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ON PSEUDO BH-ALGEBRAS

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Abstract. As a generalization of BH-algebras, the notion of pseudo BH-algebra is introduced, and some of their properties are investigated. The notions of pseudo ideals, pseudo atoms, pseudo strong ideals, and pseudo homomorphisms in pseudo BH-algebras are introduced. Characterizations of their properties are provided. We show that every pseudo homomorphic image and preimage of a pseudo ideal is also a pseudo ideal. Any pseudo ideal of a pseudo BH-algebra can be decomposed into the union of some sets. The notion of pseudo complicated BH-algebra is introduced and some related properties are obtained.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3,4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. BCK-algebras have several connections with other areas of investigation, such as: lattice ordered groups, MV-algebras, Wajsberg algebras, and implicative commutative semigroups. J. M. Font et al. [1] have discussed Wajsberg algebras which are term-equivalent to MV-algebras. D. Mundici [9] proved MV-algebras are categorically equivalent to bounded commutative BCK-algebra, and J. Meng [8] proved that implicative commutative semigroups are equivalent to a class of BCK-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim [7] introduced the notion of a BH-algebra, which is a generalization of BCK/BCI-algebras. E. H. Roh and S. Y. Kim [11] estimated the number of BH^*-subalgebras of order i in a transitive BH^*-algebras by using Hao's method. G. Georgescu and A. Iorgulescu [2] introduced the notion of a pseudo BCK-algebra.

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Y. B. Jun characterized pseudo BCK-algebras. He found conditions for a pseudo BCK-algebra to be $\wedge$-semi-lattice ordered. Y. B. Jun, H.S. Kim, J. Neggers [5] introduced the notion of a pseudo d-algebra as a generalization of the idea of a d-algebra.

In this paper, we introduce the notion of pseudo BH-algebra as a generalization of BH-algebra and investigate some of their properties. We also define the notions of pseudo ideals, pseudo atoms, pseudo strong ideals, and pseudo homomorphisms in pseudo BH-algebras and provide characterizations of their properties in pseudo BH-algebras. We show that every pseudo homomorphic image and preimage of a pseudo ideal is also a pseudo ideal. Any pseudo ideal of a pseudo BH-algebra can be decomposed into the union of some sets. We introduced the notion of pseudo complicated BH-algebra and obtain some related properties.

2. Preliminaries

By a *BH-algebra* ([7]), we mean an algebra $(X;*,0)$ of type (2,0) satisfying the following conditions:

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

For brevity, we also call X a BH-algebra. In X we can define a binary operation " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. A non-empty subset S of a BH-algebra X is called a *subalgebra* of X if, for any $x, y \in S$, $x * y \in S$, i.e., S is a closed under binary operation.

Definition 2.1([7]). A non-empty subset A of a BH-algebra X is called an *ideal* of X if it satisfies:

(I1) $0 \in A$,
(I2) $x * y \in A$ and $y \in A$ imply $x \in A$, $\forall x, y \in X$.

An ideal A of a BH-algebra X is said to be a *translation ideal* of X if it satisfies:

(I3) $x*y \in A$ and $y*x \in A$ imply $(x*z)*(y*z) \in A$ and $(z*x)*(z*y) \in A$, $\forall x, y, z \in X$.

Obviously, $\{0\}$ and X are ideals of X. We will call $\{0\}$ and X a *trivial ideal* and a *improper ideal*, respectively. An ideal A is said to be *proper* if $A \neq X$.

Definition 2.2([11])**.** A BH-algebra X is called a *BH^*-algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Example 2.3([7])**.** Let $X := \{0, a, b, c\}$ be a BH-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	1	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

Then $A := \{0, 1\}$ is a translation ideal of X.

Definition 2.4. A non-empty subset A of a BH-algebra X is called a *strong ideal* of X if it satisfies (I1) and

(I4) $(x * y) * z \in A$ and $y \in A$ imply $x * z \in A$, $\forall x, y \in X$.

Example 2.5. (1) Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Then X is a BH-algebra which is not a BCK/BCI-algebra, since $(4 * (4 * 5)) * 5 = (4 * 1) * 5 = 4 * 5 = 1 \neq 0$. Let $S := \{0, 1, 2, 3, 4\}$. It is easy to see that S is a subalgebra and a strong ideal of X.

(2) Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Then X is a BH-algebra, which is not a BCK/BCI-algebra, since $((4 * 2) * (4 * 3)) * (3 * 2) = (4 * 3) * 1 = 3 * 1 = 3 \neq 0$. Let $A := \{0, 1, 2, 3\}$. Then A is a subalgebra, not an ideal, and not a strong ideal of X, since $4 * 3 = 3 \in A$, but $4 \notin A$. Let $B := \{0, 1\}$. It is easy to show that

B is a subalgebra, an ideal of X, and not a strong ideal of X, since $(4 * 1) * 2 = 3 * 2 = 1 \in B$, but $4 * 2 = 4 \notin B$.

3. Pseudo BH-algebras

Definition 3.1. A *pseudo BH-algebra* is a non-empty set X with a constant 0 and two binary operations "$*$" and "$\diamond$" satisfying the following axioms:

(P1) $x * x = x \diamond x = 0$;
(P2) $x * 0 = x \diamond 0 = x$;
(P3) $x * y = y \diamond x = 0$ imply $x = y$ for all $x, y \in X$.

For brevity, we also call X a pseudo BH-algebra. In X we can define a binary operation " $\preceq$ " by $x \preceq y$ if and only if $x * y = 0$ if and only if $x \diamond y = 0$. Note that if $(X; *, 0)$ is a BH-algebra, then letting $x \diamond y := x * y$, produces a pseudo BH-algebra $(X; *, \diamond, 0)$. Hence every BH-algebra is a pseudo BH-algebra in a natural way.

Definition 3.2. Let $(X; *, \diamond, 0)$ be a pseudo BH-algebra and let $\emptyset \neq I \subseteq X$. I is called a *pseudo subalgebra* of X if $x * y, x \diamond y \in I$ whenever $x, y \in I$. I is called a *pseudo ideal* of X if it satisfies

(PI1) $0 \in I$;
(PI2) $x * y, x \diamond y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Example 3.3. (1) Let $X := \{0, a, b, c\}$ be a set with the following Cayley tables:

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	0
b	b	0	0	b
c	c	0	c	0

$\diamond$	0	a	b	c
0	0	0	0	c
a	a	0	a	c
b	b	a	0	0
c	c	c	0	0

Then it is easy to show that $(X; *, 0)$ and $(X; \diamond, 0)$ are not BH-algebras, but $(X; *, \diamond, 0)$ is a pseudo BH-algebra. Let $I := \{0, a\}$. Then I is a pseudo subalgebra of X, but not a pseudo ideal of X since $b * a = 0, b \diamond a = a$, and $0, a \in I$, but $b \notin I$.
(2) Let $X := \{0, a, b, c\}$ be a set with the following Cayley tables:

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	0
b	b	b	0	b
c	c	0	a	0

$\diamond$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	0
c	c	c	0	0

Then it is easy to show that $(X;*,0)$ and $(X;\diamond,0)$ are not BH-algebras, but $(X;*,\diamond,0)$ is a pseudo BH-algebra. If we let $J := \{0,a,c\}$, then it is both a pseudo subalgebra of X and a pseudo ideal of X.

Proposition 3.4. *Let I be a pseudo ideal of a pseudo BH-algebra X. If $x \in I$ and $y \preceq x$, then $y \in I$.*

Proof. Assume that $x \in I$ and $y \preceq x$. Then $y * x = 0$ and $y \diamond x = 0$. By (PI1) and (PI2), we have $y \in I$. □

Definition 3.5. A pseudo BH-algebra $(X;*,0)$ is called a *pseudo BH^*-algebra* if it satisfies the identities $(x*y)\diamond x = 0$ and $(x \diamond y) * x = 0$ for all $x, y \in X$.

Theorem 3.6. *For any element of a pseudo BH^*-algebra X, the initial section $\downarrow a := \{x \in X | x \preceq a\}$ is a pseudo ideal of X if and only if the following implications hold:*
(i) $\forall x,y,z \in X, x*y \preceq z, y \preceq z \implies x \preceq z$.
(ii) $\forall x,y,z \in X, x\diamond y \preceq z, y \preceq z \implies x \preceq z$.

Proof. Assume that for each $a \in X$, $\downarrow a$ is a pseudo ideal of X. Let $x,y,z \in X$ be such that $x*y \preceq z, x\diamond y \preceq z$ and $y \preceq z$. Then $x*y \in \downarrow z, x\diamond y \in \downarrow z$ and $y \in \downarrow z$. Since $\downarrow z$ is a pseudo ideal of X, it follows from (PI2) that $x \in \downarrow z$, i.e., $x \preceq z$.

Conversely, consider $\downarrow z$ for any $z \in X$. Obviously, $0 \in \downarrow z$. For every $y \in \downarrow z$, let $a*y \preceq z, a\diamond y \preceq z$. Then $a*y \in \downarrow z$ and $a \diamond y \in \downarrow z$. Since $y \in \downarrow z$, it follows from hypothesis that $a \preceq z$, i.e., $a \in \downarrow z$. Hence $\downarrow z$ is a pseudo ideal of X for every $z \in X$. □

Proposition 3.7. *If J is a pseudo ideal of a pseudo BH-algebra X, then*
(i) $\forall x,y,z \in X, x,y \in J, z*y \preceq x \implies z \in J$.
(ii) $\forall a,b,c \in X, a,b \in J, c\diamond b \preceq a \implies c \in J$.

Proof. Suppose that J is a pseudo ideal of X and let $x,y,z \in X$ be such that $x,y \in J$ and $z*y \preceq x$. Then $(z*y)\diamond x = 0 \in J$. Since $x \in J$ and J is a pseudo ideal of X, we have $z*y \in J$. Since $y \in J$ and J is a pseudo ideal of X, we obtain $z \in J$. Thus (i) is valid.

Now let $a, b, c \in X$ be such that $a, b \in J$ and $c \diamond b \preceq a$. Then $(c \diamond b) * a = 0 \in J$ and so $c \diamond b \in J$. Since $b \in J$ and J is a pseudo ideal of X, we have $c \in J$. Thus (ii) is true. □

Theorem 3.8 *Let I be a non-empty subset of a pseudo BH^*-algebra X. Then I is a pseudo ideal of X if and only if for all $x, y \in I$ and $z \in X$, $z \diamond x \preceq y$, $z * x \preceq y$ imply $z \in I$.*

Proof. Suppose that I is a pseudo ideal of X and $z \diamond x \preceq y, z * x \preceq y$ for all $x, y \in I$ and $z \in X$. It follows from Proposition 3.4 that $z \diamond x \in I$ and $z * x \in I$. Using (PI2), we have $z \in I$.

Conversely, let $x \in I$, since $0 \diamond x \preceq x$ and $0 * x \preceq x$, we have $0 \in I$. Let $x * y, x \diamond y \in I$ and $y \in I$. Since $x \diamond y \preceq x \diamond y$ and $x * y \preceq x * y$, we have $x \in I$. Thus I is a pseudo ideal of X. □

Proposition 3.9. *For any pseudo BH^*-algebra X, the set*

$$K(X) := \{x \in X \mid 0 \preceq x\}$$

is a pseudo subalgebra of X.

Proof. Let $x, y \in K(X)$. Then $0 \preceq x$ and $0 \preceq y$. Hence $0 = 0 * y \preceq x * y$ and $0 = 0 \diamond y \preceq x \diamond y$ so that $x * y, x \diamond y \in K(X)$. Thus $K(X)$ is a pseudo subalgebra of X. □

Example 3.10. In Example 3.3 (2), $K(X) = \{0, a, b\}$ is a pseudo subalgebra of X, but not a pseudo ideal of X since $c \diamond b = 0, c * b = a$, and $b \in K(X)$, but $c \notin K(X)$.

Proposition 3.11. *Let A be a pseudo ideal of a pseudo BH-algebra X. If B is a pseudo ideal of A, then it is a pseudo ideal of X.*

Proof. Since B is a pseudo ideal of A, we have $0 \in B$. Let $y, x*y, x\diamond y \in B$ for some $x \in X$. If $x \in A$, then $x \in B$, since B is a pseudo ideal of A. If $x \in X - A$, then $y, x * y, x \diamond y \in B \subseteq A$ and so $x \in A$ because A is a pseudo ideal of X. Thus $x \in B$ since B is a pseudo ideal of A. This competes the proof. □

Definition 3.12. An element w of a pseudo BH-algebra X is called a *pseudo atom* if for every $x \in X$, $x \preceq w$ implies $x = w$.

Obviously, 0 is a pseudo atom of X.

Proposition 3.13. *Let X be a pseudo BH-algebra. If an element w of X satisfies the identity $y * (y \diamond (w * x)) = w * x$ for all $x, y \in X$, then w is a pseudo atom of X.*

Proof. Let $y \in X$ be such that $y \preceq w$. Then $w = w*0 = y*(y\diamond(w*0)) = y * (y \diamond w) = y * 0 = y$. Hence w is a pseudo atom of X. □

Lemma 3.14. *A non-zero element $a \in X$ is a pseudo atom of X if $\{0, a\}$ is a pseudo ideal of X.*

Proof. Straightforward. □

Lemma 3.15. *If every non-zero element of a BH^*-algebra X is a pseudo atom, then any pseudo subalgebra of X is a pseudo ideal of X.*

Proof. Let S be a pseudo subalgebra of X and let $x, y * x, y \diamond x \in S$. Since $y * x \preceq y$ and $y \diamond x \preceq y$ for all $x, y \in X$ and y is an atom of Y, we have $y * x = y, y \diamond x = y \in S$. Thus S is a pseudo ideal of X. □

From above Lemmas we obtain the following Theorem.

Theorem 3.16. *A BH^*-algebra contains only pseudo atoms if and only if its pseudo subalgebra is a pseudo ideal.*

Definition 3.17. A non-empty subset A of a pseudo BH-algebra X is called a *pseudo strong ideal* of X if it satisfies (PI1) and

(PI3) $(x * y) \diamond z, y \in A$ imply $x * z \in A$;
(PI3′) $(x \diamond y) * z, y \in A$ imply $x \diamond z \in A$ for all $x, y, z \in X$.

Note that if X is a pseudo BH-algebra satisfying $x * y = x \diamond y$ for all $x, y \in X$, then the notions of a pseudo strong ideal and a strong ideal coincide.

Proposition 3.18. *In a pseudo BH-algebra, any pseudo strong ideal is a pseudo ideal.*

Proof. Putting $z := 0$ in (PI3) and (PI3′), we have $x * y, x \diamond y, y \in A$ imply $x \in A$. □

Proposition 3.19. *In a BH^*-algebra X, any pseudo ideal is a pseudo subalgebra.*

Proof. Let A be a pseudo ideal of X. Then $0 \in A$ and $(x * y) \diamond x = (x \diamond y) * x = 0$ for any $x, y \in X$. Then for any $x \in A$, we have $(x * y) \diamond x, (x \diamond y) * x \in A$, which implies $x * y, x \diamond y \in A$. □

Corollary 3.20. *Any pseudo strong ideal of BH^*-algebra is a pseudo subalgebra.*

Example 3.21. In Example 3.3 (2), $J := \{0, a, c\}$ is a pseudo ideal of X but not a pseudo strong ideal of X, since $(b * a) \diamond c = b \diamond c = 0$, and $(b \diamond c) * a = 0 * a = 0, a \in J$, but $b * c = b, b \diamond a = b \notin J$.

We provide conditions for a pseudo subalgebra to be a pseudo strong ideal.

Proposition 3.22. *Let X be a pseudo BH-algebra. Then a pseudo subalgebra of X is a pseudo strong ideal of X if and only if $\forall x, y, z \in X, x \in A, y * z, y \diamond z \in X - A$ imply $(y * x) \diamond z, (y \diamond x) * z \in X - A$.*

Proof. Assume that a pseudo subalgebra A of X is a pseudo strong ideal of X and let $x, y, z \in X$ be such that $x \in A$ and $y * z, y \diamond z \in X - A$. If $(y * x) \diamond z \notin X - A$, then $(y * x) \diamond z \in A$. Since A is a pseudo strong ideal of X and $x \in A$, we have $y * z \in J$. This is a contradiction. If $(y \diamond x) * z \notin X - A$, then $(y \diamond x) * z \in A$. Since A is a pseudo strong ideal of X and $x \in A$, we have $y \diamond z \in A$. This is a contradiction.

Conversely, assume that $\forall x, y, z \in X, x \in A, y * z, y \diamond z \in A$ imply $(y * x) \diamond z, (y \diamond x) * z \in X - A$. Since A is a pseudo subalgebra of X, we have $0 \in A$. For every $x \in A$, let $(y * x) \diamond z, (y \diamond x) * z \in A$. If $y * z \notin A$ or $y \diamond z \notin A$, then $(y * x) \diamond z$ or $(y \diamond x) * z \in X - A$ by assumption. This is a contradiction. Hence $y * z \in A$ and $y \diamond z \in A$. Thus A is a pseudo strong ideal of X. □

Putting $z := 0$ in Proposition 3.22, we have the following Corollary.

Corollary 3.23. *Let A be a pseudo subalgebra of a pseudo BH-algebra X. Then A is a pseudo ideal of X if and only if $\forall x, y \in X, x \in A, y \in X - A$ imply $y * x, y \diamond x \in X - A$.*

Definition 3.24. Let X and Y be a pseudo BH-algebras. A mapping $f : X \to Y$ is called a *homomorphism* of pseudo BH-algebras if $f(x*y) = f(x) * f(y)$ and $f(x \diamond y) = f(x) \diamond f(y)$ for all $x, y \in X$.

Note that if $f : X \to Y$ is a homomorphism of pseudo BH-algebras, then $f(0_X) = 0_Y$ where 0_X and 0_Y are zero elements of X and Y, respectively.

Theorem 3.25. *Let $f : X \to Y$ be a homomorphism of pseudo BH-algebras. If B is a pseudo strong ideal of Y, then $f^{-1}(B)$ is a pseudo strong ideal of X.*

Proof. Assume that B is a pseudo strong ideal of Y. Obviously, $0_X \in f^{-1}(B)$. Let $x, y, z \in X$ be such that $(x * y) \diamond z, (x \diamond y) * z, y \in f^{-1}(B)$. Then $(f(x) * f(y)) \diamond f(z) = f((x * y) \diamond z), f(y) \in B$. Sine B is a pseudo strong ideal of Y, it follows from (PI3) and (PI3′) that $f(x * z) = f(x) * f(z), f(x \diamond z) = f(x) \diamond f(z) \in B$ so that $x * z, x \diamond z \in f^{-1}(B)$. Hence $f^{-1}(B)$ is a pseudo strong ideal of X. □

Theorem 3.26. *Let $f : X \to Y$ be a homomorphism of pseudo BH-algebras.*
(i) If B is a pseudo ideal of Y, then $f^{-1}(B)$ is a pseudo ideal of X.

(ii) If f is surjective and I is a pseudo ideal of X, then $f(I)$ is a pseudo ideal of Y.

Proof. (i) Straightforward.
(ii) Assume that f is surjective and let I be a pseudo ideal of X. Obviously, $0_Y \in f(I)$. For every $y \in f(I)$, let $a, b \in Y$ be such that $a * y \in f(I), b \diamond y \in f(I)$. Then there exist $x_*, x_\diamond \in I$ such that $f(x_*) = a * y$ and $f(x_\diamond) = b \diamond y$. Since $y \in f(I)$, there exists $x_y \in I$ such that $f(x_y) = y$. Also f is surjective, there exist $x_a, x_b \in X$ such that $f(x_a) = a$ and $f(x_b) = b$. Hence $f(x_a * x_y) = f(x_a) * f(x_y) = a * y \in f(I)$ and $f(x_b \diamond x_y) = f(x_b) \diamond f(x_y) = b \diamond y \in f(I)$, which imply that $x_a * x_y \in I$ and $x_b * x_y \in I$. Since I is a pseudo ideal of X, we get $x_a, x_b \in I$ and thus $a = f(x_a), b = f(x_b) \in f(I)$. Therefore $f(I)$ is a pseudo ideal of X □

Corollary 3.27. *Let $f : X \to Y$ be a homomorphism of pseudo BH-algebras. Then $Kerf := \{x \in X | f(x) = 0\}$ is a pseudo strong ideal(ideal) of X.*

Proof. Straightforward. □

Proposition 3.28. *Let $f : (X; *_1, \diamond_1, 0) \to (Y; *_2, \diamond_2, 0)$ be a homomorphism of pseudo BH-algebras. Then $x *_1 y, y \diamond_1 x \in Kerf$ if and only if $f(x) = f(y), \forall x \in X$.*

Proof. If $x *_1 y, y \diamond_1 x \in Kerf$, then $f(x) *_2 f(y) = f(x *_1 y) = 0$ and $f(y) \diamond_2 f(x) = f(y \diamond_1 x) = 0$. Using (P3), we have $f(x) = f(y)$.

Conversely, assume that $f(x) = f(y), \forall x \in X$. Then $f(x) *_2 f(y) = f(x *_1 y) = 0$ and $f(x) \diamond_2 f(y) = f(x \diamond_1 y) = 0$. Hence $x *_1 y, y \diamond_1 x \in Kerf$. □

Proposition 3.29. *Let $f : (X; *_1, \diamond_1, 0) \to (Y; *_2, \diamond_2, 0)$ be a homomorphism of pseudo BH-algebras. If $y \in Kerf$, then $x *_1 (x *_1 y), (x_1 * y) *_1 x, x \diamond_1 (x *_1 y), (x_1 * y) \diamond_1 x, x *_1 (x \diamond_1 y), (x_1 * y) *_1 x, x \diamond_1 (x \diamond_1 y), (x_1 \diamond_1 y) \diamond_1 x \in Kerf$.*

Proof. Straightforward. □

Lemma 3.30. *Let $f : X \to Y$ be a homomorphism of pseudo BH-algebras. Then f is a monomorphism if and only if $Kerf = \{0\}$.*

Proof. Straightforward. □

The following theorems are easy to prove, and we omit the proofs.

Theorem 3.31. *Let X, Y and Z be pseudo BH-algebras, and $h : X \to Y$ be an onto homomorphism of pseudo BH-algebras and $g : X \to Z$*

be a homomorphism of pseudo BH-algebras. If $Kerh \subseteq Kerg$, then there exists a unique homomorphism of pseudo BH-algebras $f : Y \to Z$ satisfying $f \circ h = g$.

Theorem 3.32. *Let X, Y and Z be pseudo BH-algebras, and $g : X \to Z$ be a homomorphism of pseudo BH-algebras and $h : Y \to Z$ be an one-to-one homomorphism of pseudo BH-algebras. If $Img \subseteq Imh$, then there exists a unique homomorphism of pseudo BH-algebras $f : X \to Y$ satisfying $h \circ f = g$.*

Note that the standard projections from their direct product or sum of pseudo BH-algebras to their components are homomorphism of pseudo BH-algebras with kernels having the usual form.

4. Pseudo Complicated BH-algebras

Let X be a pseudo BH-algebra. For any $a, b \in X$, we denote

$$A(a, b) := \{x \in X | (x * a) \diamond b = 0\}.$$

Theorem 4.1. *If I is a pseudo ideal of a pseudo BH-algebra X, then $I = \cup_{a,b \in I} A(a, b)$.*

Proof. Let I be a pseudo ideal of a pseudo BH-algebra X. If $a \in I$, then $(a * a) \diamond 0 = 0 \diamond 0 = 0$. Hence $a \in A(a, 0)$. It follows that

$$I \subseteq \cup_{a \in I} A(a, 0) \subseteq \cup_{a,b \in I} A(a, b).$$

Let $x \in \cup_{a,b \in I} A(a, b)$. Then there exist $a, b \in I$ such that $x \in A(a, b)$ so that $(x * a) \diamond b = 0 \in I$. Since I is a pseudo ideal of X, we have $x \in I$. Thus $\cup_{a,b \in I} A(a, b) \subseteq I$, and consequently $I = \cup_{a,b \in I} A(a, b)$, □

Corollary 4.2. *If I is a pseudo ideal of a pseudo BH-algebra X, then $I = \cup_{a \in I} A(a, 0)$.*

Proof. By Theorem 4.1, we have

$$\cup_{a \in I} A(a, 0) \subseteq \cup_{a,b \in I} A(a, b) \subseteq I.$$

If $a \in I$, then $a \in A(a, 0)$, since $(a * a) \diamond 0 = 0 \diamond 0 = 0$. Hence $I \subseteq \cup_{a \in I} A(a, 0)$. This completes the proof. □

Example 4.3. Let $X := \{0, a, b, c\}$ be as in Example 3.3(2). Set $J := \{0, a, c\}$. Then J is a pseudo ideal of X and $A(a, 0) = \{x \in X | (x*a) \diamond 0 = 0\} = J$.

Theorem 4.4. *Let I be a non-empty subset of a BH-algebra X such that $0 \in I$ and $I = \cup_{a,b \in I} A(a, b)$. Then I is a pseudo ideal of X.*

Proof. Let $x*y, x\diamond y, y \in I = \cup_{a,b\in I}A(a,b)$. Since $(x*y)\diamond(x*y) = 0$, we have $x \in A(y, x*y)$ and so $x \in I$. Hence I is a pseudo ideal of X. □

Combining Theorems 4.1 and 4.4, we have the following corollary.

Corollary 4.5. *Let X be a pseudo BH-algebra. Then I is a pseudo ideal of X if and only if $I = \cup_{a,b\in I}A(a,b)$.*

Note that $A(a,b)$ is not a pseudo ideal of X in general as seen in the following example.

Example 4.6. Let $X := \{0, a, b, c\}$ be a set with the following Cayley tables:

$*$	0	a	b	c
0	0	0	0	a
a	a	0	0	a
b	b	0	0	b
c	c	a	a	0

$\diamond$	0	a	b	c
0	0	0	0	c
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

Then it is easy to show that $(X;*,0)$ and $(X;\diamond,0)$ are not BH-algebras, but $(X;*,\diamond,0)$ is a pseudo BH-algebra. Let $I := \{0, a, b\}$. Then I is not a pseudo ideal of X since $c*b = a, c\diamond b = 0 \in I$, and $a, 0 \in I$, but $c \notin I$. Also $A(a,0) = \{x \in X | (x*a)\diamond 0 = 0\} = I$.

A pseudo ideal I of a pseudo BH-algebra X is said to be *closed* if $0*x, 0\diamond x \in I$ for any $x \in I$.

Proposition 4.7. *Let X be a pseudo BH^*-algebra. Every pseudo ideal of X is closed.*

Proof. Since $(0*x)\diamond 0 = 0*x = 0$ and $(0\diamond x)*0 = 0\diamond x = 0$ for all $x \in X$, we have $0*x = 0$ and $0\diamond x = 0$. □

Proposition 4.8. *Let I be a subset of a pseudo BH-algebra X with the following conditions:*

(i) $0 \in I$,
(ii) $x*z, x\diamond z, y*z, y\diamond z \in I$ *and* $z \in I$ *imply* $x*y, x\diamond y \in I$ *for any* $x, y, z \in X$.

Then I is a pseudo subalgebra(closed ideal) of X.

Proof. Let $x, y \in I$. By (P2), we have $x = x*0 = x\diamond 0 = 0$ and $y = y*0 = y\diamond 0$. It follows from (ii) that $x*y \in I$ and $x\diamond y \in I$. Hence I is a pseudo subalgebra of X.

Assume that I satisfies (i) and (ii). We claim that I is a pseudo closed ideal of X. Let $x*y, x\diamond y, y \in I$. Since $0*0 = 0\diamond 0, y*0 = y\diamond 0,$

and $0 \in I$, it follows from (ii) that $0 * y, 0 \diamond y \in I$ which proves that I is closed. Since $x * y, x \diamond y, 0 * y, 0 \diamond y, y \in I$, by applying (ii) again, we obtain that $x = x * 0 = x \diamond 0 \in I$, so that I is a pseudo ideal of X. □

Definition 4.9. A pseudo BH^*-algebra X is said to be *pseudo complicated* if the following condition holds:

(PC) there exist, for all $a, b \in X$,
$$a \odot b \overset{notation}{=} \max\{x | x * a \preceq b\} = \max\{x | x \diamond a \preceq b\}.$$

Note that $A(a,b)$ is a non-empty, since $0, a, b \in A(a,b)$, where X is a pseudo BH^*-algebra.

Proposition 4.10. *In a pseudo complicated BH^*-algebra, the following hold:*

(i) $z \preceq x \odot y \Leftrightarrow z * x \preceq y \Leftrightarrow z \diamond x \preceq y$.
(ii) $a \preceq a \odot b$ *and* $b \preceq a \odot b$.
(iii) $a \odot 0 = a = 0 \odot a$.

Proof. Straightforward. □

Theorem 4.11. *Let A be a non-empty subset of a pseudo complicated BH^*-algebra X. If A is a pseudo ideal of X, then it satisfies the following conditions:*

(i) $(\forall x \in A)(\forall y \in X)(y \preceq x \Rightarrow y \in A)$.
(ii) $(\forall x, y \in A)(\exists z \in A)(x \preceq z, y \preceq z)$.

Proof. (i) Assume that A is a pseudo ideal of X. Let $x \in A, y \in X$ with $y \preceq x$. Then $y * x = y \diamond x = 0$. Since A is a pseudo ideal of X, $y \in A$. Thus (i) is valid.
(ii) Let $x, y \in A$. Since $(x \odot y) * x \preceq y$ and $(x \odot y) \diamond x \preceq y$ and $y \in A$, it follows from (i) that $(x \odot y) * x, (x \odot y) \diamond x \in A$. Hence $x \odot y \in A$, since $x \in A$ and A is a pseudo ideal of X. If $z := x \odot y$, then $x \preceq x \odot y$ and $y \preceq x \odot y$ by Proposition 4.10(ii). This completes the proof. □

Theorem 4.12. *Let I be a non-empty subset of a pseudo complicated BH^*-algebra. Then I is a pseudo ide al of X if and only if for all $x, y \in I$ and $z \in X$, $z \preceq x \odot y$ imply $z \in I$.*

Proof. By Theorem 3.8 and Proposition 4.10(i). □

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