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RASMUSSEN INVARIANTS OF SOME 4-STRAND PRETZEL KNOTS

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Abstract. It is known that there is an infinite family of general pretzel knots, each of which has Rasmussen s-invariant equal to the negative value of its signature invariant. For an instance, homologically σ-thin knots have this property. In contrast, we find an infinite family of 4-strand pretzel knots whose Rasmussen invariants are not equal to the negative values of signature invariants.

1. Introduction

Khovanov [7] introduced a graded homology theory for oriented knots and links, categorifying Jones polynomials. Lee [10] defined a variant of Khovanov homology and showed the existence of a spectral sequence of rational Khovanov homology converging to her rational homology. Lee also proved that her rational homology of a knot is of dimension two. Rasmussen [13] used Lee homology to define a knot invariant s that is invariant under knot concordance and additive with respect to connected sum. He showed that $s(K) = -\sigma(K)$ if K is an alternating knot, where $\sigma(K)$ denotes the signature of K.

Suzuki [14] computed Rasmussen invariants of most of 3-strand pretzel knots. Manion [11] computed rational Khovanov homologies of all non quasi-alternating 3-strand pretzel knots and links and found the Rasmussen invariants of all 3-strand pretzel knots and links.

For general pretzel knots and links, Jabuka [5] found formulas for their signatures. Since Khovanov homologically σ-thin knots have s equal to $-\sigma$, Jabuka's result gives formulas for s invariant of any quasi-alternating pretzel knot. Note that quasi-alternating pretzel knots are

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Khovanov homologically σ-thin [12] and are classified by Greene [4]. Kawamura [6] gave an estimate of Rasmussen invariant and determined s invariants of general pretzel knots with only one negatively twisted strand, in which case s are equal to $-\sigma$ and the twice values of Ozsváth–Szábo Heegaard Floer τ invariant. The authors [8] combined known crossing change formulas to compute s invariants of a family of general preztel knots with many negatively twisted strands, which are again equal to $-\sigma$ and 2τ.

Combining Jabuka's formulas for signatures and Manion's formulas for s invariants of 3-strand pretzel knots, one can easily find an infinite family of 3-strand preztel knots with $s \neq -\sigma$. In this paper, we extend this result to compute the Rasmussen invariants of infinitely many 4-strand pretzel knots that are not equal to $-\sigma$.

Theorem 1.1. *There are infinitely many 4-strand pretzel knots with* $s \neq -\sigma$*. In fact, for any odd integer* $p \geq 9$*, the Rasmussen invariant of 4-strand pretzel knot* $P(-3, -p, 5, 4)$ *is* $-p+5$*, while the negative value of its signature is* $-p+7$*.*

In order to prove this theorem, we use the long exact sequence of Khovanov homologies arisen from a link skein relation.

2. Basic definitions and backgrounds

For a finite sequence of integers, $a_1, \ldots, a_n$, let $P(a_1, \ldots, a_n)$ be the pretzel link of n strands with $|a_i|$ crossings on the i–th strand, which are right-handed if $a_i \geq 0$, or left-handed if $a_i < 0$. See Figure 1.

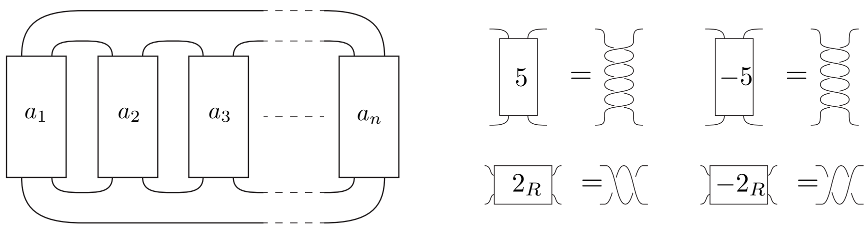


FIGURE 1. Pretzel link $P(a_1, \ldots, a_n)$ and tangle notations

A *state* of a link diagram D is a function $\mathfrak{s}$ from the set of crossings of D into the set of two numbers, $\{0, 1\}$. Given a state $\mathfrak{s}$ of D, a diagram $D_{\mathfrak{s}}$ is obtained from D by smoothing each crossing of D in either way as shown in Figure 2 corresponding to the value of $\mathfrak{s}$ at the crossing. The number of simple closed curves in $D_{\mathfrak{s}}$ is denoted by $|D_{\mathfrak{s}}|$ or simply by $|\mathfrak{s}|$. The state $\mathfrak{s}_{\mathbf{0}}$ of D is obtained from D by smoothing all the crossings in type 0 resolution. The state $\mathfrak{s}_{\mathbf{1}}$ of D is obtained in a similar way, instead, using type 1 resolution at every crossing.

FIGURE 2. Type 0 and 1 resolutions of a crossing

Let V be a graded algebra with rank 2 over the ring of integers or a field such as the field of rationals or a finite field. To each state $\mathfrak{s}$ of a diagram D, one assigns a graded algebra $\mathcal{A}(\mathfrak{s})$ that is a tensor product $V^{\otimes|\mathfrak{s}|}$ of $|\mathfrak{s}|$ copies of V. For each pair of states differing at only one crossing one can define a homomorphism between them. Combining these homomorphisms, Khovanov [7] defined a graded chain complex whose homology group is known to be *Khovanov* homology of D. He proved that his homology is a link invariant and has two gradings: homological and q gradings denoted by i and j, respectively. With respect to these gradings, the Khovanov homology of a link L is a direct sum of $KH^{i,j}(L)$, each of which is again a link invariant. It is useful to define another grading δ of Khovanov homology by $\delta = j - 2i$.

A link L is said to be *Khovanov homologically thin* over a ring R if the Khovanov homology over R are free R-modules whose nonzero elements have only two possible δ-gradings. Moreover, it is said to be *Khovanov homologically σ-thin* if it is Khovanov homologically thin with $\delta = -\sigma(L) \pm 1$. Lee [10] proved that alternating knots are Khovanov homologically σ-thin. Manolescu and Ozsváth [12] generalized this result to the quasi-alternating links. Note that the Khovanov homology of a Khovanov homologically σ-thin link is determined by its signature σ and Jones polynomial.

Lee [10] defined a variant of Khovanov homology for which the boundary maps do not preserve q gradings while those of Khovanov homology do. She then proved that the rational Khovanov homology can be considered as the E_1 term of a spectral sequence which converges to her rational homology. The rational Lee homology of a link of n components is a vector space of dimension 2^n. In particular the rational Lee homology $KH^{\mathbf{Q}}_{\mathrm{Lee}}(K)$ of a knot K is isomorphic to $\mathbf{Q} \oplus \mathbf{Q}$ generated by the homology classes of states $\mathfrak{s}_{\mathfrak{o}}$ and $\mathfrak{s}_{\bar{\mathfrak{o}}}$ that are given by the orientation $\mathfrak{o}$ of K and its reversed orientation $\bar{\mathfrak{o}}$, respectively. Both classes have homological grading $i = 0$.

Rasmussen [13] showed that the highest q grading minus the lowest q grading of nonzero elements of $KH^{\mathbf{Q}}_{\text{Lee}}(K)$ is two for a knot K. He defined $s(K)$ to be the average of the higest and the lowest q gradings of $KH^{\mathbf{Q}}_{\text{Lee}}(K)$, which is called the *Rasmussen s invariant*. Rasmussen proved that his invariant is unchaged under knot concordance and additive under connected sum and hence induces a homomophism from the knot concordance group to the group of integers. Note that Khovanov homologically σ-thin knots have $s = -\sigma$. So do quasi-alternating links. Beliakova and Wehrli [3] generalized Rasmussen invariant to links and we will mean their invariant for a link L by $s(L)$ in this paper.

The s invarinats of most 3-strand pretzel knots were computed by Suzuki [14] and those of all remaining 3-strand pretzel knots and links were computed by Manion [11]. The s invariants of general pretzel knots with only one negatively twisted strand were computed by Kawamura [6], which are turned out to be equal to $-\sigma$ and 2τ. The authors [8] also determined s invariants of an infinite family of pretzel knots other than Kawamura's examples, which are equal to $-\sigma$ and 2τ.

Jabuka [5] found formulas for the signatures of all general preztel knots. Using his formula it is easy to see that, for any odd integer $a \geq 3$, the negative value of the signature of pretzel knot $P(-a, a+2, a+1)$ is equal to 2. On the other hand, Manion's formula in [11] tells us that s invariant of $P(-a, a+2, a+1)$ is zero for any positive integer a. Thus there is an infinite family of 3-strand pretzel knots whose s invariants are not equal to $-\sigma$.

3. Facts on Khovanov homology

A crossing of a link diagram is said to be *nugatory* if there is a simple closed circle in the plane intersecting only at the crossing with the diagram. See Figure 3. A link diagram without any nugatory crossings is said to be *reduced*.

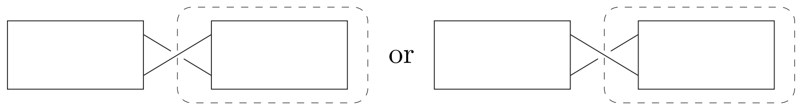


FIGURE 3. Nugatory crossings

Asaeda and Przytycki [2, Corollary 5.8] shows that if D is a nonsplit reduced k-almost alternating link diagram then D is H-(k,k)-thick. Here, a link diagram D of N crossings is said to be *H-(k_1,k_2)-thick* if $KH_{i',j'}(D)=0$ with a possible exception of i' and j' satisfying:

$$N-2|\mathfrak{s}_{\mathbf{1}}|-4k_2\le j'-2i'\le 2|\mathfrak{s}_{\mathbf{0}}|-N+4k_1,$$

where Asaeda–Przytycki's gradings i' and j' are related to gradings i and j of Khovanov in the identities: $i'=w(D)-2i$ and $j'=3w(D)-2j$, when $w(D)$ is the writhe of D, i.e., the number of positive crossings minus the number of negative crossings. So, in terms of i and j, an H-(k_1,k_2)-thick link diagram has nonzero $KH^{i,j}(D)$ only when:

$$n_+-|\mathfrak{s}_{\mathbf{0}}|-2k_1\le j-2i=\delta\le -n_-+|\mathfrak{s}_{\mathbf{1}}|+2k_2,$$

where n_+ and n_- are the numbers of positive and negative crossings in D, respectively.

Since a nonsplit reduced almost alternating diagram has the identity $|\mathfrak{s}_{\mathbf{0}}|+|\mathfrak{s}_{\mathbf{1}}|=N$, it has nonzero $KH^{i,j}(D)$ only when:

$$n_+-|\mathfrak{s}_{\mathbf{0}}|-2\le\delta\le n_+-|\mathfrak{s}_{\mathbf{0}}|+2.$$

In this case, if the diagram represents a knot, its Lee homology is generated by the states $\mathfrak{s}_{\mathfrak{o}}$ and $\mathfrak{s}_{\overline{\mathfrak{o}}}$ given by the orientation $\mathfrak{o}$ and its reversed orientation $\overline{\mathfrak{o}}$, respectively, as proved in [10]. It is easy to see that these states have the grading $i=0$. Since the Rasmussen invariant is defined to be the average of the highest and lowest j-gradings of the Lee homology, we see that the Rasmussen invariant $s(D)$ is either $n_+-|\mathfrak{s}_{\mathbf{0}}|-1$ or $n_+-|\mathfrak{s}_{\mathbf{0}}|+1$ for a nonsplit reduced almost alternating knot diagram D.

There is a long exact sequence for a triad of knot diagrams that differ only at a crossing, which we will summarize below. Khovanov [7] implicitly described this long exact sequence in his paper and Viro [16] explicitly stated it. A good reference for this is a survey paper by Turner [15].

Let D be a diagram of an oriented link L. Let D_0 and D_1 be the diagrams obtained from D by smoothing a crossing of D in type 0 and type 1 resolutions, respectively.

If the selected crossing of D is a negative crossing, then the orientation of D does not induce an orientation of D_0. We give an orientation on D_0 and let c be the number of negative crossings in D_0 minus the number of negative crossings in D. Then we have a long exact sequence:

$$\begin{aligned}\cdots\to KH^{i,j+1}(D_1)\to KH^{i,j}(D)\to KH^{i-c,j-3c-1}(D_0)\\ \to KH^{i+1,j+1}(D_1)\to KH^{i+1,j}(D)\to KH^{i+1-c,j-3c-1}(D_0)\to\cdots\end{aligned}$$

If the selected crossing of D is positive, we assign an orientation to D_1 and let c be the number of negative crossings in D_1 minus the number of negative crossings in D. We have a long exact sequence:

$$\cdots \to KH^{i-c-1,j-3c-2}(D_1) \to KH^{i,j}(D) \to KH^{i,j-1}(D_0)$$
$$\to KH^{i-c,j-3c-2}(D_1) \to KH^{i+1,j}(D) \to KH^{i+1,j-1}(D_0) \to \cdots$$

4. Proof of the theorem

We will prove Theorem 1.1 in this section. Consider a 4-strand pretzel knot $P(-a,-b,c,d)$ for any positive odd numbers a, b and c and for any positive even number d. Abe, Kishimoto and Jong [1] constructed a reduced almost alternating diagram of the pretzel knot $P(-a,-b,c,d)$ as shown in Figure 4. The diagram has $2+c+d$ positive crossings, i.e.,

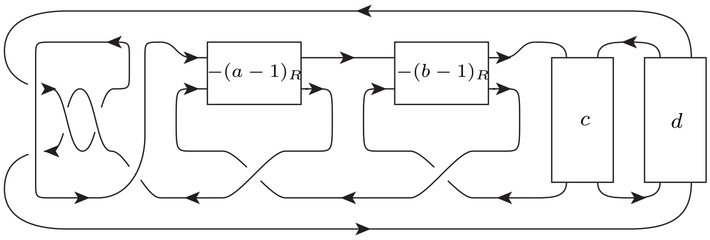


FIGURE 4. A reduced almost alternating diagram of $P(-a,-b,c,d)$

$n_+ = 2+c+d$, and the state $\mathfrak{s}_0$ has $2+a+b$ disjoint circles. Thus δ gradings of nonzero Khovanov homology of $P(-a,-b,c,d)$ are bounded by

$$-a-b+c+d-2 \leq \delta \leq -a-b+c+d+2.$$

Rasmussen [13, Proposition 5.3] showed that if the Khovanov homology of a knot K has nonzero $KH^{i,j}(K)$ only for 3 or less number of δ gradings, then the spectral sequence for the Lee homology converges after the E^4 term, and $s(K)$ can be computed as follows: Let $r^{i,j}$ be the dimension of the rational Khovanov homology $KH^{i,j}(K)$. Then the value $r = r^{0,-a-b+c+d-2} - r^{1,-a-b+c+d+2} + r^{2,-a-b+c+d+6}$ determines the Rasmussen invariant s. Namely,

$$s = \begin{cases} -a-b+c+d-1 & \text{if } r = 1 \\ -a-b+c+d+1 & \text{if } r = 0. \end{cases}$$

For an example, Figure 5 shows part of $KH^{i,j}(P(-3,-7,5,4))$, which converges to the rational Lee homology by a series of so-called knight's move cancellations. The value of r of $P(-3,-7,5,4)$ is 1 and hence $s=-2$. Here, the rational Khovanov homology is computed using the Mathematica Package KnotTheory of the Knot Atlas [9].

$j=5$							$\mathbf{Q}^4$
$j=3$					$\mathbf{Q}$	$\mathbf{Q}^4$	$\mathbf{Q}^4$
$j=1$				$\mathbf{Q}$	$\mathbf{Q}$	$\mathbf{Q}^4$	
$j=-1$			$\mathbf{Q}^2$	$\mathbf{Q}^4$	$\mathbf{Q}^4$		
$j=-3$		$\mathbf{Q}^3$	$\mathbf{Q}$	$\mathbf{Q}^2$			
$j=-5$	$\mathbf{Q}^3$	$\mathbf{Q}^3$	$\mathbf{Q}^2$				
$j=-7$	$\mathbf{Q}^3$						
	$i=-3$	-2	-1	0	1	2	3

Khovanov homology

$\longrightarrow$

$j=5$	
$j=3$	
$j=1$	
$j=-1$	$\mathbf{Q}$
$j=-3$	$\mathbf{Q}$
$j=-5$	
$j=-7$	
	$i=0$

Lee homology

FIGURE 5. The rational Khovanov and Lee homologies of $P(-3,-7,5,4)$

Now let us consider the 4-strand Pretzel knot $P(-3,-5,5,4)$. It is a slice knot since two middle strands can be cancelled out after a band move as shown in Figure 6 leaving an unknot union a 2-strand pretzel knot $P(-3,4)$ that is again an unknot. Thus, its Rasmussen invariant is zero.

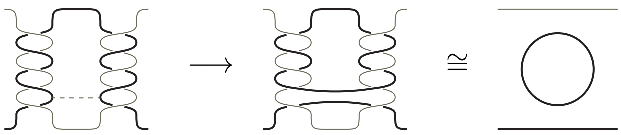
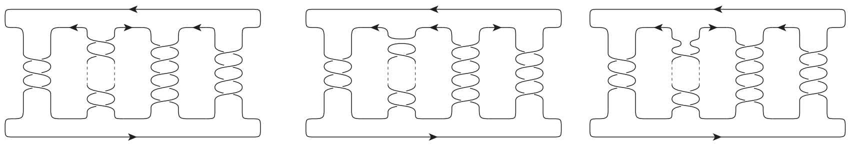


FIGURE 6. A band move cancelling two middle strands

Using this, we shall prove that $s(P(-3,-5-k,5,4))=-k$ for any integer $k\geq 0$. Here the orientation of $P_k=P(-3,-5-k,5,4)$ is given as shown in Figure 7 and its s invariant is the Beliakova–Wehrli invariant if k is odd. To do this it suffices to show that the value $r(P_k)=r^{0,-1-k}(P_k)-r^{1,3-k}(P_k)+r^{2,7-k}(P_k)$ is equal to 1.

Let D be a standard pretzel diagram of $P(-3,-5-(k+1),5,4)$ as in Figure 7 and choose a crossing in its second strand. It is easy to see that D_1 is the standard pretzel diagram of $P(-3,-5-k,5,4)$ and D_0 is a standard pretzel diagram of $P(-3,5,4)$ with $5+k$ number of negative half twists between two strands which can be unkinked by a series of



$D = P(-3, -5-(k+1), 5, 4)$ $D_0 = P(-3, 5, 4)$ $D_1 = P(-3, -5-k, 5, 4)$

FIGURE 7. Skein related pretzel links

Reidemeister move one as shown in Figure 7. Since the selected crossing is negative, we give D_0 an orientation as in Figure 7. The numbers of negative crossings in D_0 and D are 7 and $9+k$, respectively, and hence $c = -2-k$. Now we have a long exact sequence:

$$\cdots \to KH^{i+k+1, j+3k+5}(D_0) \to KH^{i,j+1}(D_1) \to KH^{i,j}(D)$$
$$\to KH^{i+k+2, j+3k+5}(D_0) \to \cdots$$

The rational Khovanov homology of $D_0 \cong P(-3, 5, 4)$ can be computed by the work of Manion [11]. Since $P(-3, 5, 4)$ is isotopic to $P(-3, 4, 5)$, we can compute the Khovanov homology of $P(-3, 4, 5)$ from Manion's formula $Kh(P(-3, 4, 5)) = q^{-13}t^{-7}L_{-3.4.5} \oplus q^{-1}t^{0}U_{-3,4,5}$ whose nonzero groups are all depicted in Figure 8.

$j=1$							$\mathbf{Q}$	$\mathbf{Q}$
$j=-1$								$\mathbf{Q}$
$j=-3$					$\mathbf{Q}$	$\mathbf{Q}$		
$j=-5$				$\mathbf{Q}$				
$j=-7$				$\mathbf{Q}$				
$j=-9$		$\mathbf{Q}$	$\mathbf{Q}$					
$j=-11$								
$j=-13$	$\mathbf{Q}$							
	$i=-7$	-6	-5	-4	-3	-2	-1	0

FIGURE 8. The rational Khovanov homology of $P(-3, 4, 5)$

In particular, observe that

$$KH^{i,j}(D_0) = 0 \text{ for } j > 1.$$

If $j+3k+5 > 1$, i.e., $j > -4-3k$, the long exact sequence becomes

$$0 \to KH^{i,j+1}(D_1) \to KH^{i,j}(D) \to 0$$

and we have isomorphisms: for any i and $j > -4-3k$,

$$KH^{i,j}(P(-3, -5-(k+1), 5, 4)) \cong KH^{i,j+1}(P(-3, -5-k, 5, 4)).$$

Since $6-k > 2-k > -2-k > -4-3k$ for any $k \geq 0$, the isomorphisms hold when $(i,j) = (0,-2-k)$, $(1, 2-k)$ and $(2, 6-k)$. Thus, for all $k \geq 0$,

$$\begin{aligned} r(P_{k+1}) &= r^{0,-2-k}(P_{k+1}) - r^{1,2-k}(P_{k+1}) + r^{2,6-k}(P_{k+1}) \\ &= r^{0,-1-k}(P_k) - r^{1,3-k}(P_k) + r^{2,7-k}(P_k) \\ &= r(P_k). \end{aligned}$$

Since $P_0 = P(-3,-5,5,4)$ is a slice knot as shown early, its Rasmussen invariant should be 0, so $r(P_0) = 1$. Now we can inductively see that $r(P_k) = 1$ for all $k \geq 0$. This proves that $s(P_k) = -k$ for any integer $k \geq 0$.

Jabuka [5, Theorem 1.18] determined the signature invariants of all pretzel knots. His formula concerning $P(-3,-5-k,5,4)$, k even, is the following: Given a pretzel knot $K = P(a_1,\ldots,a_n)$ with n even, a_n nonzero and even, and $a_1,\ldots,a_{n-1}$ odd, we have

$$-\sigma(K) = \left(\sum_{i=1}^{n} \operatorname{Sign}(a_i)\cdot(|a_i|-1)\right) + \operatorname{Sign}\left(\sum_{i=1}^{n} \frac{1}{a_i}\right).$$

For $P_k = P(-3,-5-k,5,4)$ with $k \geq 0$ even, we have

$$\begin{aligned} -\sigma(P_k) &= -2-(4+k)+4+3+\operatorname{Sign}\Big(-\tfrac{1}{3} - \tfrac{1}{5+k} + \tfrac{1}{5} + \tfrac{1}{4}\Big) \\ &= 1-k+\operatorname{Sign}(7k-25) \\ &= \begin{cases} -k & \text{if } k \leq 3 \\ -k+2 & \text{if } k \geq 4. \end{cases} \end{aligned}$$

This shows that, for any even number $k \geq 4$, $P(-3,-5-k,5,4)$ have the Rasmussen s–invariant $-k$ that is not equal to $-k+2$, the negative value of its knot signature.

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