

SYSTEM OF GENERALIZED NONLINEAR MIXED VARIATIONAL INCLUSIONS INVOLVING RELAXED COCOERCIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. We considered a new *system of generalized nonlinear mixed variational inclusions* in Hilbert spaces and define an iterative method for finding the approximate solutions of this class of *system of generalized nonlinear mixed variational inclusions*. We also established that the approximate solutions obtained by our algorithm converges to the exact solutions of a new *system of generalized nonlinear mixed variational inclusions*.

1. Introduction

It is well known that the variational inequality theory and complementarity problems are very powerful tools of the current mathematical technology. In recent years, classical variational inequality and complementarity problems have been extended and generalized into a large variety of problems arising in mechanics, physics, optimization and control theory etc., see [3, 4, 6, 10, 11]. Hassouni and Moudafi [12] introduced and studied a class of mixed type variational inequalities with single-valued mappings which was called variational inclusions. Verma [22, 23, 24] studied some system of variational inequalities with single-valued mappings and suggested some iterative algorithms to compute approximate solutions of these systems in Hilbert spaces. As an application of system of variational inclusions Pang [20] showed that the traffic equilibrium problem, the Nash equilibrium problem and the general equilibrium problem, can be modeled as a system of variational inequalities and inclusions, see [5, 9, 16, 19].

Inspired and motivated by recent research works in this field, see [1, 2, 7, 8, 13, 14, 15, 17, 21, 25], we introduce a new *system of generalized nonlinear mixed variational inclusions* and suggest an iterative algorithm. By the definition of relaxed cocoersive mapping and resolvent operator techniques, we find

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the exact solutions of our *systems of generalized nonlinear mixed variational inclusions* involving different nonlinear operators and fixed point problems in Hilbert spaces.

2. Basic Foundation

Throughout this paper, \mathcal{H} is a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\| \cdot \|$. Let $CB(\mathcal{H})$ denote the family of all nonempty closed bounded subsets of \mathcal{H} . Let $M_i : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be the maximal monotone mappings, $T_i : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be the set-valued mappings and $A_i, B_i, g_i : \mathcal{H} \rightarrow \mathcal{H}$ be the nonlinear single-valued mappings for $i = 1, 2, 3$. We consider the following problem of finding $x, y, z \in \mathcal{H}$ such that $u_1 \in T_1y, u_2 \in T_2z, u_3 \in T_3x$ and

$$\begin{aligned}
 & 0 \in g_1(x) - g_1(y) + r_1(A_1(g_1(y)) - B_1(u_1)) + r_1M_1(g_1(x)), \\
 & 0 \in g_2(y) - g_2(z) + r_2(A_2(g_2(z)) - B_2(u_2)) + r_2M_2(g_2(y)), \\
 (1) \quad & 0 \in g_3(z) - g_3(x) + r_3(A_3(g_3(x)) - B_3(u_3)) + r_3M_3(g_3(z)),
 \end{aligned}$$

which is called the *system of generalized nonlinear mixed variational inclusions* with a solution set (x, y, z, u_i) , denoted by $SGNMVID(\mathfrak{A}, \mathfrak{B}, \mathfrak{T}, \mathfrak{g})$ of the problem (1) for $i = 1, 2, 3$.

Definition 1. [4] If $M : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ is a maximal monotone mapping, then for any fixed $r > 0$, the mapping $J_M^r : \mathcal{H} \rightarrow \mathcal{H}$ defined by

$$J_M^r(x) = (I + rM)^{-1}(x), \quad \forall x \in \mathcal{H}, \tag{2}$$

is called the resolvent operator of index r of M , where I is the identity mapping on \mathcal{H} . Furthermore the resolvent operator J_M^r is single-valued and non-expansive, *i.e.*,

$$\|J_M^r(x) - J_M^r(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H}. \tag{3}$$

Lemma 2.1. [18] *Let D be a Hausdorff metric in $CB(\mathcal{H})$ and $A, B \in CB(\mathcal{H})$, and $\epsilon > 0$ be any real number. Then for $a \in A$, there exists $b \in B$ such that*

$$\|a - b\| \leq (1 + \epsilon) D(A, B).$$

Definition 2. Let $T : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a set-valued mapping and $x, y \in \mathcal{H}$. T is said to be

(i) μ -cocoercive if there exists a constant $\mu > 0$ such that

$$\langle u_1 - u_2, x - y \rangle \geq \mu \|u_1 - u_2\|^2, \quad \forall u_1 \in Tx, u_2 \in Ty$$

(ii) relaxed μ -cocoercive if there exists a constant $\mu > 0$ such that

$$\langle u_1 - u_2, x - y \rangle \geq -\mu \|u_1 - u_2\|^2, \quad \forall u_1 \in Tx, u_2 \in Ty$$

Definition 3. Let $T : \mathcal{H} \rightarrow CB(\mathcal{H})$ be a set-valued mapping. T is said to be λ -Lipschitz continuous if $D(Tx, Ty) \leq \lambda \cdot \|x - y\|$ for $x, y \in \mathcal{H}$.

Remark 1. The same definitions could be obtained for the single-valued mapping T in Definition 2 and Definition 3.

Definition 4. A single-valued mapping $B : \mathcal{H} \rightarrow \mathcal{H}$ is said to be relaxed μ -cocoercive with respect to $T : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ if

$$\langle Bu_1 - Bu_2, x - y \rangle \geq -\mu \|Bu_1 - Bu_2\|^2 \text{ for } x, y \in \mathcal{H} \text{ and } u_1 \in Tx, u_2 \in Ty.$$

3. Main Results

First we give the following lemma, the proof of which is a direct consequence of Definition 1, hence is omitted.

Lemma 3.1. (x, y, z, u_i) is the solution set of system of generalized nonlinear mixed variational inclusions (1) if and only if it satisfies

$$\begin{aligned} g_1(x) &= J_{M_1}^{r_1} [g_1(y) - r_1(A_1(g_1(y)) - B_1(u_1))], \\ g_2(y) &= J_{M_2}^{r_2} [g_2(z) - r_2(A_2(g_2(z)) - B_2(u_2))], \\ (4) \quad g_3(z) &= J_{M_3}^{r_3} [g_3(x) - r_3(A_3(g_3(x)) - B_3(u_3))]. \end{aligned}$$

where $r_i > 0$ and $J_{M_i}^{r_i} = (I + r_i M_i)^{-1}$ is a resolvent operator for $i = 1, 2, 3$.

The preceding lemma allows us to suggest the following iterative algorithm for the system (1).

Algorithm 3.2. Let $\{\epsilon_n\}$ be a sequence of nonnegative real numbers with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. For any given $x_0, y_0, z_0 \in \mathcal{H}$, compute $\{x_n\}, \{y_n\}, \{z_n\} \subset \mathcal{H}, \{u_{n,1}\} \subset \bigcup_{n=0}^{\infty} T_1 y_n, \{u_{n,2}\} \subset \bigcup_{n=0}^{\infty} T_2 z_n$ and $\{u_{n,3}\} \subset \bigcup_{n=0}^{\infty} T_3 x_n$, generated by the following iterative processes;

$$\begin{aligned} x_{n+1} &= x_n - g_1(x_n) + J_{M_1}^{r_1} [g_1(y_n) - r_1(A_1(g_1(y_n)) - B_1(u_{n,1}))]; \\ u_{n,1} \in T_1(y_n), u_{n-1,1} \in T_1(y_{n-1}) : &\|u_{n,1} - u_{n-1,1}\| \leq (1 + \epsilon_n)D(T_1(y_n), T_1(y_{n-1})), \\ g_2(y_n) &= J_{M_2}^{r_2} [g_2(z_n) - r_2(A_2(g_2(z_n)) - B_2(u_{n,2}))]; \\ u_{n,2} \in T_2(z_n), u_{n-1,2} \in T_2(z_{n-1}) : &\|u_{n,2} - u_{n-1,2}\| \leq (1 + \epsilon_n)D(T_2(z_n), T_2(z_{n-1})), \\ g_3(z_n) &= J_{M_3}^{r_3} [g_3(x_n) - r_3(A_3(g_3(x_n)) - B_3(u_{n,3}))]; \\ u_{n,3} \in T_3(x_n), u_{n-1,3} \in T_3(x_{n-1}) : &\|u_{n,3} - u_{n-1,3}\| \leq (1 + \epsilon_n)D(T_3(x_n), T_3(x_{n-1})), \\ &r_1, r_2, r_3 > 0 \text{ and } n = 0, 1, 2, \dots \end{aligned} \tag{5}$$

We apply Algorithm 3.2 to prove the following convergence theorem.

Theorem 3.3. Let \mathcal{H} be a real Hilbert space and $M_i : \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be the maximal monotone mapping ($i = 1, 2, 3$). Let $T_i : \mathcal{H} \rightarrow CB(\mathcal{H})$ be the η_i -Lipschitz continuous, $B_i : \mathcal{H} \rightarrow \mathcal{H}$ be σ_i -Lipschitz continuous and B_1 be relaxed ξ_1 -cocoercive with respect to T_1 ($i = 1, 2, 3$). Let $g_i : \mathcal{H} \rightarrow \mathcal{H}$ be μ_i -cocoercive and λ_i -Lipschitz continuous, and $A_i : \mathcal{H} \rightarrow \mathcal{H}$ be ρ_i -Lipschitz continuous ($i = 1, 2, 3$). Suppose that for all $x, y \in \mathcal{H}$, we have

$$\|J_{M_i}^{r_i}(x) - J_{M_i}^{r_i}(y)\| \leq \|x - y\| \quad (i = 1, 2, 3). \tag{6}$$

Put

$$p_1 = \sqrt{1 - 2\mu_1 \lambda_1^2 + \lambda_1^2} < 1. \tag{7}$$

If

$$(i) \quad \left| r_1 - \frac{\xi_1 \sigma_1^2 \eta_1^2 - \rho_1 \lambda_1 (1 - 2p_1)}{\sigma_1^2 \eta_1^2 - \rho_1^2 \lambda_1^2} \right| < \frac{\sqrt{(\xi_1 \sigma_1^2 \eta_1^2 - \rho_1 \lambda_1 (1 - 2p_1))^2 - (\sigma_1^2 \eta_1^2 - \rho_1^2 \lambda_1^2) 4p_1 (1 - p_1)}}{\sigma_1^2 \eta_1^2 - \rho_1^2 \lambda_1^2},$$

$$\xi_1 \sigma_1^2 \eta_1^2 > \rho_1 \lambda_1 (1 - 2p_1) + \sqrt{(\sigma_1^2 \eta_1^2 - \rho_1^2 \lambda_1^2) 4p_1 (1 - p_1)},$$

$$\xi_1 \sigma_1^2 \eta_1^2 > \rho_1 \lambda_1 (1 - 2p_1)$$

$$\sigma_1^2 \eta_1^2 > \rho_1^2 \lambda_1^2$$

$$p_1 < \frac{1}{2},$$

(ii)

$$\left| r_2 - \frac{\lambda_2 (\mu_2 \lambda_2 - 1)}{\rho_2 \lambda_2 + \sigma_2 \eta_2} \right| < 0,$$

(iii)

$$\left| r_3 - \frac{\lambda_3 (\mu_3 \lambda_3 - 1)}{\rho_3 \lambda_3 + \sigma_3 \eta_3} \right| < 0,$$

(iv)

$$\text{and } \theta < 1$$

where

$$\theta = (\theta_0 + (\theta_0 + \theta_1 + r_1 \rho_1 \lambda_1) \frac{1}{\mu_2 \mu_3 \lambda_2^2 \lambda_3^2} \theta_2 \theta_3) \text{ with} \tag{8}$$

$$\theta_0 = p_1,$$

$$\theta_1 = \sqrt{1 - 2r_1 \xi_1 \sigma_1^2 \eta_1^2 + r_1^2 \sigma_1^2 \eta_1^2},$$

$$\theta_2 = \lambda_2 + r_2 (\rho_2 \lambda_2 + \sigma_2 \eta_2),$$

$$\theta_3 = \lambda_3 + r_3 (\rho_3 \lambda_3 + \sigma_3 \eta_3).$$

Then $SGNMVID(\mathfrak{A}, \mathfrak{B}, \mathfrak{T}, \mathfrak{g}) \neq \emptyset$. Moreover, the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{u_{n,1}\}, \{u_{n,2}\}$ and $\{u_{n,3}\}$ generated by Algorithm 3.2 converge strongly to $x^*, y^*, z^*, u_1^* \in T_1(y^*), u_2^* \in T_2(z^*)$ and $u_3^* \in T_3(x^*)$, respectively, where $(x^*, y^*, z^*, u_1^*, u_2^*, u_3^*) \in SGNMVID(\mathfrak{A}, \mathfrak{B}, \mathfrak{T}, \mathfrak{g})$.

Proof. From Algorithm 3.2 and using the non-expansiveness of the resolvent operator, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|x_n - g_1(x_n) + J_{M_1}^{r_1} [g_1(y_n) - r_1(A_1(g_1(y_n)) - B_1(u_{n,1}))] - \{x_{n-1} - g_1(x_{n-1}) \\ &\quad + J_{M_1}^{r_1} [g_1(y_{n-1}) - r_1(A_1(g_1(y_{n-1})) - B_1(u_{n-1,1}))]\}\| \\ &\leq \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\| + \|g_1(y_n) - g_1(y_{n-1}) - r_1(A_1(g_1(y_n)) - A_1(g_1(y_{n-1}))) \\ &\quad + r_1(B_1(u_{n,1}) - B_1(u_{n-1,1}))\| \\ &\leq \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\| + \|y_n - y_{n-1} - (g_1(y_n) - g_1(y_{n-1}))\| \\ &\quad + \|y_n - y_{n-1} + r_1(B_1(u_{n,1}) - B_1(u_{n-1,1}))\| + r_1 \|A_1(g_1(y_n)) - A_1(g_1(y_{n-1}))\|. \end{aligned} \tag{9}$$

On the other hand, by using the λ_1 -Lipschitz continuity and μ_1 -cocoercivity of g_1 , we have

$$\begin{aligned} \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|^2 &\leq \|x_n - x_{n-1}\|^2 - 2\langle g_1(x_n) - g_1(x_{n-1}), x_n - x_{n-1} \rangle \\ &\quad + \|g_1(x_n) - g_1(x_{n-1})\|^2 \\ &\leq \|x_n - x_{n-1}\|^2 - 2\mu_1\lambda_1^2\|x_n - x_{n-1}\|^2 + \lambda_1^2\|x_n - x_{n-1}\|^2 \\ &\leq p_1^2\|x_n - x_{n-1}\|^2, \end{aligned} \tag{10}$$

and

$$\|y_n - y_{n-1} - (g_1(y_n) - g_1(y_{n-1}))\|^2 \leq p_1^2\|y_n - y_{n-1}\|^2, \tag{11}$$

where $p_1 = \sqrt{1 - 2\mu_1\lambda_1^2 + \lambda_1^2}$.

Since B_1 is σ_1 -Lipschitz continuous and T_1 is η_1 -Lipschitz continuous, we have

$$\begin{aligned} \|B_1(u_{n,1}) - B_1(u_{n-1,1})\| &\leq \sigma_1\|u_{n,1} - u_{n-1,1}\| \\ &\leq \sigma_1 D(T_1(y_n), T_1(y_{n-1})) \\ &\leq \sigma_1(1 + \epsilon_n)\eta_1\|y_n - y_{n-1}\|. \end{aligned} \tag{12}$$

Since B_1 is relaxed ξ_1 -cocoercive with respect to T_1 , from (12) we have

$$\begin{aligned} &\|y_n - y_{n-1} + r_1(B_1(u_{n,1}) - B_1(u_{n-1,1}))\|^2 \\ &= \|y_n - y_{n-1}\|^2 + 2r_1\langle B_1(u_{n,1}) - B_1(u_{n-1,1}), y_n - y_{n-1} \rangle + r_1^2\|B_1(u_{n,1}) - B_1(u_{n-1,1})\|^2 \\ &\leq \|y_n - y_{n-1}\|^2 - 2r_1\xi_1\|B_1(u_{n,1}) - B_1(u_{n-1,1})\|^2 + r_1^2\|B_1(u_{n,1}) - B_1(u_{n-1,1})\|^2 \\ &\leq \|y_n - y_{n-1}\|^2 - 2r_1\xi_1\sigma_1^2(1 + \epsilon_n)^2\eta_1^2\|y_n - y_{n-1}\|^2 + r_1^2\sigma_1^2(1 + \epsilon_n)^2\eta_1^2\|y_n - y_{n-1}\|^2 \\ &\leq (1 - 2r_1\xi_1\sigma_1^2(1 + \epsilon_n)^2\eta_1^2 + r_1^2\sigma_1^2(1 + \epsilon_n)^2\eta_1^2)\|y_n - y_{n-1}\|^2. \end{aligned} \tag{13}$$

By the Lipschitz continuities of A_1 and g_1 with constants ρ_1 and λ_1 , respectively, we have

$$\begin{aligned} \|A_1(g_1(y_n)) - A_1(g_1(y_{n-1}))\| &\leq \rho_1\|g_1(y_n) - g_1(y_{n-1})\| \\ (14) \qquad \qquad \qquad &\leq \rho_1\lambda_1\|y_n - y_{n-1}\|. \end{aligned}$$

From (10) to (14), (9) becomes

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\leq p_1\|x_n - x_{n-1}\| + (p_1 + \sqrt{1 - 2r_1\xi_1\sigma_1^2\eta_1^2(1 + \epsilon_n)^2 + r_1^2\sigma_1^2(1 + \epsilon_n)^2\eta_1^2 + r_1\rho_1\lambda_1}) \\ (15) \quad &\times \|y_n - y_{n-1}\|. \end{aligned}$$

Now

$$\begin{aligned} \|g_2(y_n) - g_2(y_{n-1})\|\|y_n - y_{n-1}\| &\geq \langle g_2(y_n) - g_2(y_{n-1}), y_n - y_{n-1} \rangle \\ &\geq \mu_2\|g_2(y_n) - g_2(y_{n-1})\|^2 \\ (16) \qquad \qquad \qquad &\geq \mu_2\lambda_2^2\|y_n - y_{n-1}\|^2. \end{aligned}$$

Hence

$$\|y_n - y_{n-1}\| \leq \frac{1}{\mu_2\lambda_2^2}\|g_2(y_n) - g_2(y_{n-1})\|$$

$$\begin{aligned}
 &\leq \frac{1}{\mu_2\lambda_2^2} \|J_{M_2}^{r_2} [g_2(z_n) - r_2(A_2(g_2(z_n)) - B_2(u_{n,2}))] \\
 &\quad - J_{M_2}^{r_2} [g_2(z_{n-1}) - r_2(A_2(g_2(z_{n-1})) - B_2(u_{n-1,2}))]\| \\
 &\leq \frac{1}{\mu_2\lambda_2^2} \|g_2(z_n) - g_2(z_{n-1}) - r_2(A_2(g_2(z_n)) - A_2(g_2(z_{n-1}))) + r_2(B_2(u_{n,2}) - B_2(u_{n-1,2}))\| \\
 &\leq \frac{1}{\mu_2\lambda_2^2} [\|g_2(z_n) - g_2(z_{n-1})\| + r_2\|A_2(g_2(z_n)) - A_2(g_2(z_{n-1}))\| + r_2\|B_2(u_{n,2}) - B_2(u_{n-1,2})\|] \\
 &\leq \frac{1}{\mu_2\lambda_2^2} [\lambda_2\|z_n - z_{n-1}\| + r_2\lambda_2\rho_2\|z_n - z_{n-1}\| + r_2\sigma_2\|u_{n,2} - u_{n-1,2}\|] \\
 &\leq \frac{1}{\mu_2\lambda_2^2} [\lambda_2\|z_n - z_{n-1}\| + r_2\lambda_2\rho_2\|z_n - z_{n-1}\| + r_2\sigma_2(1 + \epsilon_n)\eta_2\|z_n - z_{n-1}\|] \\
 &\leq \frac{1}{\mu_2\lambda_2^2} [\lambda_2 + r_2(\lambda_2\rho_2 + \sigma_2(1 + \epsilon_n)\eta_2)]\|z_n - z_{n-1}\|
 \end{aligned}$$

Consequently,

$$\|y_n - y_{n-1}\| \leq \frac{1}{\mu_2\lambda_2^2} \theta_{n,2} \|z_n - z_{n-1}\|, \tag{17}$$

where $\theta_{n,2} = \lambda_2 + r_2(\lambda_2\rho_2 + \sigma_2(1 + \epsilon_n)\eta_2)$.

Again

$$\begin{aligned}
 (18) \quad \|g_3(z_n) - g_3(z_{n-1})\| \|z_n - z_{n-1}\| &\geq \langle g_3(z_n) - g_3(z_{n-1}), z_n - z_{n-1} \rangle \\
 &\geq \mu_3 \|g_3(z_n) - g_3(z_{n-1})\|^2 \\
 &\geq \mu_3 \lambda_3^2 \|z_n - z_{n-1}\|^2.
 \end{aligned}$$

That implies that

$$\begin{aligned}
 \|z_n - z_{n-1}\| &\leq \frac{1}{\mu_3\lambda_3^2} \|g_3(z_n) - g_3(z_{n-1})\| \\
 &\leq \frac{1}{\mu_3\lambda_3^2} \|J_{M_3}^{r_3} [g_3(x_n) - r_3(A_3(g_3(x_n)) - B_3(u_{n,3}))] \\
 &\quad - J_{M_3}^{r_3} [g_3(x_{n-1}) - r_3(A_3(g_3(x_{n-1})) - B_3(u_{n-1,3}))]\| \\
 &\leq \frac{1}{\mu_3\lambda_3^2} \|g_3(x_n) - g_3(x_{n-1}) - r_3(A_3(g_3(x_n)) - A_3(g_3(x_{n-1}))) + r_3(B_3(u_{n,3}) - B_3(u_{n-1,3}))\| \\
 &\leq \frac{1}{\mu_3\lambda_3^2} [\|g_3(x_n) - g_3(x_{n-1})\| + r_3\|A_3(g_3(x_n)) - A_3(g_3(x_{n-1}))\| + r_3\|B_3(u_{n,3}) - B_3(u_{n-1,3})\|] \\
 &\leq \frac{1}{\mu_3\lambda_3^2} [\lambda_3\|x_n - x_{n-1}\| + r_3\lambda_3\rho_3\|x_n - x_{n-1}\| + r_3\sigma_3\|u_{n,3} - u_{n-1,3}\|] \\
 &\leq \frac{1}{\mu_3\lambda_3^2} [\lambda_3\|x_n - x_{n-1}\| + r_3\lambda_3\rho_3\|x_n - x_{n-1}\| + r_3\sigma_3(1 + \epsilon_n)\eta_3\|x_n - x_{n-1}\|] \\
 &\leq \frac{1}{\mu_3\lambda_3^2} [\lambda_3 + r_3(\lambda_3\rho_3 + \sigma_3(1 + \epsilon_n)\eta_3)]\|x_n - x_{n-1}\|
 \end{aligned}$$

Hence

$$\|z_n - z_{n-1}\| \leq \frac{1}{\mu_3\lambda_3^2} \theta_{n,3} \|x_n - x_{n-1}\|, \tag{19}$$

where $\theta_{n,3} = \lambda_3 + r_3(\lambda_3\rho_3 + \sigma_3(1 + \epsilon_n)\eta_3)$.
 Combining (17) and (19), we have

$$\|y_n - y_{n-1}\| \leq \frac{1}{\mu_2\mu_3\lambda_2^2\lambda_3^2}\theta_{n,2}\theta_{n,3}\|x_n - x_{n-1}\|. \tag{20}$$

From (15) to (20), we have

$$\|x_{n+1} - x_n\| \leq [\theta_0 + (\theta_0 + \theta_{n,1} + r_1\rho_1\lambda_1)\frac{1}{\mu_2\mu_3\lambda_2^2\lambda_3^2}\theta_{n,2}\theta_{n,3}]\|x_n - x_{n-1}\|, \tag{21}$$

where

$$\begin{aligned} \theta_0 &= p_1 = \sqrt{1 - 2\mu_1\lambda_1^2 + \lambda_1^2}, \\ \theta_{n,1} &= \sqrt{1 - 2r_1\xi_1\sigma_1^2\eta_1^2(1 + \epsilon_n)^2 + r_1^2\sigma_1^2\eta_1^2(1 + \epsilon_n)^2}, \\ \theta_{n,2} &= \lambda_2 + r_2(\rho_2\lambda_2 + \sigma_2\eta_2(1 + \epsilon_n)) \text{ and} \\ \theta_{n,3} &= \lambda_3 + r_3(\rho_3\lambda_3 + \sigma_3\eta_3(1 + \epsilon_n)). \end{aligned}$$

Hence

$$\|x_{n+1} - x_n\| \leq \theta_n\|x_n - x_{n-1}\|. \tag{22}$$

where

$$\theta_n = [\theta_0 + (\theta_0 + \theta_{n,1} + r_1\rho_1\lambda_1)\frac{1}{\mu_2\mu_3\lambda_2^2\lambda_3^2}\theta_{n,2}\theta_{n,3}].$$

Letting

$$\theta = (\theta_0 + (\theta_0 + \theta_1 + r_1\rho_1\lambda_1)\frac{1}{\mu_2\mu_3\lambda_2^2\lambda_3^2}\theta_2\theta_3),$$

where

$$\begin{aligned} \theta_0 &= p_1 = \sqrt{1 - 2\mu_1\lambda_1^2 + \lambda_1^2}, \\ \theta_1 &= \sqrt{1 - 2r_1\xi_1\sigma_1^2\eta_1^2 + r_1^2\sigma_1^2\eta_1^2}, \\ \theta_2 &= \lambda_2 + r_2(\rho_2\lambda_2 + \sigma_2\eta_2) \\ \theta_3 &= \lambda_3 + r_3(\rho_3\lambda_3 + \sigma_3\eta_3), \end{aligned}$$

from (22) we have

$$\|x_{n+1} - x_n\| \leq \theta\|x_n - x_{n-1}\|. \tag{23}$$

By the condition (iv), we have $\theta < 1$. It follows from (23) that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$, which implies that $\{x_n\}$ is a Cauchy sequence in \mathcal{H} . By (20), it follows that $\{y_n\}$ is a Cauchy sequence in \mathcal{H} and from (19) $\{z_n\}$ is also a Cauchy sequence in \mathcal{H} . Moreover, since T_i is η_i -Lipschitz continuous we have that $\{u_{n,i}\}$ is also a Cauchy sequence in \mathcal{H} for $i = 1, 2, 3$. Thus there exist x^*, y^*, z^* and $u_i^* \in \mathcal{H}$ such that $x_n \rightarrow x^*, y_n \rightarrow y^*, z_n \rightarrow z^*$ and $u_{n,i} \rightarrow u_i^*$ as $n \rightarrow \infty$ for $i = 1, 2, 3$. Since $g_i, A_i, B_i, T_i, J_{M_i}^{r_i}$ are continuous for $i = 1, 2, 3$ in (5) we have

$$\begin{aligned} g_3(z^*) &= J_{M_3}^{r_3}[g_3(x^*) - r_3(A_3(g_3(x^*)) - B_3(u_3^*))], \\ g_2(y^*) &= J_{M_2}^{r_2}[g_2(z^*) - r_2(A_2(g_2(z^*)) - B_2(u_2^*))], \\ x^* &= x^* - g_1(x^*) - J_{M_1}^{r_1}[g_1(y^*) - r_1(A_1(g_1(y^*)) - B_1(u_1^*))]. \end{aligned}$$

Hence from (1), (4) and (5), it follows that $(x^*, y^*, z^*, u_i^*) \in \text{SGNMVID}(\mathfrak{A}, \mathfrak{B}, \mathfrak{T}, \mathfrak{g})$ is a solution of the system of generalized nonlinear mixed variational inclusions (1). Finally, we prove that $u_1^* \in T_1(y^*)$. Indeed we have

$$\begin{aligned} d(u_1^*, T_1(y^*)) &= \inf\{\|u_1^* - w\| : w \in T_1(y^*)\} \\ &\leq \|u_1^* - u_{n,1}\| + d(u_{n,1}, T_1(y^*)) \\ &\leq \|u_1^* - u_{n,1}\| + D(T_1(y_n), T_1(y^*)) \\ &\leq \|u_1^* - u_{n,1}\| + \eta_1(1 + \epsilon_n)\|y_n - y^*\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus $d(u_1^*, T_1(y^*)) = 0$. Since $T_1(y^*) \in CB(\mathcal{H})$, we must have $u_1^* \in T_1(y^*)$. Similarly, we can show that $u_2^* \in T_2(z^*)$, $u_3^* \in T_3(x^*)$. This complete the proof. \square

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