AN ESTIMATE OF HEMPEL DISTANCE
FOR BRIDGE SPHERES

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Abstract. Tomova [8] gave an upper bound for the distance of a bridge surface for a knot with two different bridge positions in a 3-manifold. In this paper, we show that the result of Tomova [8, Theorem 10.3] can be improved in the case when there are two different bridge spheres for a link in $S^3$.

1. Introduction

Hempel [4] introduced the concept of distance of a Heegaard surface, and it is shown by many authors that it well represents various complexities of 3-manifolds. For example, Hartshorn [2] showed that the Euler characteristic of an incompressible surface in a 3-manifold bounds the distance of its Heegaard splittings, and Scharlemann and Tomova [7] showed that the Euler characteristic of any Heegaard splitting of a 3-manifold similarly bounds the distance of any non-isotopic Heegaard splitting.

The above concept and results have been extended to bridge surfaces for knots and links in closed 3-manifolds, and have been studied by several authors. For example, Bachman and Schleimer [1] proved that Hartshorn’s results can be extended to the distance of a bridge surface for a knot in a closed orientable 3-manifold, and also Tomova [8] proved that Scharlemann and Tomova’s results can be extended to the distance of a bridge surface for a knot in a closed orientable 3-manifold. Moreover, Johnson and Tomova [6] proved that Tomova’s result can be extended to a bridge surface for a tangle in a compact 3-manifold. Recently Jang [5] showed that for a link in a closed orientable 3-manifold, the result of Bachman and Schleimer [1] can be improved in the case when there exist essential meridional spheres.

In this paper, we find a property of essential simple closed curves disjoint from the disk complex of a 3-ball containing trivial arcs (for detail, see Lemma 2.1). This allows us to improve the result of Tomova [8, Theorem 10.3] in the case when there are two different bridge spheres for a link in $S^3$. 

Received March 6, 2013; Revised August 10, 2013.

2010 Mathematics Subject Classification. 57M27.

Key words and phrases. Heegaard splitting, bridge decomposition, distance.

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Theorem 1.1. Suppose $L$ is a link in $S^3$ and $P$ is a bridge sphere for $L$ with $|P \cap L| \geq 6$. If $Q$ is another bridge sphere for $L$ such that $Q$ is not equivalent to $P$, then $d(P, L) \leq |Q \cap L| - 2$, where $d(P, L)$ denotes the distance of the bridge sphere $P$.

By the above theorem, we can improve a result of Tomova [8, Corollary 10.7] in the case of bridge sphere.

Corollary 1.2. If $P$ is a bridge sphere for a link $L$ such that $d(P, L) > |P \cap L| - 2$, then the minimal bridge sphere for $L$ is unique up to isotopy transverse to $P \cap L$.

2. Definitions and notations

Let $M$ be a closed orientable 3-manifold, $\gamma$ a union of mutually disjoint arcs or simple closed curves properly embedded in $M$, $F$ a surface embedded in $M$, which is in general position with respect to $\gamma$. A surface $D$ in $M$ is a $\gamma$-disk, if $D$ is a disk intersecting $\gamma$ in at most one transverse point. Let $\ell(\subset F)$ be a simple closed curve with $\ell \cap \gamma = \emptyset$. We say that $\ell$ is $\gamma$-inessential if $\ell$ bounds a $\gamma$-disk in $F$, and $\ell$ is $\gamma$-essential if it is not $\gamma$-inessential. We say that a surface $D$ is a $\gamma$-compressing disk for $F$ if; $D$ is a $\gamma$-disk, and $D \cap F = \partial D$, and $\partial D$ is a $\gamma$-essential simple closed curve in $F$. Let $F_1, F_2$ be surfaces in $M$ which are in general position with respect to $\gamma$. We say that $F_1$ and $F_2$ are $\gamma$-parallel if they co-bound a 3-manifold homeomorphic to $F_1 \times [0, 1]$ intersecting $\gamma$ in vertical arcs, where $F_1 = F_1 \times \{0\}$ and $F_2 = F_1 \times \{1\}$. We say that $F_1$ and $F_2$ are $\gamma$-isotopic if there exists an isotopy from $F_1$ to $F_2$ so that $F_1$ remains transverse to $\gamma$ throughout the isotopy.

2.1. Handlebodies containing trivial arcs

Let $H$ be a genus-$g(\geq 0)$ handlebody. If $g > 1$, spine $\Sigma_H$ of $H$ is a 1-complex contained in the interior of $H$, which is a strong deformation retract of $H$, where each vertex of $\Sigma_H$ has valence three. Note that for a genus-0 handlebody (the 3-ball), we let the spine be a point in the interior of the 3-ball, and for a genus-1 handlebody (solid torus), we let the spine be a core circle of the solid torus. We say that a set of $n$ arcs $\{t_1, \ldots, t_n\}$ properly embedded in $H$ is a set of trivial $n$ arcs if $t_1 \cup \cdots \cup t_n$ is parallel to $\partial H$. Let $H$ be a handlebody and $\tau = \{t_1, \ldots, t_n\}$ a set of trivial $n$ arcs in $H$. Then $\tau$ can be isotoped in $H$ so that the projection from $\partial H \times [0, 1)$ to $[0, 1)$ has exactly one critical point in each $t_i$. For the pair $(H, \tau)$, we let the spine $\Sigma_{(H, \tau)}$ be the union of $\Sigma_H$ together with a collection of vertical arcs $\alpha_1, \ldots, \alpha_n$, where one endpoint of each $\alpha_i$ lies on the critical point of $t_i$, and the other endpoint lies on $\Sigma_H$. 
2.2. Bridge decompositions

It is well known that every closed orientable 3-manifold $M$ has a genus-$g$ Heegaard splitting for some $g \geq 0$, i.e., $M = A \cup P B$, where $A$ and $B$ are genus-$g$ handlebodies in $M$ such that $M = A \cup B$ and $A \cap B = \partial A = \partial B = P$. Let $L$ be a link in $M$. We say that $(A, \tau_A) \cup_P (B, \tau_B)$ is a $(g, n)$-bridge decomposition (or bridge decomposition for short) for the pair $(M, L)$ if $P$ separates $(M, L)$ into two components $(A, \tau_A)$ and $(B, \tau_B)$ where $\tau_A = L \cap A$ (resp. $\tau_B = L \cap B$) is a set of trivial $n$ arcs in $A$ (resp. $B$). Then we say that $P$ is a $(g, n)$-bridge surface (or a bridge surface for short). It is known that each $(M, L)$ has a $(g, n)$-bridge decomposition for some $g$ and $n$. (For a detailed discussion, see [3, Lemma 2.1].)

Given a $(g, n)$-bridge decomposition $(A, \tau_A) \cup_P (B, \tau_B)$ for $(M, L)$, there are three ways to create new bridge surfaces for $(M, L)$: (1) adding dual one-handles disjoint from $L$ (stabilizing), (2) adding dual one-handles where one of them has an arc of $L$ as its core (meridionally stabilizing), and (3) introducing a pair of a canceling minimum and maximum for $L$ (perturbing) (for details, see [8, Figure 15]). We say that another bridge surface $Q$ is equivalent to $P$ if $Q$ is $L$-isotopic to a copy of $P$ which may have been stabilized, meridionally stabilized and perturbed.

2.3. Sweep-outs

Let $L$ be a link in a closed orientable 3-manifold $M$. Suppose $(A, \tau_A) \cup_P (B, \tau_B)$ is a bridge decomposition for $(M, L)$. From the definition of a spine, one can construct a map $f : M \to [-1, 1]$ such that $f^{-1}(-1)$ is a spine of $(A, \tau_A)$, $f^{-1}(1)$ is a spine of $(B, \tau_B)$ and $f^{-1}(s)$ is a surface which is $L$-parallel to the bridge surface $P$ for each $s \in [-1, 1]$. This map is called a sweep-out induced from $(A, \tau_A) \cup_P (B, \tau_B)$. For each $s \in (-1, 1)$, we put $P_s = f^{-1}(s)$, $A_s = f^{-1}([-1, s])$ and $B_s = f^{-1}([s, 1])$.

2.4. Curve complexes

Let $S$ be a compact orientable surface with genus $g$ and $p$ punctures. The curve complex $\mathcal{C}(S)$ is defined as follows: the vertices of $\mathcal{C}(S)$ are the isotopy classes of essential simple closed curves on $S$, and a collections of $k+1$ vertices form a $k$-simplex of $\mathcal{C}(S)$ if they can be realized by disjoint curves in $S$. For two vertices $x, y$ of $\mathcal{C}(S)$, we define the distance $d(x, y)$ between $x$ and $y$ as the minimal number of 1-simplices of a simplicial path in $\mathcal{C}(S)$ joining $x$ and $y$. Let $X, Y$ be subsets of the vertices of $\mathcal{C}(S)$. Then we define $d(X, Y) = \min\{d(x, y) \mid x \in X, y \in Y\}$.

Let $H$ be a handlebody and $\tau$ a set of trivial $n$ arcs in $H$. Then $D(H \setminus \tau)$ denotes the subset of $\mathcal{C}(\partial H \setminus \partial \tau)$ consisting of the vertices with representatives bounding disks in $H \setminus \tau$. Suppose that $M$ is a closed orientable 3-manifold containing a link $L$, and $(A, \tau_A) \cup_P (B, \tau_B)$ is a bridge decomposition for $(M, L)$. 


Then the distance \( d(P, L) \) of \((A, \tau_A) \cup_P (B, \tau_B)\) is defined by \( d(D(A \setminus \tau_A), D(B \setminus \tau_B)) \).

Let \( B^3 \) be a 3-ball, and \( \tau \) a set of trivial \( n \) arcs in \( B^3 \) with \( n \geq 3 \).

**Lemma 2.1.** Let \( D \) be a \( \gamma \)-compressing disk in \( B^3 \). Then for a \( \tau \)-essential simple closed curve \( \ell \subset \partial(B^3 \setminus \partial \tau) \) which is disjoint from \( \partial D \), \( d(D(B^3 \setminus \partial \tau), \ell) \leq 1 \).

**Proof.** If \( D \cap \tau = \emptyset \), then clearly Lemma 2.1 holds. Suppose \( D \cap \tau \) consists of one point. Then \( D \) separates \((B^3, \tau)\) into two components \((B^3_1, \tau_1)\) and \((B^3_2, \tau_2)\) where \( \ell \subset \partial B^3_1 \setminus \partial \tau_1 \). Since \( D \) is a \( \tau \)-compressing disk, \( \tau_2 \) consists of at least two components. Hence there is a compressing disk \( D' \subset B^3_2 \) for \( \partial B^3_2 \setminus \partial \tau_2 \). Further, by \( \tau_2 \)-isotopy of \( B^3_2 \), we may suppose that \( D' \) is disjoint from the image of \( D \). Hence we may regard \( D' \) a compressing disk in \( B^3 \setminus \tau \). Since \( \ell \) is disjoint from \( D' \), we have \( d(\partial D', \ell) \leq 1 \), which implies \( d(D(B^3 \setminus \tau), \ell) \leq 1 \). \( \square \)

![Figure 1](image)

**3. Proof of main results**

Let \( L \) be a link in \( S^3 \). Let \((A, \tau_A) \cup_P (B, \tau_B)\) be a \((0, n_1)\)-bridge decomposition \((n_1 \geq 3)\) for \((S^3, L)\), and \( f \) a sweep-out induced from \((A, \tau_A) \cup_P (B, \tau_B)\). Let \( \pi_P \) be the projection map from \( P \times (-1, 1) \) to \( P \).

Suppose that \( Q \) is a \((0, n_2)\)-bridge surface \((n_2 \geq 3)\) for \((S^3, L)\) which is not equivalent to \( P \). Then, by [6, Theorem 3.1 and Lemma 4.3], \( Q \) can be \( L \)-isotoped so that \( f|_Q \) is Morse, and for every \( s \in (-1, 1) \), \( P_s \cap Q \) contains a curve that is \( L \)-essential in \( P_s \). Moreover, as in the proof of [6, Theorem 4.2], either \( d(P, L) \leq 1 \) or there exists an interval \([s_-, s_+]\), where \( s_- < s_+ \) are critical values for \( f|_Q \) such that

1. for every \( s \in (s_-, s_+) \), each component of \( P_s \cap Q \) which is \( L \)-essential in \( P_s \) does not bound a disk in \( Q \), and
2. for a small \( \epsilon \), \( P_{s_- - \epsilon} \cap Q \) contains a curve that is \( L \)-essential in \( P_{s_- - \epsilon} \) and bounds a disk in \( A_{s_- - \epsilon} \), and \( P_{s_+ + \epsilon} \cap Q \) contains a curve that is \( L \)-essential in \( P_{s_+ + \epsilon} \) and bounds a disk in \( B_{s_+ + \epsilon} \).
Hence we have:

\[
(*) \quad \text{if } d(P, L) \geq 2, \text{ there exists an interval } [s_-, s_+] \text{ satisfying }
\]
the conditions (1) and (2).

We show that in (\(*)\), the conclusion can be improved if \(d(P, L) > 2\). Namely:

**Lemma 3.1.** Let \(Q\) be as above. If \(d(P, L) > 2\), there exists a subinterval \([s_-, s_+] \subset [s_-, s_+], \) where \(s_- < s_+\) are critical values for \(f|_Q\) such that

(i) for every \(s' \in (s_-, s_+), \) each component of \(P_s \cap Q\) which is \(L\)-essential in \(P_s\) does not bound an \(L\)-disk in \(Q\), and

(ii) for a small \(\epsilon, P_{s_- - \epsilon} \cap Q\) contains a curve that is \(L\)-essential in \(P_{s_- - \epsilon}\)
and bounds an \(L\)-disk in \(A_{s_- - \epsilon}\), and \(P_{s_+ + \epsilon} \cap Q\) contains a curve that is
\(L\)-essential in \(P_{s_+ + \epsilon}\) and bounds an \(L\)-disk in \(B_{s_+ + \epsilon}\).

Proof. Suppose, for a contradiction, that for every \(s \in (s_-, s_+)\), there exists a component of \(P_s \cap Q\) which is \(L\)-essential in \(P_s\) and bounds a \(L\)-disk in \(Q\). Note that a disk in \(Q\) is an \(L\)-disk. Hence by the above condition (1), we obtain that there exists a critical value \(s^*\) (possibly \(s^* = s_-\) or \(s_+)\) such that for a small \(\epsilon, P_{s^* - \epsilon} \cap Q\) contains a curve bounding a \(L\)-disk \(D_A\)
in \(A_{s^* - \epsilon}\), and \(P_{s^* + \epsilon} \cap Q\) contains a curve bounding a \(L\)-disk \(D_B\) in \(B_{s^* + \epsilon}\).

Note that \(d(\pi_P(\partial D_A), \pi_P(\partial D_B)) \leq 1\). (Recall that \(\pi_P\) be the projection map from \(P \times (-1, 1)\) to \(P\)) Hence by regarding \(D_A\) as \(D\) and \(\pi_P(\partial D_B)\) as the \(f\) in Lemma 2.1, we have \(d(\partial D(A \ \tau_A), \pi_P(\partial D_B)) \leq 1\). Analogously we have \(d(\partial D(B \ \tau_B), \pi_P(\partial D_B)) \leq 1\). These together with a triangle inequality
\(d_P(D(A \ \tau_A), D(B \ \tau_B)) \leq d(\partial D(A \ \tau_A), \pi_P(\partial D_B)) + d(\pi_P(\partial D_B), D(B \ \tau_B))\)
imply that \(d(P, L) \leq 2\), a contradiction. Hence, there exists a component of \(P_s \cap Q\) which is \(L\)-essential in \(P_s\) and does not bound a \(L\)-disk in \(Q\). It is easy to see that this implies Lemma 3.1(i) holds. The conclusion (ii) follows from the conclusion (i) and the above condition (1).

\(\square\)

**Proof of Theorem 1.1.** By Lemma 3.1, either (I) \(d(P, L) \leq 2\) or (II) there exists an interval \([s_-, s_+]\), where \(s_- < s_+\) are critical values for \(f|_Q\) satisfying (i) and (ii) of Lemma 3.1. If \(d(P, L) \leq 2\), then since \(n_2 \leq 3\), the conclusion of Theorem 1.1 holds. Hence we consider the case (II). Let \(C\) be the union of the components of \(P_{s_+ + \epsilon} \cap Q\) and \(P_{s_- - \epsilon} \cap Q\) which are \(L\)-essential on \(Q\). Since \(Q\) is connected, there is a component, say \(Q', \) of \(Q \ \tau C\) such that \(\partial Q' \cap P_{s_- - \epsilon} \neq \emptyset\) and \(\partial Q' \cap P_{s_+ + \epsilon} \neq \emptyset\). Note that each component of \(\partial Q' \cap P_{s_+ + \epsilon}\) and \(\partial Q' \cap P_{s_- - \epsilon}\) bounds an at least twice punctured disk in \((Q \ \tau L)\) because each component of \(\partial Q' \cap P_{s_+ - \epsilon}\) and \(\partial Q' \cap P_{s_- - \epsilon}\) is \(L\)-essential on \(Q\). Hence \(\chi(Q' \ \tau L) \geq \chi(Q \ \tau L) + 2\). Let \(c_-\) (resp. \(c_+\)) be a component of \(\partial Q' \cap P_{s_+ + \epsilon}\) (resp. \(\partial Q' \cap P_{s_- - \epsilon}\)). Hence by using arguments as in the proof of [6, Theorem 4.2], \(d(\pi_P(c_-), \pi_P(c_+)) \leq -\chi(Q' \ \tau L)\). By (ii) of Lemma 3.1, \(\pi_P(c_-)\) (resp. \(\pi_P(c_+)\)) is disjoint from an \(L\)-compressing disk in \(A\) (resp. \(B\)). By Lemma 2.1,
\[ d(\mathcal{D}(A \setminus \tau_A), \pi_P(c_{\pm})) \leq 1 \text{ and } d(\pi_P(c_{\pm}), \mathcal{D}(B \setminus \tau_B)) \leq 1. \] Hence, we have
\[ d(P, L) \leq d(\mathcal{D}(A \setminus \tau_A), \pi_P(c_{\pm})) + d(\pi_P(c_{\pm}), \mathcal{D}(B \setminus \tau_B)) \]
\[ \leq 1 - \chi(Q' \setminus L) + 1 \]
\[ = -\chi(Q \setminus L). \]

This completes the proof of Theorem 1.1.

**Proof of Corollary 1.2.** Let \( Q \) be a minimal bridge sphere for a link \( L \). Suppose that \( Q \) not equivalent to \( P \). Then, by Theorem 1.1, \[ d(P, L) \leq |Q \setminus L| - 2 = |P \setminus L| - 2, \] a contradiction.

**Acknowledgments.** I would like to thank Professor Tsuyoshi Kobayashi for many helpful advices and comments. I would also like to thank Yeonhee Jang for her valuable advices and informations. I gratefully appreciate the financial support from Japan Society for the Promotion of Science (JSPS). Finally I would like to thank the referee for careful reading of the first version of the paper.

**References**


