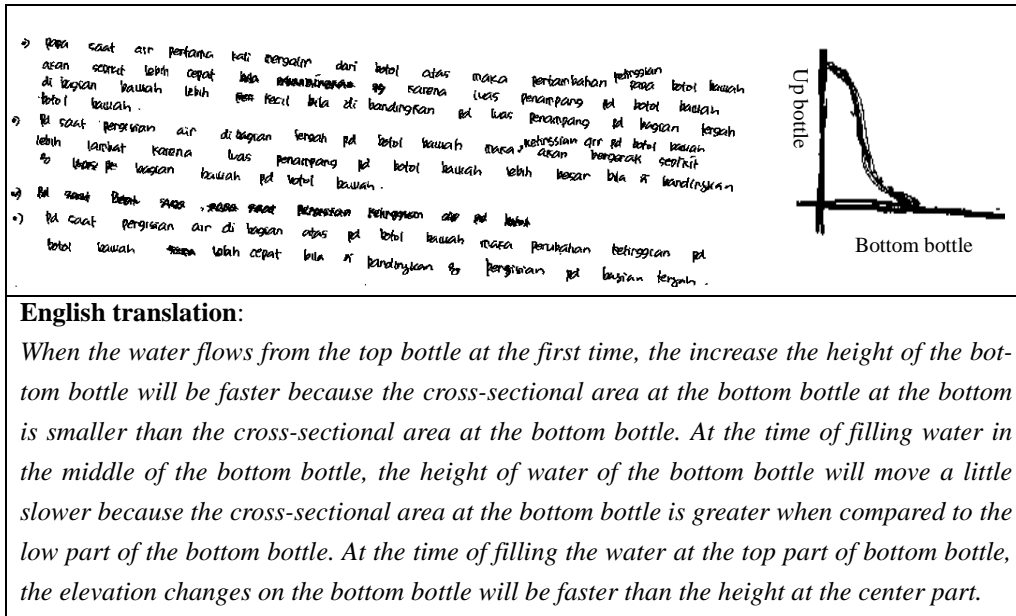


Figure 13. The Graph that was produced after the Reflection

With this answer, S2 was satisfied and was certain of the truth. This means that equilibrium occurred in his thinking process. Therefore, the S2’s thinking process before the reflection (that produced the wrong answer) did not reflect the actual thinking process (pseudo).

3.3. Incompatibility of Thinking sub-Structure Use in the Process of Assimilation or Accommodation

Pseudo thinking process started with the incompatibility of the thinking sub-structure use in the assimilation or accommodation process, the students have already had the scheme in accordance with the structure of the problem, however the students used wrong scheme to interpret the problem. After doing the reflection, there was awareness to straighten out, and then to change the wrong answer to become the true one (in accordance with the structure of the problem).

The occurrence of the pseudo thinking covariational reasoning process in relation to the incompatibility of the sub-structure use in the assimilation or accommodation process can be illustrated in the following.

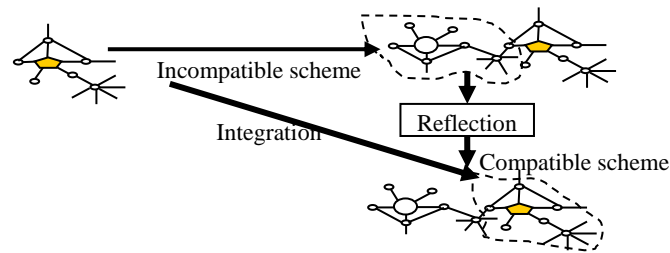


Figure 14. Incompatibility of the Thinking Substructure

In the case of incompatibility of sub-structure use, the student (hereinafter mentioned as S3) has constructed the complete thinking sub-structure related to the structure of the problem, however the one implemented to solve the problem was not the thinking sub-structure constructed. So, it is to produce an inappropriate answer. This happened, especially when S3 described “the increase is increasingly fast (after achieving the middle point)” with the sunken graph above. In this case, S3 was careless with what he did, the S3 thinking process was dominated by “the important this is getting the answer” habit. When S3 got the answer, he no longer did the reflection.

Given an opportunity for reflection, finally S3 realized that the answer (the graph) produced was inappropriate with what he thinks about at the first place, therefore he needed to straighten things out. By carrying out the assimilation and accommodation again, S3 could get the correct answer. S3 thinking process is illustrated in Figure 15.

In facing the problem of the co-variation, S3 knew some problems and some relations. Therefore the problem and relation assimilation took place. However, the thinking structure produced in the assimilation was not sufficient to solve the problem. It leads S3’s thinking process to continue to accommodation and analytical.

In this case, S3 has managed to construct the sub-structure “the water level change in the upper bottle was regular”, and then S3 confronted the problem of the ball-shaped lower bottle. S3 experienced disequilibrium (“curiosity”) about the ball characteristic. Because the shape of the bottle is a ball, then the area of its profile will keep on changing from the bottom to the top. Further, S3 formed a new sub-structure related to the water height decrease in the upper bottle and the increase in the lower. When the water level in the upper bottle decreases, then the volume of water in the lower bottle will increase, and at the same time increases its height. S3 looked for the solution strategy by dividing the problem (the shape of the ball) to become two parts: from the bottom to the middle and from the middle to the top. With a certain volume decrease in the upper bottle, then the increase in the bottom is bigger than it is in the middle. In this case, the S3’s thinking process is accommodation strategy. S3 determined a certain volume decrease in the upper bottle, and observed the height change in the lower bottle (at the bottom and in the mid-

dle). S3 implemented it into the form of concavity graph. That the form of his graph is concave upward shows that the ball-shaped lower bottle affects the graph from fast to slow. In this case, the S3's thinking process is an accommodation strategy.

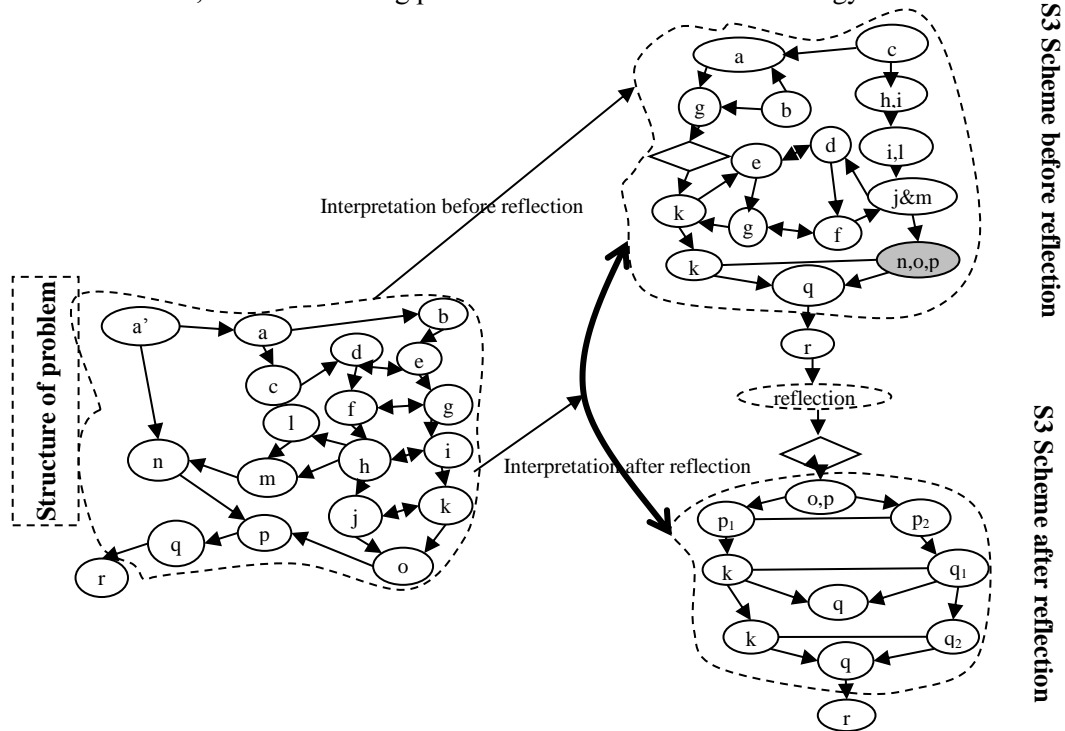


Figure 15. The Occurrence of S3 Pseudo Thinking Process

Getting the graph structure (for the lower bottle, from the bottom up to the middle), then S3 generalized the graph form for all of the events “from the bottom up to the top”. Particularly, S3 was not thorough, after getting the answer S3 felt satisfied, there were no efforts for reflection. Therefore, the answer received was inappropriate (wrong). The results of the S3's work before reflection are depicted in the Figure 16 as follows.

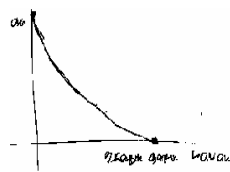


Figure 16. S3 Answer before Reflection

Given the opportunity for reflection, S3 went through suspected the answer again. S3 tried to make the function graph (in the form of sketches) as it is seen in Figure 17, and finally succeeded in making the graph in compliance to the problem structure, as it is pre-

sented in Figure 18

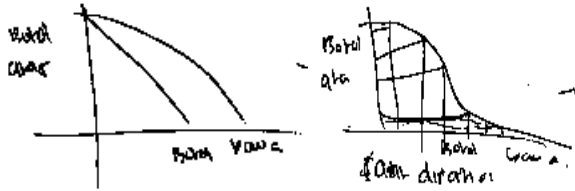


Figure 17. Process of Sketches Graph



Figure 18. S3 Answer after Reflection

4. DISCUSSION

From the research findings, it can be concluded that the occurrence of thinking process of the pseudo covariational reasoning was initiated from imperfection of the assimilation or accommodation process, it results in the imperfection of the sub-thinking structure formation. The imperfection the assimilation or accommodation can occur in three forms:

- (1) Incompleteness the thinking sub-structure in the assimilation process,
- (2) Incompleteness of thinking sub-structure in the accommodation process, and
- (3) Incompatibility of the thinking sub-structure use in the assimilation or accommodation process.

This study also reveals that there are three characteristics of the occurrence of the pseudo covariational reasoning thinking process:

- (1) The existence of imperfection the thinking sub-structure used in generalizing the solution,
- (2) The reflection process is not maximized, and
- (3) The existence of the awareness up to the straightening the wrong solution process out.

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