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NEW LAPLACE TRANSFORMS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION $_2F_2$

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Abstract. This paper is in continuation of the paper very recently published [New Laplace transforms of Kummer's confluent hypergeometric functions, Math. Comp. Modelling, 55 (2012), 1068-1071]. In this paper, our main objective is to show one can obtain so far unknown Laplace transforms of three rather general cases of generalized hypergeometric function $_2F_2(x)$ by employing generalized Watson's, Dixon's and Whipple's summation theorems for the series $_3F_2$ obtained earlier in a series of three research papers by Lavoie *et al.* [5, 6, 7]. The results established in this paper may be useful in theoretical physics, engineering and mathematics.

1. Introduction

Very recently, the Kim *et al.* [4] have obtained explicit expressions of Kummer's confluent hypergeometric functions:

$$\int_0^\infty e^{-st}\ t^{b-1}\ {}_1F_1\left[\begin{matrix} a \\ \frac{1}{2}(a+b+i+1) \end{matrix} ;\ \frac{1}{2}ts\right]\ dt \tag{1.1}$$

$$\int_0^\infty e^{-st}\ t^{-a+i}\ {}_1F_1\left[\begin{matrix} a \\ c \end{matrix} ;\ \frac{1}{2}ts\right]\ dt \tag{1.2}$$

and

$$\int_0^\infty e^{-st}\ t^{b-1}\ {}_1F_1\left[\begin{matrix} a \\ 1+a-b+i \end{matrix} ;\ -ts\right]\ dt \tag{1.3}$$

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each for $i = 0, \pm 1, \ldots, \pm 5$.

By employing respectively generalized classical Gauss's second, Bailey's and Kummer's summation theorems for the series ${}_2F_1$ obtained earlier by Lavoie *et al.* [5, 6, 7].

In the present investigation, we aim to obtain three general Laplace transforms of generalized hypergeometric function ${}_2F_2\left[\begin{matrix} a, & b \\ c, & d \end{matrix}; x\right]$. For this we shall need the following generalized summation theorems obtained earlier in a series of three research papers by Lavoie *et al* [5, 6, 7].

Generalized Watson's summation theorem [5]

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+i+1), 2c+j \end{matrix}; 1\right] \\
&= \mathcal{A}_{i,j}\ \frac{2^{a+b+i-2}\ \Gamma\big(\frac{1}{2}(a+b+i+1)\big)\Gamma\big(c+[\frac{j}{2}]+\frac{1}{2}\big)\Gamma\big(c-\frac{1}{2}(a+b+|i+j|-j-1)\big)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\
&\quad\times\Bigg\{\mathcal{B}_{i,j}\ \frac{\Gamma\big(\frac{1}{2}a+\frac{1}{4}(1-(-1)^i)\big)\Gamma(\frac{b}{2})}{\Gamma\big(c-\frac{1}{2}a+\frac{1}{2}+[\frac{j}{2}]-\frac{1}{4}(-1)^j(1-c-1)^i\big)\Gamma\big(c-\frac{b}{2}+\frac{1}{2}+[\frac{j}{2}]\big)} \\
&\quad+\mathcal{C}_{i,j}\ \frac{\Gamma\big(\frac{1}{2}a+\frac{1}{4}\big(1+(-1)^i\big)\big)\Gamma\big(\frac{1}{2}b+\frac{1}{2}\big)}{\Gamma\Big(c-\frac{1}{2}a+[\frac{j+1}{2}]+\frac{(-1)^j}{4}(1-(-1)^i)\Big)\Gamma\big(c-\frac{1}{2}b+[\frac{j+1}{2}]\big)}\Bigg\}
\end{aligned}
\tag{1.4}
$$

provided $\Re e(a+b-2c) < i+2j+1$ with $i, j = 0, \pm 1, \pm 2$. The coefficients $\mathcal{A}_{i,j}$, $\mathcal{B}_{i,j}$ and $\mathcal{C}_{i,j}$ are given in the tables in [5].

Generalized Dixon's summation theorem [6]

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b+i, 1+a-c+i+j \end{matrix}; 1\right] \\
&= \frac{2^{-2c+i+j}\ \Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma\big(b-\frac{1}{2}i+\frac{1}{2}|i|\big)\Gamma\big(c-\frac{1}{2}(i+j+|i+j|)\big)}{\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)\Gamma(b)\Gamma(c)} \\
&\quad\times\Bigg\{\mathcal{D}_{i,j}\ \frac{\Gamma\big(\frac{1}{2}a-c+\frac{1}{2}+[\frac{i+j+1}{2}]\big)\Gamma\big(\frac{1}{2}a-b-c+1+i+[\frac{j+1}{2}]\big)}{\Gamma\big(\frac{1}{2}a+\frac{1}{2}\big)\Gamma\big(\frac{1}{2}a-b+1+[\frac{i}{2}]\big)} \\
&\quad+\mathcal{E}_{i,j}\ \frac{\Gamma\big(\frac{1}{2}a-c+1+[\frac{i+j}{2}]\big)\Gamma\big(\frac{1}{2}a-b-c+\frac{3}{2}+i+[\frac{j}{2}]\big)}{\Gamma\big(\frac{1}{2}a\big)\Gamma\big(\frac{1}{2}a-b+\frac{1}{2}+[\frac{i+1}{2}]\big)}\Bigg\}
\end{aligned}
\tag{1.5}
$$

provided $\Re e(a-2b-2c) > -2-2i-j$ with $i = -3, -2, -1, 0, 1, 2$; $j = 0, 1, 2, 3$.

Here, and in what follows, $[x]$ is the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients $\mathcal{D}_{i,j}$ and $\mathcal{E}_{i,j}$ are given in the tables in [6].

Also, if $f_{i,j}$ is the left hand side of (1.5), the natural symmetry

$$f_{i,j}(a,b,c) = f_{i+j,-j}(a,b,c)$$

makes it possible to extend the result to $j = -1, -2, -3$.

Generalized Whipple's summation theorem [7]

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ e,\ f \end{matrix}; 1\right] \\
&= \frac{\Gamma(e)\Gamma(f)\Gamma\big(c-\frac{1}{2}(j+|j|)\big)\Gamma\big(e-c-\frac{1}{2}(i+|i|)\big)\Gamma\big(a-\frac{1}{2}(i+j+|i+j|)\big)}{2^{2a-i-j}\Gamma(e-a)\Gamma(f-a)\Gamma(e-c)\Gamma(a)\Gamma(c)} \\
&\times \Bigg\{ \mathcal{F}_{i,j}\, \frac{\Gamma\big(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{4}(1-(-1)^i)\big)\Gamma\big(\frac{1}{2}f-\frac{1}{2}a\big)}{\Gamma\big(\frac{1}{2}e+\frac{1}{2}a-\frac{1}{2}i+[-\frac{j}{2}]\big)\Gamma\big(\frac{1}{2}f+\frac{1}{2}a-\frac{1}{2}i+\frac{1}{4}(-1)^j((-1)^i-1)+[-\frac{j}{2}]\big)} \\
&+ \mathcal{G}_{i,j} \frac{\Gamma\big(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{4}(1+(-1)^i)\big)\Gamma\big(\frac{1}{2}f-\frac{1}{2}a+\frac{1}{2}\big)}{\Gamma\big(\frac{1}{2}e+\frac{1}{2}a-\frac{1}{2}-\frac{1}{2}i+[\frac{-j+1}{2}]\big)\Gamma\Big(\frac{f+a-1-i}{2}+\frac{(-1)^j(1-(-1)^i)}{4}+[\frac{-j+1}{2}]\Big)} \Bigg\}
\end{aligned} \tag{1.6}$$

for $i, j = 0, \pm 1, \pm 2, \pm 3$ with $a + b = 1 + i + j$ and $e + f = 1 + 2c + i$. The coefficients $\mathcal{F}_{i,j}$ and $\mathcal{G}_{i,j}$ are given in the tables in [7].

For $i = j = 0$, the results (1.4), (1.5) and (1.6) respectively reduce to the following classical Watson's, Dixon's and Whipple's summation theorems.

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(a+b+1),\ 2c \end{matrix}; 1\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}
\end{aligned} \tag{1.7}$$

provided $\Re e(2c - a - b) > -1$,

$$\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix}; 1\right] \\
&= \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)}
\end{aligned} \tag{1.8}$$

provided $\Re e(a-2b-2c) > -2$ and

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ b,\ c \\ e,\ f \end{matrix} ; 1\right] \\
(1.9)\qquad &= \frac{2^{1-2c}\ \pi\ \Gamma(e)\Gamma(f)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}f\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}e\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}f\right)}
\end{aligned}
$$

with $a+b=1$ and $e+f=1+2c$ provided $\Re e(e+f-a-b-c) > 0$.

2. Three general Laplace transforms of ${}_2F_2\left[a, b\ ;\ c, d\ ;\ x\right]$

The generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters is defined by (see, for example [10, Chapter 4]; see also [12, pp. 71-72])

$$
\begin{aligned}
(2.1)\qquad {}_pF_q\left[\begin{matrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{matrix} ; z\right] &= {}_pF_q\left[\alpha_1, \ldots, \alpha_p\ ;\ \beta_1, \ldots \beta_q\ ;\ z\right] \\
&= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},
\end{aligned}
$$

$$
\Big(p \leq q \text{ and } |z| < \infty\ ;\ p = q+1 \text{ and } |z| < 1\ ;
$$

$$
p = q+1,\ |z| = 1 \text{ and } \Re e(w) > 0\Big)
$$

where

$$
w := \sum_{j=1}^{q} b_i - \sum_{j=1}^{p} a_i
$$

and $(\alpha)_n$ denotes the Pochhammer symbol defined in terms of the Gamma function by

$$
(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & (n \in \mathbb{N};\ \alpha \in \mathbb{C}) \\ 1 & (n = 0;\ \alpha \in \mathbb{C}\backslash\{0\}). \end{cases}
$$

On the other hand, we define the Laplace transform of a function $f(t)$ of a real variable t as the integral $g(s)$ over a range of the complex parameters s, by the integral

$$
(2.4)\qquad g(s) = L\{f(t); s\} = \int_0^{\infty} e^{-st}\ f(t)\ dt
$$

provided this integral exists in the Lebesgue sense. For more details about the Laplace transforms, we refer [1] or [2].

Now, keeping in mind, the following well-known result

$$\int_0^\infty e^{-st}\, t^{\alpha-1}\, dt = \Gamma(\alpha)\, s^{-\alpha} \tag{2.5}$$

provided $\Re e(s) > 0$ and $\Re e(\alpha) > 0$, for (2.1) with $p \le q$, we have the following Laplace transform of a generalized function ${}_pF_q$ [3, section 9.23(17)]:

$$\begin{aligned}\int_0^\infty e^{-st}\, t^{v-1}\, {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots b_q \end{matrix} ;\ wt\right] dt \\ = \Gamma(v)\, s^{-v}\, {}_{p+1}F_q\left[\begin{matrix} v, a_1, \dots, a_p \\ b_1, \dots b_q \end{matrix} ;\ \frac{w}{s}\right]\end{aligned} \tag{2.6}$$

provided (i) if $p < q$, $\Re e(v) > 0$, $\Re e(s) > 0$ and w is arbitrary or (ii) if $p = q > 0$, $\Re e(v) > 0$ and $\Re e(s) > \Re e(w)$.

Further, it is not out of place to mention here that the interchanging the order of summation and integration (in the proof of (2.6)) is easily seen to be justified due to the uniform convergence of the series (2.1). In particular, where $p = q = 2$, for generalized hypergeometric function ${}_2F_2$, we conclude that its Laplace transform

$$\begin{aligned}\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ d,\ e \end{matrix} ;\ wt\right] dt \\ = \Gamma(c)\, s^{-c}\, {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix} ;\ \frac{w}{s}\right]\end{aligned} \tag{2.7}$$

is given in terms of generalized hypergeometric function ${}_3F_2$.

Clearly (2.7) is the most general case. Here, our aim is to given three new and very general transforms of ${}_2F_2$, they are not listed in any of the standard tables of Laplace transforms (Erdelyi *et al.* [3], Oberhettinger and Bady [8] and Prudnikov *et al.* [9]) and we have failed to find them in the literature. For this, in (2.7), if we set $w = s$ and either and $d = \frac{1}{2}(a+b+i+1)$, $e = 2c + j$ for $i, j = 0, \pm 1, \pm 2$ or $d = 1 + a - b + i$ and $e = 1 + a - c + i + j$ for $i, j = 0, \pm 1, \pm 2, \pm 3$ or $a + b = 1 + i + j$ and $d + e = 2c + 1 + i$ for $i, j = 0, \pm 1, \pm 2, \pm 3$ then the resulting series ${}_3F_2(1)$, appearing on the right-hand side of (2.7), can be summed by using the summation formulas, we have the following three general results.

$$\begin{aligned}
(2.8)\quad &\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix}\,;\ st\right]\, dt \\
&= \mathcal{A}_{i,j}\ \frac{s^{-c}\ 2^{a+b+i-2}\ \Gamma(c)\Gamma\left(\frac{a+b+i+1}{2}\right)\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(c-\frac{(a+b+|i+j|-j-1)}{2}\right)}{\Gamma(\frac{1}{2})\Gamma(a)\Gamma(b)} \\
&\times\left\{\mathcal{B}_{i,j}\ \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1-(-1)^i\right)\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{1}{4}(-1)^j(1-(-1)^j)\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}+\left[\frac{j}{2}\right]\right)}\right. \\
&\left.+\ \mathcal{C}_{i,j}\ \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{1}{2}b+\left[\frac{j+1}{2}\right]\right)}\right\}
\end{aligned}$$

for $i, j = 0, \pm1, \pm2$, provided $\Re e(c) > 0$ and $\Re e(s) > 0$ and

$$\begin{aligned}
(2.9)\quad &\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ 1+a-b+i,\ 1+a-c+i+j\end{matrix}\,;\ st\right]\, dt \\
&= \frac{s^{-c}\ 2^{-2c+i+j}\Gamma(c)\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\ \Gamma\left(b-\frac{(i+|i|)}{2}\right)\Gamma\left(c-\frac{i+j+|i+j|}{2}\right)}{\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)\ \Gamma(b)} \\
&\times\left\{\mathcal{D}_{i,j}\ \frac{\Gamma\left(\frac{1}{2}a-c+\frac{1}{2}+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(\frac{1}{2}a-b-c+1+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+1+\left[\frac{i}{2}\right]\right)}\right. \\
&\left.+\ \mathcal{E}_{i,j}\ \frac{\Gamma\left(\frac{1}{2}a-c+1+\left[\frac{i+j}{2}\right]\right)\ \Gamma\left(\frac{1}{2}a-b-c+\frac{3}{2}+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{1}{2}a\right)\ \Gamma\left(\frac{1}{2}a-b+\frac{1}{2}+\left[\frac{i+1}{2}\right]\right)}\right\}
\end{aligned}$$

provided $\Re e(a-2b-2c) > -2-2i-j$ with $i = 0, \pm1, \pm2 \pm 3$ and $j = 0, 1, 2, 3$. The coefficients $\mathcal{D}_{i,j}$ and $\mathcal{E}_{i,j}$ are given in the tables in [6].

And

$$\begin{aligned}
(2.10)\quad &\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ d,\ e\end{matrix}\,;\ st\right]\, dt \\
&= \frac{s^{-c}\ \Gamma(c)\Gamma(d)\Gamma(e)\Gamma\left(c-\frac{1}{2}(j+|j|)\right)\ \Gamma\left(d-c-\frac{1}{2}(i+|i|)\right)\ \Gamma\left(a-\frac{1}{2}(i+j+|i+j|)\right)}{2^{2a-i-j}\ \Gamma(d-a)\Gamma(e-a)\Gamma(d-c)\Gamma(a)\Gamma(c)} \\
&\times\left\{\mathcal{F}_{i,j}\ \frac{\Gamma\left(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{4}\left(1-(-1)^i\right)\right)\ \Gamma\left(\frac{1}{2}e-\frac{1}{2}a\right)}{\Gamma\left(\frac{1}{2}d+\frac{1}{2}a-\frac{1}{2}i+\left[-\frac{j}{2}\right]\right)\Gamma\left(\frac{1}{2}e+\frac{1}{2}a-\frac{1}{2}i+\frac{1}{4}(-1)^j((-1)^i-1)+\left[\frac{-j}{2}\right]\right)}\right. \\
&\left.+\ \mathcal{G}_{i,j}\ \frac{\Gamma\left(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)\ \Gamma\left(\frac{1}{2}e-\frac{1}{2}a+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}d+\frac{1}{2}a-\frac{1}{2}-\frac{1}{2}i+\left[\frac{-j+1}{2}\right]\right)\Gamma\left(\frac{e-a-1-i}{2}+\frac{(-1)^j(1-(-1)^i)}{4}+\left[\frac{-j+1}{2}\right]\right)}\right\}
\end{aligned}$$

with $a+b = 1+i+j$, $d+e = 2c+1+i$ for $i, j = 0, \pm1, \pm2, \pm3$. The coefficients $\mathcal{A}_{i,j}$, $\mathcal{B}_{i,j}$, $\mathcal{C}_{i,j}$ and $\mathcal{D}_{i,j}$ are the same as those given in the tables in [5, 6] while $\mathcal{E}_{i,j}$ and $\mathcal{F}_{i,j}$ in (2.10) can be obtained from the tables of $\mathcal{E}_{i,j}$ and $\mathcal{F}_{i,j}$ by simple changing e to d and f to e respectively.

Moreover, in (2.8), (2.9) and (2.10) if we take $i = j = 0$, we respectively get

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ \frac{1}{2}(a+b+1),\ 2c \end{matrix};\ st\right] dt = \frac{s^{-c}\ \Gamma\left(\frac{1}{2}\right)\ \Gamma(c)\ \Gamma\left(c+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\ \Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} \tag{2.11}$$

provided $\Re e(2c - a - b) > -1$,

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ 1+a-b,\ 1+a-c \end{matrix};\ st\right] dt = \frac{s^{-c}\ \Gamma(c)\ \Gamma\left(1+\frac{1}{2}a\right)\ \Gamma(1+a-b)\ \Gamma(1+a-c)\ \Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\ \Gamma\left(1+\frac{1}{2}a-b\right)\ \Gamma\left(1+\frac{1}{2}a-c\right)\ \Gamma(1+a-b-c)} \tag{2.12}$$

provided $\Re e(a - 2b - 2c) > -2$ and

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_2F_2\left[\begin{matrix} a,\ b \\ d,\ e \end{matrix};\ st\right] dt = \frac{s^{-c}\ \pi\ \Gamma(c)\ \Gamma(d)\ \Gamma(e)}{2^{2c-1}\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}d\right)\ \Gamma\left(\frac{1}{2}a+\frac{1}{2}e\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}d\right)\ \Gamma\left(\frac{1}{2}b+\frac{1}{2}e\right)} \tag{2.13}$$

provided $\Re e(c) > 0$ and $\Re e(d + e - a - b - c) > 0$ with $a + b = 1$ and $d + e = 2c + 1$.

Similarly other elementary results can also be obtained from our main three general results.

Concluding remarks

In this research paper, we have obtained Laplace transforms of three rather general cases of generalized hypergeometric functions ${}_2F_2(x)$ by employing generalized Watson's, Dixon's and Whipple's summation theorems available in the literature. From these three general cases, on specializing the values to i and j, one can obtain more than 85 results. These results may be useful in theoretical physics, engineering and mathematics.

References

[1] B. Davies, *Integral transforms and their applications*, third ed. Springer, New york, 2002.

[2] G. Doetsch, *Introduction to the theory and applications of Laplace transformation*, Springer, New york, 1974.

[3] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Tables of Integral transforms*, Vol. I, II, McGraw Hill, New york, 1954.
[4] Y.S. Kim, A.K. Rathie and D. Cvijovic, *New Laplace transforms of Kummer's confluent hypergeometric functions*, Math. Comp. Modelling **55** (2012), 1068-1071.
[5] J.L. Lavoie, F. Grondin and A.K. Rathie, *Generalizations of Watson's theorem on the sum of a* ${}_3F_2$, Indian J. Math. **34** (1992), 23-32.
[6] J.L. Lavoie, F. Grondin, A.K. Rathie and K. Arora, *Generalizations of Dixon's theorem on the sum of a* ${}_3F_2$, Math. Comp. **62** (1994), 267-276.
[7] J.L. Lavoie, F. Grondin and A.K. Rathie, *Generalizations of Whipple's theorem on the sum of a* ${}_3F_2$, J. Comp. **72** (1996), 293-300.
[8] F. Oberhettinger and L. Badi, *Tables of Laplace transforms*, Springer, Berlin, 1973.
[9] A.P. Prudnikov, Yu.A. Brychkov and O. I. Marichev, *Integrals and series: Direct Laplace transforms*, Vol. 4, Gordon and Breach science publishers, New York, 1992.
[10] E.D. Rainville, *Special functions*, Macmillan, New York, 1960.
[11] L.J. Slater, *Generalized hypergeometric functions*, Cambridge University press, Cambridge, 1966.
[12] H.M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated series and Integrals*, Elsvier Science Publishers, Amsterdam, London and New York, (2012).

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