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ON INTERVAL-VALUED FUZZY LATTICES

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Abstract. We discuss the relationship between interval-valued fuzzy ideals and interval-valued fuzzy congruence on a distributive lattice L and show that for a generalized Boolean algebra the lattice of interval-valued fuzzy ideals is isomorphic to the lattice of interval-valued fuzzy congruences. Finally we consider the products of interval-valued fuzzy ideals and obtain a necessary and sufficient condition for an interval-valued fuzzy ideal on the direct sum of lattices to be representable as a direct sum of interval-valued fuzzy ideals on each lattice.

1. Introduction

In 1975, Zadeh[12] introduced the concept of interval-valued fuzzy sets as a generalization of fuzzy sets introduced by himself[11]. After then, Biswas[1] applied the notion of interval-valued fuzzy sets to group theory. Moreover, Gorzalczany[4] applied it to a method of inference in approximate reasoning, and Mondal and Samanta[9] applied it to topology. Recently, Hur et al.[6] introduced the concept of an interval-valued fuzzy relations and obtained some of it's properties. Also, Choi et al.[3] applied it to topology in the sense of Šostak, Kang and Hur[7] applied it to algebra.

In this paper, we discuss the relationship between interval-valued fuzzy ideals and interval-valued fuzzy congruence on a distributive lattice L and show that for a generalized Boolean algebra the lattice of interval-valued fuzzy ideals is isomorphic to the lattice of interval-valued fuzzy congruences. Finally we consider the products of interval-valued fuzzy

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ideals and obtain a necessary and sufficient condition for an interval-valued fuzzy ideal on the direct sum of lattices to be representable as a direct sum of interval-valued fuzzy ideals on each lattice.

2. Preliminaries

We will some concepts and two results needed in the later sections. Throughout this paper, $L = (L, +, \cdot)$ denotes a lattice and $D(I)^L$ denotes the set of all interval-valued fuzzy sets in L (i.e., of all mappings from L into $(I, \vee, \wedge)$, where I denotes the unit interval $[0, 1]$ and $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$).

Let $D(I)$ be the set of all closed subintervals of the unit interval $I = [0, 1]$. The elements of $D(I)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denoted, $\mathbf{0} = [0, 0]$, $\mathbf{1} = [1, 1]$, and $a = [a, a]$ for every $a \in (0, 1)$. We also note that

(*i*) $(\forall M, N \in D(I))(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$,
(*ii*) $(\forall M, N \in D(I))(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(I)$, the complement of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^U, 1 - M^L]$ (See [9]).

Definition 2.1[4,12]. A mapping $A : X \to D(I)$ is called an *interval-valued fuzzy set* (in short, *IVFS*) in X, denoted by $A = [A^L, A^U]$, if $A^L, A^U \in I^X$ such that $A^L \leq A^U$, i.e., $A^L(x) \leq A^U(x)$ for each $x \in X$, where $A^L(x)$ [resp. $A^U(x)$] is called the *lower*[resp. *upper*] *end point* of x to A. For any $[a, b] \in D(I)$, the interval-valued fuzzy set A in X defined by $A(x) = [A^L(x), A^U(x)] = [a, b]$ for each $x \in X$ is denoted by $\widetilde{[a, b]}$ and if $a = b$, then the IVFS$\widetilde{[a, b]}$ is denoted by simply $\tilde{a}$. In particular, $\tilde{0}$ and $\tilde{1}$ denote the *interval-valued fuzzy empty set* and the *interval-valued fuzzy whole set* in$X$, respectively.

We will denote the set of IVFSs in X as $D(I)^X$. It is clear that set $A = [A^L, A^U] \in D(I)^X$ for each $A \in I^X$.

Definition 2.2[9]. An IVFS A is called an *interval-valued fuzzy point*(in short, *IVFP*) in X with the support $x \in X$ and the value $[a, b] \in D(I)$ with $b > 0$, denoted by $A = x_{[a,b]}$, if for each $y \in X$,

$$A(y) = \begin{cases} [a,b], & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if $b = a$, then $x_{[a,b]}$ is denoted by x_a.

We will denote the set of all IVFPs in X as $\text{IVF}_P(X)$.

Definition 2.3[9]. Let $A, B \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(i) $A \subset B$ iff $A^L \leq B^L$ and $A^U \leq B^U$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^c = [1 - A^U, 1 - A^L]$.

(iv) $A \cup B = [A^L \vee B^L, A^U \vee B^U]$.

(iv)′ $\bigcup_{\alpha\in\Gamma} A_\alpha = [\bigvee_{\alpha\in\Gamma} A^L_\alpha, \bigvee_{\alpha\in\Gamma} A^U_\alpha]$.

(v) $A \cap B = [A^L \wedge B^L, A^U \wedge B^U]$.

(v)′ $\bigcap_{\alpha\in\Gamma} A_\alpha = [\bigwedge_{\alpha\in\Gamma} A^L_\alpha, \bigwedge_{\alpha\in\Gamma} A^U_\alpha]$.

Result 2.A[9, Theorem 1]. Let $A, B, C \in D(I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I)^X$. Then:

(a) $\tilde{0} \subset A \subset \tilde{1}$.

(b) $A \cup B = B \cup A, \quad A \cap B = B \cap A$.

(c) $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$.

(d) $A, B \subset A \cup B, \quad A \cap B \subset A, B$.

(e) $A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma}(A \cap A_\alpha)$.

(f) $A \cup (\bigcap_{\alpha\in\Gamma} a_\alpha) = \bigcap_{\alpha\in\Gamma}(A \cup A_\alpha)$.

(g) $(\tilde{0})^c = \tilde{1}, \quad (\tilde{1})^c = \tilde{0}$.

(h) $(A^c)^c = A$.

(i) $(\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A^c_\alpha, \quad (\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A^c_\alpha$.

Definition 2.4[7]. An interval-valued fuzzy set A in G is called an *interal-valued fuzzy subgroupoid*(in short, *IVGP*) in G if

$$A^L(xy) \geq A^L(x) \wedge A^L(y) \quad \text{and} \quad A^U(xy) \geq A^U(x) \wedge A^U(y), \forall x, y \in G.$$

It is clear that $\tilde{0}, \tilde{1} \in \text{IVGP}(G)$. We will denote the IVGPs in G as $\text{IVGP}(G)$.

Definition 2.5[10]. Let $A \in I^L$. Then A is called a *fuzzy sublattice* of L if it satisfies the following conditions: For all $x, y \in L$,

(i) $A(x + y) \geq A(x) \wedge A(y)$.

(ii) $A(x \cdot y) \geq A(x) \wedge A(y)$.

Definition 2.6[10]. Let A be a fuzzy sublattice of L. Then A is called a *fuzzy ideal* [resp. *filter*] of L if $x \leq y$ in L implies $A(x) \geq A(y)$ [resp. $A(x) \leq A(y)$].

Definition 2.7[10]. Let A be a fuzzy ideal [resp. filter] of L. Then A is called a *fuzzy prime ideal* [resp. *filter*] of L if $A(x \cdot y) \leq A(x) \vee A(y)$ [resp. $A(x + y) \leq A(x) \vee A(y)$] for all $x, y \in L$.

3. Interval-valued fuzzy sublattices, ideals and filters

We discuss here some basic results. In the process, some well-known basic concepts of lattice theory are extended to the inverval-valued fuzzy setting. After systematically introducing the notions of interval-valued fuzzy ideal [resp. filter, prime ideal and prime filter], we provide their characterizations.

In this section, $A \in D(I)^L$ is said to be *monotonic* [resp. *antimonotonic*] if $A^L(x) \leq A^L(y)$ and $A^U(x) \leq A^U(y)$ [resp. $A^L(x) \geq A^L(y)$ and $A^U(x) \geq A^U(y)$] whenever $x \leq y$ in L.

Theorem 3.1. Let $A \in D(I)^L$. Then the following are equivalent : For any $x, y \in L$,

(a) A is antimonotonic.
(b) $A^L(x \cdot y) \geq A^L(x) \vee A^L(y)$ and $A^U(x \cdot y) \geq A^U(x) \vee A^U(y)$.
(c) $A^L(x + y) \leq A^L(x) \wedge A^L(y)$ and $A^U(x + y) \leq A^U(x) \wedge A^U(y)$.

Proof. (a) $\Rightarrow$ (b) : Suppose the condition (a) holds.

Let $x, y \in L$. Then clearly $x \cdot y \leq x$ and $x \cdot y \leq y$. Thus, by the condition (a),

$$A^L(x \cdot y) \geq A^L(x), \quad A^L(x \cdot y) \geq A^L(y)$$

and

$$A^U(x \cdot y) \geq A^U(x), \quad A^U(x \cdot y) \geq A^U(y).$$

So $A^L(x \cdot y) \geq A^L(x) \vee A^L(y)$ and $A^U(x \cdot y) \geq A^U(x) \vee A^U(y)$.

(b) $\Rightarrow$ (a) : Suppose the condition (b) holds. Let $x, y \in L$ such that $x \leq y$ in L. Then clearly $x \cdot y = x$. Thus, by the condition (b),

$$A^L(x \cdot y) = A^L(x) \geq A^L(x) \vee A^L(y)$$

and

$$A^U(x \cdot y) = A^U(x) = A^U(x) \vee A^U(y).$$

So $A^L(x) \geq A^L(y)$ and $A^U(x) \geq A^U(y)$.

(a) $\Rightarrow$ (c) : Suppose the condition (a) holds. Let $x, y \in L$. Then clearly $x \leq x+y$ and $y \leq x+y$. Thus, by the condition (a),

$$A^L(x) \geq A^L(x+y), \quad A^L(y) \geq A^L(x+y)$$

and

$$A^U(x) \geq A^U(x+y), \quad A^U(y) \geq a^U(x+y).$$

So $A^L(x+y) \leq A^L(x) \wedge A^L(y)$ and $A^U(x+y) \leq A^U(x) \wedge A^U(y)$.

(c) $\Rightarrow$ (a): Suppose the condition (c) holds. Let $x, y \in L$ such that $x \leq y$. Then clearly $x+y=y$. Thus, by the condition (c),

$$A^L(y) = A^L(x+y) \leq A^L(x) \wedge A^L(y)$$

and

$$A^U(y) = A^U(x+y) \leq A^U(x) \wedge A^U(y).$$

So $A^L(y) \leq A^L(x)$ and $A^U(y) \leq A^U(x)$. Hence A is antimonotonic. This completes the proof. □

Definition 3.2. Let $A \in D(I)^L$. Then A is called an *interval-valued fuzzy sublattice* (in short, *IVFL*) of L if it satisfies the following conditions: For any $x, y \in L$,

(i) $A^L(x+y) \geq A^L(x) \wedge A^L(y), \quad A^U(x+y) \geq A^U(x) \wedge A^U(y)$.

(ii) $A^L(x \cdot y) \geq A^L(x) \wedge A^L(y), \quad A^U(x \cdot y) \geq A^U(x) \wedge A^U(y)$.

We will denote the set of all IVFLs of L as IVFL(L). From Definitions 2.5 and 3.2, it is clear that if $A \in$ IVFL(L), then A^L and A^U are fuzzy sublattices of L.

Definition 3.3. Let $A \in$ IVFL(L). Then A is called an *interval-valued fuzzy ideal* (in short, *IVFI*) of L if it satisfies any one of the conditions of Theorem 3.1.

We will denote the set of all IVFIs of I as IVFI(L). From Definitions 2.6 and 3.3, it is obvious that if $A \in$ IVFI(L), then A^L and A^U are fuzzy ideals of L.

The following is the dual of Theorem 3.1.

Theorem 3.4[The dual of Theorem 3.1]. Let $A \in D(I)^L$. Then the following are equivalent : For any $x, y \in L$,

(a) A is monotonic.

(b) $A^L(x+y) \geq A^L(x) \vee A^L(y)$ and $A^U(x+y) \geq A^U(x) \vee A^U(y)$.
(c) $A^L(x \cdot y) \leq A^L(x) \wedge A^L(y)$ and $A^U(x \cdot y) \leq A^U(x) \wedge A^U(y)$.

Definition 3.5. Let $A \in \text{IVFL}(L)$. Then A is called an *interval-valued fuzzy filter* (in short, *IVFF*) of L if it is satisfies any one of the conditions of Theorem 3.7.

We will denote the set of all IVFFs of L on $\text{IVFF}(L)$. From Definitions 2.6 and 3.5, it is clear that if $A \in \text{IVFF}(L)$, then A^L and A^U are fuzzy filters of L.

Theorem 3.6. Let $A \in D(I)^L$. Then A is an IVFF [resp. IVFI] if and only if

$$A^L(xy) = A^L(x) \wedge A^L(y) \quad \text{and} \quad A^U(xy) = A^U(x) \wedge A^U(y)$$

[resp. $A^L(x+y) = A^L(x) \wedge A^L(y)$ and $A^U(x+y) = A^U(x) \wedge A^U(y)$] for all $x, y \in L$, i.e., A^L and A^U are homomorphisms from $(L, \cdot)$ [resp. $(L, +)$] into $(I, \wedge)$.

Proof. ($\Rightarrow$): Suppose A is an IVFF of L. Let $x, y \in L$ such that $x \leq y$. Then

$$\begin{aligned} A^L(xy) &\geq A^L(x) \wedge A^L(y) \quad [\text{Since}\, A \,\text{is an IVFS}] \\ &= A^L(x) \quad [\text{Since}\, x \leq y \,\text{in}\, L \,\text{and}\, A \,\text{is an IVFF}] \\ &\geq A^L(xy). \quad [\text{Since}\, x \geq xy \,\text{in}\, L \,\text{and}\, A \,\text{is an IVFF}] \end{aligned}$$

Thus $A^L(xy) = A^L(x) \wedge A^L(y)$. Similarly, we have that $A^U(xy) = A^U(x) \wedge A^U(y)$.

Suppose A is an IVFI. Then, by the similar arguments, we have that

$$A^L(x+y) = A^L(x) \wedge A^L(y) \quad \text{and} \quad A^U(x+y) = A^U(x) \wedge A^U(y)$$

for all $x, y \in L$.

($\Leftarrow$): Suppose $A^L(xy) = A^L(x) \wedge A^L(y)$ and $A^U(xy) = A^U(x) \wedge A^U(y)$ for all $x, y \in L$. Let $x, y \in L$ such that $x \leq y$. Then

$$\begin{aligned} A^L(x) &= A^L(xy) \quad [\text{Since}\, x = x \cdot y] \\ &= A^L(x) \wedge A^L(y). \quad [\text{By the hypothesis}] \end{aligned}$$

Thus $A^L(x) \leq A^L(y)$.

Similarly, we have that $A^U(x) \leq A^U(y)$ for any $x, y \in L$ with $x \leq y$. So A is an IVFF of L.

Now suppose $A^L(x+y) = A^L(x) \wedge A^L(y)$ and $A^U(x+y) \leq A^U(x) \wedge A^U(y)$ for all $x, y \in L$. Then, by the similar arguments, we can see that A is an IVFI of L. This completes the proof. □

Definition 3.7. Let $A \in$ IVFI(L) [resp. IVFF(L)]. Then A is called an *interval-valued fuzzy prime ideal* (in short, *IVFPI*) [resp. *filter* (in short, *IVFPF*)] of L it satisfies the following conditions : For any $x, y \in L$,

$$A^L(x \cdot y) \leq A^L(xy) \vee A^L(y) \quad \text{and} \quad A^U(x \cdot y) \leq A^U(x) \vee A^U(y)$$

$$\text{[resp.} \quad A^L(x+y) \leq A^L(x) \vee A^L(y) \quad \text{and} \quad A^U(x+y) \leq A^U(x) \vee A^U(y)].$$

We will denote the set of all IVFPIs [resp. IVFPFs] ad IVFPI(L) [resp. IVFPF(L)]. From Definitions 2.6 and 3.7, it is clear that $A \in$ IVFPI(L) [resp. IVFPF(L)], then A^L and A^U are fuzzy prime ideals [resp. filters] of L.

The following is the immediate results of Theorem 3.1 and Definition 3.7.

Theorem 3.8. Let $A \in$ IVFI(L). Then the following are equivalent: For any $x, y \in L$,

(a) $A \in$ IVFPI(L).

(b) $A^L(x \cdot y) = A^L(x) \vee A^L(y)$ and $A^U(x \cdot y) = A^U(x) \vee A^U(y)$.

(c) $A(x \cdot y) = A(x)$ or $A(y)$.

The following is the immediate result of Theorem 3.4 and Definition 3.7.

Theorem 3.9. Let $A \in$ IVFF(L). Then the following are equivalent: For any $x, y \in L$,

(a) $A \in$ IVFPF(L).

(b) $A^L(x+y) = A^L(x) \vee A^L(y)$ and $A^U(x+y) = A^U(x) \vee A^U(y)$.

(c) $A(x+y) = A(x)$ or $A(y)$.

The following is the immediate result of Theorems 3.6, 3.8 and 3.9.

Corollary 3.10. Let $A \in D(I)^L$. Then $A \in$ IVFPI(L) [resp. IVFPF(L)] if and only if A^L and A^U are homomorphisms from $(L, +, \cdot)$ into $(I, \wedge, \vee)$ [resp. $(I, \vee, \wedge)$].

The following is the immediate result of Definitions 2.3 (iii), 3.3 and 3.5.

Theorem 3.11. Let $A \in$ IVFPI(L) if and only if $A^c \in$ IVFPF(L).

4. Level sublattice ideals and filters

In this section, we introduce the concept of level subsets and establishes the fact that it is going to play an important role in the theory of interval-valued fuzzy lattices, as is the case in the theories of interval-valued fuzzy groups and interval-valued fuzzy rings.

Definition 4.1. Let A be an IVFS in a set X and let $[r,s] \in D(I)$. Then the set

$$A_{[r,s]} = \{x \in X : A^L(x) \geq r \quad \text{and} \quad A^U(x) \geq s\}$$

is called a *level subset* of A.

Result 4.A[8, Proposition 2.4]. Let A be an IVFS in a set X and let $[r_1,s_1]$, $[r_2,s_2] \in D(I)$. Then $[r_1,s_1] \leq [r_2,s_2]$ in $D(I)$ if and only if $A_{[r_2,s_2]} \subset A_{[r_1,s_1]}$.

Theorem 4.2. Let $A \in D(I)^L$. Then $A \in \text{IVFL}(L)$ if and only if $A_{[r,s]}$ is a sublattice of L for each $[r,s] \in \text{ImA}$.

Equivalently, $A \in \text{IVFL}(L)$ if and only if each nonempty level subset $A_{[r,s]}$ is a sublattice of L. In this case, $A_{[r,s]}$ is called a *level sublattice* of L.

Proof. We prove here the second assertion of the theorem. The proof of the first is the same, except for trivial modification.

($\Rightarrow$): Suppose $A \in \text{IVFL}(L)$. For each $[r,s] \in D(I)$, let $A_{[r,s]}$ be nonempty level subset of L and let $x,y \in A_{[r,s]}$. Then $A^L(x) \geq r, A^L(y) \geq r$ and $A^U(x) \geq s, A^U(y) \geq s$, Since $A \in \text{IVFL}(L)$.

$$A^L(x+y) \geq A^L(x) \wedge A^L(x) \wedge A^L(y) \geq r, \quad A^U(x+y) \geq A^U(x) \wedge A^U(y) \geq s$$

and

$$A^L(x \cdot y) \geq A^L(x) \wedge A^L(y) \geq r, \quad A^U(x \cdot y) \geq A^U(x) \wedge A^U(y) \geq s.$$

Thus $x+y \in A_{[r,s]}$ and $x \cdot y \in A_{[r,s]}$. So $A_{[r,s]}$ is a sublattice of L.

($\Leftarrow$): Suppose the necessary condition holds. For any $x,y \in L$, let $A(x) = [r_1,s_1]$ and let $A(y) = [r_2,s_2]$. Without loss of generality, we can assume that $[r_1,s_1] \leq [r_2,s_2]$.
Then, by Result 4.A,

$$A_{[r_2,s_2]} \subset A_{[r_1,s_1]}.$$

Since $A(x) = [r_1, s_1]$ and $A(y) = [r_2, s_2]$, $x \in A_{[r_1,s_1]}$ and $y \in A_{[r_2,s_2]}$. Thus $x, y \in A_{[r_1,s_1]}$. Since $A_{[r_1,s_1]}$ is a sublattice of L,

$$x + y \in A_{[r_1,s_1]} \quad \text{and} \quad x \cdot y \in A_{[r_1,s_1]}.$$

So

$$A^L(x+y) \geq r_1, \quad A^U(x+y) \geq s_1$$

and

$$A^L(x \cdot y) \geq r_1, \quad A^U(x \cdot y) \geq s_1.$$

Hence

$$A^L(x+y) \geq r_1 = A^L(x) \wedge A^L(y), \quad A^U(x+y) \geq s_1 = A^U(x) \wedge A^U(y)$$

and

$$A^L(x \cdot y) \geq r_1 = A^L(x) \wedge A^L(y), \quad A^U(x \cdot y) \geq s_1 = A^U(x) \wedge A^U(y).$$

Therefore $A \in \mathrm{IVFL}(L)$. This completes the proof. □

It is easy to show that the set of level sublattices of an IVFS of a lattice forms a chain. However, in contrast to the chain level subgroups of an interval-valued subgroup, the chain of level sublattices of an IVFS may not contain a least element.

In interval-valued fuzzy group theory, it is well known that an interval-valued fuzzy subgroup of a group attains its supremum at the identity of the given group. However, the situation is different in the case of an IVFS of a lattice, since an IVFS may neither attain its supremum nor its infimum at any element of the given lattice. The following examples make the situation clear.

The following examples are the modifications of Examples 3.12, 3.13 and 3.14 in [1].

Example 4.3. Let $L = \mathbb{N}$, the chain of natural numbers. We define a mapping $A : L \to D(I)$ as follows: For each $x \in L$,

$$A(x) = [1 - \frac{1}{n+1}, 1 - \frac{1}{n+1}], \quad \text{for all} \quad x \in \frac{(2^n)}{(2^{n+1})}$$

for each fixed nonnegative integer n, where (2^n) denotes the set of all those positive integers which are multiple of 2^n. Then clearly $A \in D(I)^L$. Furthermore, by Theorem 4.2, $A \in \mathrm{IVFL}(L)$ with the following chain of level sublattices:

$$\cdots \subset (2^3) \subset (2^2) \subset (2) \subset L,$$

where "1" is the least element of L and $A(0) = [0, 0]$. But $\bigvee_{x \in L} A(x) = [1, 1]$ and it is not attained any where by A. □

Example 4.4. Let $L = \mathbb{N} \times \mathbb{N}$, the Cartesian product of the chain of natural numbers with it self. Define subsets L_i of L as follows:

$$L_1 = \{(2,2),(2,3),(3,2),(3,3)\}$$

$$L_n = \{x \in L : x \leq (n+1, n+1)\}$$

for each positive integer $n \geq 2$. Now define a mapping $A : L \to D(I)$ as follows : For each $x \in L$,

$$A(x) = \begin{cases} [1,1] & \text{if} \quad x \in L_1, \\ [\frac{1}{n}, \frac{1}{n}] & \text{if} \quad x \in \frac{L_n}{L_{n-1}}, \end{cases}$$

where $n \in \mathbb{N}$ and $n \geq 2$.
Then clearly $A \in \text{IVFL}(L)$ by Theorem 4.2. Furthermore, $\bigvee_{x \in L} A(x) = [1, 1]$ which is not attained at the least element $(1, 1)$ of L by A and infimum is not attained anywhere. □

Example 4.5. Let $\mathbb{N}$ be the set of natural numbers and let

$$L = \{\emptyset, \mathbb{N}\} \cup \{\{n\} : n \in \mathbb{N}\}.$$

Then clearly L is a lattice under the ordering of set inclusion with $\emptyset$ as its least element and $\mathbb{N}$ the greatest element. Consider all the finite sublattices of L of the form:

$$L_1 = \{\emptyset, \mathbb{N}\},$$

$$L_n = \{\emptyset, \mathbb{N}\} \cup \{\{i\} : i \leq n-1\},$$

for each $n \in \mathbb{N}$ and $n \geq 2$.

We define two mappings $A, B : L \to D(I)$ as follows: For each $x \in L$,

$$A(x) = \begin{cases} [1,1] & \text{if} \quad x \in L_1, \\ [\frac{1}{n}, \frac{1}{n}] & \text{if} \quad x \in \frac{L_n}{L_{n-1}}, \end{cases}$$

$$B(x) = \begin{cases} [0,0] & \text{if} \quad x \in L_1, \\ [1-\frac{1}{n}, 1-\frac{1}{n}] & \text{if} \quad x \in \frac{L_n}{L_{n-1}}, \end{cases}$$

Then we can easily see that $A, B \in \text{IVFL}(L)$. Moreover, it is easy to show that A does not attain its infimum, whereas B does not attain its supremum. □

The following result is straightforward.

Theorem 4.6. Let $A \in \text{IVFL}(L)$. Then $A \in \text{IVFI}(L)$ [resp. $\text{IVFF}(L)$] if and only if $A_{[r,s]}$ is an ideal [resp. a filter] of L, for each $[r,s] \in \text{Im}A$. In this case, $A_{[r,s]}$ is called a *level ideal* [resp. *filter*] of L.

Theorem 4.7. Let $A \in \text{IVFI}(L)$ [resp. $\text{IVFF}(L)$]. Then $A \in \text{IVFPI}(L)$ [resp. $\text{IVFPF}(L)$] if and only if $A_{[r,s]}$ is prime, for each $[r,s] \in \text{Im}A$.

Proof. ($\Rightarrow$): Suppose $A \in \text{IVFPI}(L)$. For each $[r,s] \in \text{Im}A$ and for any $a, b \in L$, let $a \cdot b \in A_{[r,s]}$. Then clearly $A^L(a \cdot b) \geq r$ and $A^U(a \cdot b) \geq s$. By Theorem 3.8,

$$A(a \cdot b) = A(a) \quad \text{or} \quad A(a \cdot b) = A(b).$$

Thus

$$A^L(a) \geq r, \quad A^U(a) \geq s$$

or

$$A^L(b) \geq r, \quad A^U(b) \geq s.$$

So $a \in A_{[r,s]}$ or $b \in A_{[r,s]}$. Hence $A_{[r,s]}$ is prime.

($\Leftarrow$): Suppose each level ideal $A_{[r,s]}$ is prime. Assume that $A \notin \text{IVFPI}(L)$. Then, by Theorem 3.8, there exists $a, b \in L$ such that

$$A(a \cdot b) \neq A(a) \quad \text{and} \quad A(a \cdot b) \neq A(b).$$

Since $A \in \text{IVFI}(L)$,

$$A^L(a \cdot b) > A^L(a), \quad A^U(a \cdot b) > A^U(a)$$

and

$$A^L(a \cdot b) > A^L(b), \quad A^U(a \cdot b) > A^U(b)$$

Let $A(a \cdot b) = [r,s]$. Then clearly $a \cdot b \in A_{[r,s]}$. But $a \notin A_{[r,s]}$ and $b \notin A_{[r,s]}$. This contradicts the fact that $A_{[r,s]}$ is a prime ideal of L. So $A \in \text{IVFPI}(L)$.

By the similar arguments, we can see that $A \in \text{IVFPF}(L)$ if and only if each level filter $A_{[r,s]}$ is prime. This completes the proof. □

Proposition 4.8. Let $\{A_\alpha\}_{\alpha \in \Gamma}$ be any family of IVFLs or IVFIs [resp. IVFFs] of L. Then $\bigcap_{\alpha \in \Gamma} A_\alpha \in \text{IVFL}(L)$ or $\text{IVFI}(L)$ [resp. $\text{IVFF}(L)$].

Proof. We omit. □

Definition 4.9. Let $A \in D(I)^L$. Then, the least IVFL or IVFI [resp. IVFF] of L containing A is called the interval-valued fuzzy sublattice or interval-valued fuzzy ideal [resp. filter] *generated* by A.

From Proposition 4.8, it is obvious that A is a nonzero interval-valued fuzzy set in a lattice, then a least interval-valued fuzzy sublattice containing A exists. In the future, we will denote an IVFS generated by a nonzero interval-valued fuzzy set A as $[A]$, and the same notation $[A_{[r,s]}]$ is used for a sublattice generated by a non-interval-valued fuzzy subset $A_{[r,s]}$. However the same will be clear from the context. Similar are the cases of interval-valued fuzzy ideal [resp. filter] which is denoted by $(A]$ [resp. $[A)$].

We recall from lattice theory that the sublattice generated by any subset H of a lattice L consists of the lattice polynomial functions of the elements of H. That is, if $[H]$ is a sublattice generated by a subset H of L, then $[H] = \{a : a = P(h_0, h_1, \cdots, h_{n-1}), n \geq 1, h_i \in H\}$, for some n-ary polynomial P (See [5]).

In the following result, we construct an IVFL generated by an IVFS in a specified way.

Proposition 4.10. Let $A \in D(I)^L$. We define a mapping $A^* : L \to D(I)$ as follows : For each $x \in L$,

$$A^*(x) = [r, s], \quad \text{if}\, x \in [A_{[r,s]}]\, \text{and}\, x \notin [A_{[t,w]}],$$

where $[t, w] > [r, s]$ for $[t, w], [r, s] \in \text{Im}A$.
Then A^* is the IVFL generated by A, i.e., $A^* = [A]$.

Proof. **Claim 1**: $A^*_{[r,s]} = [A_{[r,s]}]$, for $[r, s] \in \text{Im}A^*$.

Let $[r, s] \in \text{Im}A^*$ and let $x \in A^*_{[r,s]}$. Then either $A^*(x) = [r, s]$ or $(A^*)^L(x) > r$ and $(A^*)^U(x) > s$.
Thus, by the definition of A^*,

$$x \in [A_{[r,s]}] \quad \text{for}\, A^*(x) = [r, s].$$

For $(A^*)^L(x) > r$ and $(A^*)^U(x) > s$, let $A^*(x) = [t, w]$. Then, by the definition of A^*,

$$x \in [A_{[t,w]}] \quad \text{and}\, A_{[t,w]} \subset A_{[r,s]}.$$

Thus $x \in [A_{[r,s]}]$. So $A^*_{[r,s]} \subset [A_{[r,s]}]$.

Now suppose $x \notin A^*_{[r,s]}$. Then $(A^*)(x) < r$ and $(A^*)(x) < s$. Let $A^*(x) = [t, w]$. Thus,

$$x \in [A_{[t,w]}] \quad \text{and} \quad x \notin [A_{[u,v]}]\, \text{for}\, [u, v] > [t, w].$$

Since $[r, s] > [t, w]$,

$$x \in [A_{[t,w]}] \quad \text{and} \quad x \notin [A_{[r,s]}].$$

So $[A_{[r,s]}] \subset A^*_{[r,s]}$. Hence $A^*_{[r,s]} = [A_{[r,s]}]$.
From Theorem 4.2, it is clear that $A^* \in \text{IVFL}(L)$.

Claim 2: $A \subset A^*$. Assume that there exists $x \in L$ such that $A(x) = [r, s]$, and $A^L(x) > (A^*)^L(x)$ and $A^U(x) > (A^*)^U(x)$. Let $A^*(x) = [t, w]$. Then

$$x \in [A_{[t,w]}] \quad \text{and} \quad x \notin [A_{[u,v]}] \,\text{for}\, [u, v] > [t, w].$$

Since $[r, s] > [t, w]$, $x \notin [A_{[r,s]}]$. This contracts the fact that $x \in A_[r, s]$. So $A \subset A^*$.

Claim 3: A^* is the least IVFL containing A.
Let B be any IVFL of L containing A. Suppose $A^*(x) = [r, s]$. Then

$$x \in [A_{[r,s]}] \quad \text{and} \quad x \notin [A_{[t,w]}] \,\text{for}\, [t, w] > [r, s].$$

where x is a lattice polynomial function. Thus, x can be written as

$$x = P(h_1, h_2, \cdots, h_k), \quad \text{where}\, h_i \in A_{[r,s]},\ i = 1, 2, \cdots, k.$$

By induction on the rank of x and by using the definition of IVFL,

$$\begin{aligned} B^L(x) &\geq B^L(h_1) \wedge B^L(h_2) \wedge \cdots \wedge B^L(h_k) \\ &\geq A^L(h_1) \wedge A^L(h_2) \wedge \cdots \wedge A^L(h_k) \\ &\geq r. \end{aligned}$$

Similarly, we have that

$$B^U(x) \geq s.$$

So $B^L(x) \geq r = (A^*)^L(x)$ and $B^U(x) \geq s = (A^*)^U(x)$. Hence $A^* \subset B$. This completes the proof. □

The following example is the modification of Example 3.20 in [1].
Example 4.11. Let $L = \mathbb{N} \times \mathbb{N}$ be the lattice in Example 4.5. We define a mapping $A : L \to D(I)$ as follows : For each $(m, n) \in L$,

$$A((m, n)) = [\frac{1}{m+n}, \frac{1}{m+n}].$$

For each $n \in \mathbb{N}$, consider the sublattice L_n of L defined as follows :

$$L_n = \{x \in L : x \leq (n, n)\}.$$

Fig 4.1
Then, for each $r \in \mathbb{N}$, the level subsets $A_{[\frac{1}{r},\frac{1}{r}]}$ of A are given by

$$A_{[\frac{1}{2},\frac{1}{2}]} = \{(1, 1)\} = L_1,$$

$$A_{[\frac{1}{r},\frac{1}{r}]} = \{(m, n) \in L : m + n \leq r\}.$$

Also, it follows that

$$[A_{[\frac{1}{2},\frac{1}{2}]}] = A_{[\frac{1}{2},\frac{1}{2}]} = L_1,$$

$$[A_{[\frac{1}{r},\frac{1}{r}]}] = L_{r-1} \quad \text{for each}\, r \in \mathbb{N} \quad \text{(See Figure 4.2).}$$

Fig. 4.2

On the other hand, we can easily see that $A^* = [A]$ is given by for each $x \in L$,

$$A^*(x) = [\frac{1}{2}, \frac{1}{2}] \quad \text{for}\, x \in L_1$$

and

$$A^*(x) = [\frac{1}{n+1}, \frac{1}{n+1}] \quad \text{for} x \in \frac{L_n}{L_{n-1}} \quad \text{and} n \geq 2.$$

By the definition of A, $A((3,2)) = [\frac{1}{5}, \frac{1}{5}]$. But $(3,2) \in \frac{L_3}{L_2}$, $A^*((3,2)) = [\frac{1}{4}, \frac{1}{4}]$. So $A \neq A^*$ and $A \subset A^*$. □

In a similar way as in Proposition 4.10, we can obtain the IVFI [resp. IVFF] generated by an interval-valued fuzzy set. Here, we make use of the result that an ideal generated by a subset H of L will be of the form $(H] = \{a : a \leq h_1 + h_2 + \cdots + h_k, n \geq 1, h_i \in H\}$ (See [5]). Thus we have the following result:

Proposition 4.12. Let $A \in D(I)^L$. We define a mapping $A^* : L \to D(I)$ as follows : For each $x \in L$,

$$A^*(x) = [r,s] \quad \text{if}\, x \in (A_{[r,s]}] \quad \text{and} \quad x \notin (A_{[t,w]}],$$

where $[t,w] > [r,s]$ for $[r,s], [t,w] \in \text{Im}A$.
Then A is the IVFI generated by A, i.e., $A^*(A]$.

The following is the result with respect to the IVFF generated by an IVFS.

Proposition 4.13. Let $A \in D(I)^L$. We define a mapping $A^* : L \to D(I)$ as follows : For each $x \in L$,

$$A^*(x) = [r,s] \quad \text{if}\, x \in [A_{[r,s]}) \quad \text{and} \quad x \notin [A_{[t,w]}),$$

where $[t,w] > [r,s]$ for $[r,s], [t,w] \in \text{Im}A$.
Then A is the IVFF generated by A, i.e., $A^* = [A)$.

5. Interval-valued fuzzy convexity

Convex sublattices occupy an important place in lattice theory. In this section, we extend the concept of convexity to the interval-valued fuzzy setting and provide its characterizations. We show that an intersection of an IVFI and an IVFF is an interval-valued fuzzy convex sublattice, and every interval-valued fuzzy convex sublattice has a unique representation of this type. That is an interval-valued fuzzy analog of a famous result of lattice theory.

Definition 5.1. Let $A \in \mathrm{IVFL}(L)$. Then A is said to be *interval-valued fuzzy convex* (in short, *IVFC*) if for each $[a,b] \subset L$, and each $x \in [a,b]$,

$$A^L(x) \geq A^L(a) \wedge A^L(b) \quad \text{and} \quad A^U(x) \geq A^U(a) \wedge A^U(b).$$

We will denote the set of all interval-valued fuzzy convex sublattices of L as $\mathrm{IVFCL}(L)$.

Theorem 5.2. Let $A \in$IVFL(L). Then $A \in \mathrm{IVFCL}(L)$ if and only if $A_{[r,s]}$ is a convex sublattice of L for each $[r,s] \in \mathrm{Im}A$.

Proof. ($\Rightarrow$): Suppose $A \in \mathrm{IVFCL}(L)$. Let $[r,s] \in \mathrm{Im}A$ and let $[a,b]$ be any interval contained in $A_{[r,s]}$.
Then

$$A^L(a) \geq r, \quad A^U(a) \geq s$$

and

$$A^L(b) \geq r, \quad A^U(b) \geq s.$$

Thus

$$A^L(a) \wedge A^L(b) \geq r \quad \text{and} \quad A^U(a) \wedge A^U(b) \geq s.$$

Since $A \in \mathrm{IVFCL}(L)$, for all $x \in [a,b]$,

$$A^L(x) \geq A^L(a) \wedge A^L(b) \quad \text{and} \quad A^U(x) \geq A^U(a) \wedge A^U(b).$$

So

$$A^L(x) \geq r \quad \text{and} \quad A^U(x) \geq s.$$

Hence $x \in A_{[r,s]}$. Therefore $A_{[r,s]}$ is a convex sublattice of L.

($\Leftarrow$): Suppose the necessary condition holds.

Let $[a,b]$ be any interval in L and let

$$A^L(a) \wedge A^L(b) = r \quad \text{and} \quad A^U(a) \wedge A^U(b) = s.$$

Then $a \in A_{[r,s]}$ and $b \in A_{[r,s]}$. Let $x \in [a, b]$. Sice $A_{[r,s]}$ is a convex sublattice of L, $x \in A_{[r,s]}$. Thus $A^L(x) \geq r$ and $A^U(x) \geq s$. So, for each $x \in [a, b]$,

$$A^L(x) \geq A^L(a) \wedge A^L(b) \quad \text{and} \quad A^U(x) \geq A^U(a) \wedge A^U(b).$$

Hence $A \in$ IVFCL(L). This completes the proof. □

Proposition 5.3. In a lattice, every IVFI [resp. IVFF] is an IVFCL.

Proof. The proof is straightforward. □

Proposition 5.4. Let $\{A_\alpha\}_{\alpha\in\Gamma}$ be any family of IVFCLs of L. Then $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVFCL(L).

Proof. For any interval $[a, b] \subset L$, let $x \in [a, b]$. Then, by the hypothesis, for each $\alpha \in \Gamma$,

$$A_\alpha^L(x) \geq A_\alpha^L(a) \wedge A_\alpha^L(b) \quad \text{and} \quad A_\alpha^U(x) \geq A_\alpha^U(a) \wedge A_\alpha^U(b).$$

Thus

$$\begin{aligned}(\bigcap_{\alpha\in\Gamma} A_\alpha)^L(x) &= \bigwedge_{\alpha\in\Gamma} A_\alpha^L(x) \geq \bigwedge_{\alpha\in\Gamma} [A_\alpha^L(a) \wedge A_\alpha^L(b)] \\ &= (\bigwedge_{\alpha\in\Gamma} A_\alpha^L(a)) \wedge (\bigwedge_{\alpha\in\Gamma} A_\alpha^L(b)) \\ &= (\bigcap_{\alpha\in\Gamma} A_\alpha)^L(a) \wedge (\bigcap_{\alpha\in\Gamma} A_\alpha)^L(b).\end{aligned}$$

Similarly, we have that

$$(\bigcap_{\alpha\in\Gamma} A_\alpha)^U(x) \geq (\bigcap_{\alpha\in\Gamma} A_\alpha)^U(a) \wedge (\bigcap_{\alpha\in\Gamma} A_\alpha)^U(b).$$

So, $\bigcap_{\alpha\in\Gamma} A_\alpha \in$ IVFCL(L). □

Result 5.A [5, Lemma 1 in p.18]. Let L be a lattice, and let H and I be nonempty subsets of L.

(a) I is an ideal if and only if $a, b \in I$ imply that $a + b \in I$, and $a \in I, x \in L, x \leq a$ imply that $x \in I$.

(b) $I = (H]$ if and only if I is an ideal, $H \subset I$, and for all $x \in I$, there exists an integer $n \geq 1$ and there exist

$$h_0, h_2, \cdots, h_{n-1} \in H \quad \text{such that} \quad x \leq h_0 + h_1 + \cdots + h_{n-1}.$$

Result 5.B [5, Lemma 6 in p. 19]. Let I be an ideal and let D be a filter. If $I \cap D \neq \emptyset$, then $I \cap D$ is a convex sublattice, and every convex sublattice can be expressed in this form in one and only one way.

Theorem 5.5 (The generalization of Result 5.B). Let $A \in \mathrm{IVFI}(L)$ and let $D \in \mathrm{IVFF}(L)$. If $A \cap D \neq \tilde{0}$, then $A \cap D \in \mathrm{IVFCL}(L)$, and every IVFCL can be expressed in this form in one and only one way.

Proof. (i) From Propositions 5.3 and 5.4, it is clear that $A \cap D \in \mathrm{IVFCL}(L)$.

(ii) **Claim 1**: Every IVFCL can be expressed as an intersection of an IVFI and IVFF.

Let $B \in \mathrm{IVFCL}(L)$. Then clearly we have that $B \subset (B] \cap [B)$.
For each $x \in L$, let $(B](x) = [r_0, s_0]$ and let $[B)(x) = [r_1, s_1]$.
Without loss of generality, we assume that $(r_0, s_0) \leq (r_1, s_1)$.
Then, by Proposition 4.12,

$$x \in (B_{[r_0,s_0]}] \quad \text{and} \quad x \neq (B_{[r,s]}] \quad \text{for}\, [r,s] > [r_0, s_0].$$

Thus, by Result 5.A (b), there exists $y_1 \in D_{[r_0,s_0]}$ such that $x \leq y_1$.
Similarly,

$$x \in [B_{[r_1,s_1]}] \quad \text{and} \quad x \neq [B_{[r,s]}) \quad \text{for}\, [r,s] > [r_1, s_1]$$

and there exists $y_2 \in B_{[r_1,s_1]}$ such that $y_2 \leq x$. Since $B_{[r_1,s_1]} \subset B_{[r_0,s_0]}$, $y_1, y_2 \in B_{[r_0,s_0]}$. Since $B \in \mathrm{IVFCL}(L)$, by Theorem 5.2, $B_{[r_0,s_0]}$ is a convex sublattice of L. Then $x \in B_{[r_0,s_0]}$. Thus

$$B^L(x) \geq r_0 \quad \text{and} \quad B^U(x) \geq s_0.$$

So

$$B^L(x) \geq (B]^L(x) \wedge [B)^L(x) = r_0$$

and

$$B^U(x) \geq (B]^L(x) \wedge [B)^L(x) = s_0$$

Hence $(B] \cap [B) \subset B$. Therefore $B = (B] \cap [B)$.

Claim 2 : This representation is unique.

For any $B \in \mathrm{IVFCL}(L)$, let $B = A \cap D$, where $A \in \mathrm{IVFI}(L)$ and $D \in \mathrm{IVFF}(L)$. We show that

$$A = (B] \quad \text{and} \quad D = [B).$$

Since $B \subset A, (B] \subset A$. Now let $a \in L$ and let $A(a) = [r_0, s_0]$. Then clearly $a \in A_{[r_0,s_0]}$.
Also, since $B \subset A$

$$B_{[r_0,s_0]} \subset A_{[r_0,s_0]}.$$

Let $b \in B_{[r_0,s_0]}$. Since $A \in \mathrm{IVFI}(L)$, by Theorem 4.6, $A_{[r_0,s_0]}$ is an ideal of L. Then, by Result 5.A (a), we have

$$a + b \in A_{[r_0,s_0]} \quad \text{and} \quad b \in [B_{[r_0,s_0]}).$$

Since $b \leq a + b$ and $[B_{[r_0,s_0]})$ is a filter of L, by the dual of Result 5.A (a)

$$a + b \in [B_{[r_0 s_0]}).$$

On the other hand, we can easily see that $[B_{[r_0,s_0]}) = [B)_{[r_0,s_0]}$. Then

$$[B)^L(a+b) \geq r_0 \quad \text{and} \quad [B)^U(a+b) \geq s_0.$$

Since $[B) \subset D$, we have

$$D^L(a+b) \geq r_0 \quad \text{and} \quad D^U(a+b) \geq s_0.$$

Thus

$$(A \cap D)^L(a+b) = A^L(a+b) \wedge D^L(a+b) \geq r_0$$

and

$$(A \cap D)^U(a+b) = A^U(a+b) \wedge D^U(a+b) \geq s_0.$$

So

$$B^L(a+b) = (A \cap D)^L(a+b) \geq r_0$$

and

$$B^U(a+b) = (A \cap D)^U(a+b) \geq s_0.$$

Hence $a + b \in B_{[r_0,s_0]}$. Therefore $a + b \in (B_{[r_0,s_0]}]$. Since $a \leq a + b$ and $(B_{[r_0,s_0]}$ is the ideal generated by $B_{[r_0,s_0]}$, $a \in (B_{[r_0,s_0]}]$. Since $(B_{[r_0,s_0]}] = (B]_{[r_0,s_0]}$, $a \in (B]_{[r_0,s_0]}$. Thus $(B]^L(a) \geq r_0$ and $(B]^U(a) \geq s_0$. So

$$(B]^L(a) \geq A^L(a) = r_0$$

and

$$(B]^U(a) \geq A^U(a) = s_0.$$

Hence $A \subset (B]$. Therefore $(B] = A$.

By the similar arguments, we can prove that $[B) = D$. Hence the uniqueness holds. This complete the proof. □

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