Honam Mathematical J. **37** (2015), No. 2, pp. 265–267
http://dx.doi.org/10.5831/HMJ.2015.37.2.265

POWERS OF THE AUGMENTATION IDEAL OF A CYCLIC GROUP RING

Joongul Lee

Abstract. Let G be a cyclic group of prime power order. There is a natural embedding of $\mathbb{Z}[G]$ into a product of rings of integers of cyclotomic fields. In this paper we determine the image of powers of the augmentation ideal.

Fix a prime p and a positive integer k. Let G be a cyclic group of order p^k with generator σ. We denote the integral group ring of G by $\mathbb{Z}[G]$, and its augmentation ideal by I_G. Let H be the subgroup of G of order p. Let us choose a complex character χ_k of G with order p^k, and for $0 \leq i \leq k-1$ we inductively define $\chi_i = \chi_{i+1}^p$ so that the order of χ_i is p^i. We may view χ_i as complex characters of G/H for $i < k$. We also set $\zeta_i = \chi_i(\sigma)$ and $\lambda_i = \zeta_i - 1$, and extend χ_i to a ring homomorphism from $\mathbb{Z}[G]$ to $\mathbb{Z}[\zeta_i]$ by linearity. We note that λ_i generates the unique maximal ideal of $\mathbb{Z}[\zeta_i]$ over p which is totally ramified. (cf. [1])

Consider

$$\Phi : \mathbb{Z}[G] \longrightarrow \prod_{i=0}^{k} \mathbb{Z}[\zeta_i]$$
$$\Phi(\alpha) = (\chi_0(\alpha), \ldots, \chi_k(\alpha)).$$

Φ is an injective ring homomorphism, and the image of Φ is determined in [2]. The goal of this paper is to determine $\Phi(I_G^n)$ for a positive integer n. We start with a simple observation.

Proposition 1. *The following statements are equivalent.*

1. $\alpha \in I_G^n$.
2. $\alpha = \beta(\sigma - 1)^n$ *for some* $\beta \in \mathbb{Z}[G]$.
3. $\Phi(\alpha) = (0, \beta_1\lambda_1^n, \ldots, \beta_k\lambda_k^n)$ *for some* $(\beta_0, \ldots, \beta_k) \in \operatorname{Im}\Phi$.

Received March 11, 2015. Accepted April 23, 2015.
2010 Mathematics Subject Classification. 16S34, 20C05.
Key words and phrases. Group rings.
This work was supported by 2014 Hongik University Research Fund.

Proof. It is self-evident once we observe that

$$\Phi(\sigma - 1) = (0, \lambda_1, \lambda_2, \ldots, \lambda_k).$$

□

Therefore $(0, \beta_1\lambda_1^n, \ldots, \beta_k\lambda_k^n)$ is in $\Phi(I_G^n)$ if and only if we can find $\beta \in \mathbb{Z}[G]$ satisfying $\chi_i(\beta) = \beta_i$ for $1 \leq i \leq k$.

Let $\phi_i(x)$ be the p^i-th cyclotomic polynomial for $0 \leq i \leq k$, and consider the following elements in $\mathbb{Z}[G]$

$$\eta_i = \phi_i(\sigma) = \begin{cases} \sigma - 1 & \text{if } i = 0 \\ \sum_{j=0}^{p-1} \sigma^{jp^{i-1}} & \text{otherwise,} \end{cases}$$

and

$$\theta_i = \prod_{j=1}^{i} \eta_j = \sum_{j=0}^{p^i-1} \sigma^j.$$

Proposition 2.

(a) *Each* $\chi_i : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[\zeta_i]$ *is surjective.*
(b) $\operatorname{Ker}\chi_i$ *is the principal ideal of* $\mathbb{Z}[G]$ *generated by* η_i*.*
(c) $\cap_{j=1}^{i} \operatorname{Ker}\chi_j$ *is the principal ideal of* $\mathbb{Z}[G]$ *generated by* θ_i*.*

Proof. (a) is clear. For (b) and (c), note that there is a natural isomorphism from $\mathbb{Z}[x]/(x^{p^k} - 1)$ to $\mathbb{Z}[G]$ that sends x to σ, and from $\mathbb{Z}[x]/(\phi_i(x))$ to $\mathbb{Z}[\zeta_i]$ sending x to ζ_i, therefore we may regard the occuring maps as natural projections among quotients of polynomial rings with integer coefficients. The result follows from the fact that $\mathbb{Z}[x]$ is a unique factorization ring and that $\phi_i(x)$ is irreducible in $\mathbb{Z}[x]$. □

We observe that

$$\chi_{i+1}(\theta_i) = \sum_{j=0}^{p^i-1} \zeta_{i+1}^j = \lambda_1/\lambda_{i+1}$$

which generates the same ideal of $\mathbb{Z}[\zeta_{i+1}]$ as $\lambda_{i+1}^{p^i-1}$.

Given $\beta_i \in \mathbb{Z}[\zeta_i]$ for $1 \leq i \leq k$, let us find an equivalence condition for the existence of an element $\beta \in \mathbb{Z}[G]$ satisfying $\chi_i(\beta) = \beta_i$ for $1 \leq i \leq k$. First assume there exists such β. If γ is another elements of $\mathbb{Z}[G]$ such that $\chi_j(\gamma) = \beta_j$ for $1 \leq j \leq i$, then $\beta - \gamma$ belongs to the principal ideal of $\mathbb{Z}[G]$ generated by θ_i as proved in Proposition 2(c), and therefore

$$\chi_{i+1}(\gamma) \equiv \beta_{i+1} \pmod{\lambda_{i+1}^{p^i-1}}.$$

Conversely, if there exists an element γ such that $\chi_j(\gamma) = \beta_j$ for $1 \leq j \leq i$ and $\chi_{i+1}(\gamma) \equiv \beta_{i+1} \pmod{\lambda_{i+1}^{p^i-1}}$, then we may write

$$\chi_{i+1}(\gamma) = \beta_{i+1} + c\lambda_1/\lambda_{i+1}$$

for some $c \in \mathbb{Z}[\zeta_{i+1}]$. Choose $\xi \in \mathbb{Z}[G]$ such that $\chi_{i+1}(\xi) = c$, then the element $\beta = \gamma - \xi\theta_i$ would satisfy $\chi_j(\beta) = \beta_j$ for $1 \leq j \leq i+1$. If we can inductively perform this step up to $i = k-1$, then we would have found the desired element β. We summarize the above discussion in the following theorem.

Theorem 3. *Let $\alpha = (\alpha_0, \ldots, \alpha_k)$ be an element of $\prod_{i=0}^{k} \mathbb{Z}[\zeta_i]$. α is in $\Phi(I_G^n)$ if and only if the following conditions hold;*

(i) $\alpha_0 = 0$, *and* $\alpha_i = \beta_i \lambda_i^n$ *for some* $\beta_i \in \mathbb{Z}[\zeta_i]$ $(1 \leq i \leq k)$,

(ii) *For each i, if we inductively find $\gamma \in \mathbb{Z}[G]$ such that $\chi_j(\gamma) = \beta_j$ for $1 \leq j \leq i$, then $\chi_{i+1}(\gamma) \equiv \beta_{i+1} \pmod{\lambda_{i+1}^{p^i-1}}$.*

Remark. The method we adopted in this paper is only slightly different from that of [2]. Here we utilize the fact that $\alpha_0 = 0$ to obtain a more prcise congruence relation. Also we work with concrete elements in $\mathbb{Z}[G]$ which hopefully explains better where the congruence relation is coming from.

References

[1] S. Lang, *Algebraic Number Theory*, Second edition, Springer-Verlag, New York, 1994.

[2] J. Lee, *Characterization of a cyclic group ring in terms of character values*, Commun. Korean Math. Soc. **30(1)** (2015), 1-5.

Joongul Lee
Department of Mathematics Education, Hongik University,
Seoul 121-791, Korea.
E-mail: jglee@hongik.ac.kr