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DIRICHLET BOUNDARY VALUE PROBLEM FOR A CLASS OF THE NONCOOPERATIVE ELLIPTIC SYSTEM

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Abstract. This paper is devoted to investigate the existence of the solutions for a class of the noncooperative elliptic system involving critical Sobolev exponents. We show the existence of the negative solution for the problem. We show the existence of the unique negative solution for the system of the linear part of the problem under some conditions, which is also the negative solution of the nonlinear problem. We also consider the eigenvalue problem of the matrix.

1. Introduction

Let Ω be a bounded domain of R^n with smooth boundary $\partial\Omega$, $n \geq 3$, α, β, γ be real constants and ϕ_i, $i \geq 1$, be the normalized eigenfunctions associated to eigenvalues λ_i of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u|_{\partial\Omega} = 0$. We note that ϕ_1 is the positive normalized eigenfunction associated to λ_1. In this paper we investigate the existence of the solutions for the following class of the systems of the noncooperative elliptic

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equations with Dirichlet boundary condition

$$\begin{cases} -\Delta u &= \alpha u + \beta v + \frac{2p}{p+q} u_+^{p-1} v_+^q + \bar{f} \qquad \text{in } \Omega, \\ \Delta v &= \beta u + \gamma v + \frac{2q}{p+q} u_+^p v_+^{q-1} + \bar{g} \qquad \text{in } \Omega, \\ \quad u &= v = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u_+ = \max\{u, 0\}$, p, $q > 1$ are real constants, $p + q = 2^*$, $2^* = \frac{2n}{n-2}$, $n \geq 3$ and we may write $\bar{f}$, $\bar{g}$ as

$$\bar{f} = \tau\phi_1 + f, \qquad \bar{g} = t\phi_1 + g,$$

where τ, t are real constants and f, $g \in L^2(\Omega)$ with

$$\int_\Omega f\phi_1 = \int_\Omega g\phi_1 = 0.$$

Our problems are characterized as Ambrosetti-Prodi type problems. Since the pioneering work on the subject in [1], these problem have been investigated in many ways. For a survey on the scalar case we recommend the paper [3] and the references therein. In the last decades, some works on the matter were published focusing some other obstacles added to this kind of nonlinearities problems having critical growth and the case involving systems. Ambrosetti-Prodi type problems for the critical growth case were studied in [4] and in [5]. For systems, these problems were firstly investigated in [7], and lately, in [2].

Let $H_0^1(\Omega)$ be the Sobolev space and $H = H_0^1(\Omega) \times H_0^1(\Omega)$. Then H is a Hilbert space endowed with the norm

$$\|(u, v)\|_H^2 = \|u\|^2_{H_0^1(\Omega)} + \|v\|^2_{H_0^1(\Omega)}.$$

We are looking for the weak solutions of (1.1) in H. The weak solutions in H satisfy

$$\begin{aligned} \int_\Omega [(-\Delta u, \Delta v) \cdot (z, w) - (\alpha u + \beta v, \beta u + \gamma v) \cdot (z, w) \\ -(\tau\phi_1 + f, t\phi_1 + g) \cdot (z, w)]dx = 0 \quad \forall (z, w) \in H. \end{aligned} \tag{1.2}$$

We observe that the weak solutions of (1.1) coincide with the critical points of the the associated functional

$$J : H \to R \in C^{1.1},$$

$$J(u, v) = Q_{\alpha\beta\gamma}(u, v) - \int_\Omega [\frac{2}{p+q} u_+^p v_+^q + \tau\phi_1 u + fu + t\phi_1 v + gv]dx, \tag{1.3}$$

where

$$Q_{\alpha\beta\gamma}(u,v) = \frac{1}{2}\int_\Omega [|\nabla u|^2 - |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx.$$

Let B be $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in M_{2\times 2}(R)$ and $\mu^+_{\lambda_i}$ and $\mu^-_{\lambda_i}$ be the eigenvalues of the matrix
$\begin{pmatrix} \lambda_i - \alpha & -\beta \\ -\beta & -\lambda_i - \gamma \end{pmatrix} \in M_{2\times 2}(R)$, i. e.,

$$\mu^+_{\lambda_i} = \frac{1}{2}\{-\gamma - \alpha + \sqrt{(-\gamma-\alpha)^2 - 4\{(\lambda_i - \alpha)(-\lambda_i - \gamma) - \beta^2\}}\},$$

$$\mu^-_{\lambda_i} = \frac{1}{2}\{-\gamma - \alpha - \sqrt{((-\gamma-\alpha))^2 - 4\{(\lambda_i - \alpha)(-\lambda_i - \gamma) - \beta^2\}}\}.$$

We note that if $-\gamma = \alpha$, then $\mu^+_{\lambda_i} = \mu^-_{\lambda_i} = 0$,

$$\text{if } \; -\gamma > \alpha \text{ and } 4\{(\lambda_1 - \alpha)(-\lambda_1 - \gamma) - \beta^2\} > 0, \text{ then } 0 < \mu^-_{\lambda_1} < \mu^+_{\lambda_1},$$

$$\text{if } \; -\gamma < \alpha \text{ and } 4\{(\lambda_1 - \alpha)(-\lambda_1 - \gamma) - \beta^2\} > 0, \text{ then } \mu^-_{\lambda_1} < \mu^+_{\lambda_1} < 0,$$

and

$$\text{if } \; 4\{(\lambda_i - \alpha)(-\lambda_i - \gamma) - \beta^2\} < 0, \; i \geq 2, \text{ then } \mu^-_{\lambda_i} < 0 < \mu^+_{\lambda_i}.$$

In this paper we show the existence of the unique negative solution for the system of the linear part of the problem under some conditions, which is also the negative solution of (1.1). We also consider some property of the eigenvalue problem of the matrix.

The outline of the proofs of main results are as follows: In section 2, we show the existence of the unique negative solution of the linear part of (1.1), which is also a negative solution under the conditions ($B1$)-($B3$) and some condition for τ and $t$so. In section 3, we obtain some result for the eigenvalue problem of the matrix.

2. A negative solution

LEMMA 2.1. *Assume that the conditions* ($B1$), ($B2$) *and* ($B3$) *hold. Let* $N_{\alpha\beta\gamma} : H \to H$ *be the operator defined by* $N_{\alpha\beta\gamma}(u,v) = (-\Delta u - \alpha u - \beta v, \Delta v - \beta u - \gamma v)$. *Then the operator*

$$N^{-1}_{\alpha\beta\gamma} : H \to H$$

is well defined and continuous, and the system

$$\begin{cases} -\Delta u &= \alpha u + \beta v + f \quad \text{in } \Omega, \\ \Delta v &= \beta u + \gamma v + g, \quad \text{in } \Omega, \\ u &= v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

has a unique solution (u_{fg}, v_{fg}), *which is of the form*

$$u_{fg} = \sum_m \left(\frac{(-\lambda_m - \gamma) f_m + \beta g_m}{(\lambda_m - \alpha)(-\lambda_m - \gamma) - \beta^2}\right)\phi_m,$$

$$v_{fg} = \sum_m \left(\frac{(\lambda_m - \alpha) g_m + \beta f_m}{(\lambda_m - \alpha)(-\lambda_m - gamma) - \beta^2}\right)\phi_m,$$

where $f = \sum_m f_m \phi_m$ *with* $\sum_m f_m^2 < +\infty$ *and* $g = \sum_m g_m \phi_m$ *with* $\sum_m g_m^2 < +\infty$.

Proof. let us take (f, g) in H. We can write $f = \sum_m f_m \phi_m$ with $\sum_m f_m^2 < +\infty$ and $g = \sum_m g_m \phi_m$ with $\sum_m g_m^2 < +\infty$. We define, for m integers,

$$u_m = \frac{(-\lambda_m - \gamma) f_m + \beta g_m}{(\lambda_m - \alpha)(-\lambda_m - \gamma) - \beta^2}, \qquad v_m = \frac{(\lambda_m - \alpha) g_m + \beta f_m}{(\lambda_m - \alpha)(-\lambda_m - \gamma) - \beta^2},$$

which make sense since $(\lambda_m - \alpha)(-\lambda_m - \gamma) - \beta^2 \neq 0$ for every m. We have

$$|u_m| \leq \frac{C}{|\lambda_m|}(|f_m| + |g_m|),$$

from which it follows that

$$\lambda_m^2 u_m^2 \leq C_1 (f_m^2 + g_m^2)$$

for suitable constants C, C_1 not depending on m. We apply the same inequality for v_m. So if $u_{fg} = \sum_m u_m \phi_m$, $v_{fg} = \sum_m v_m \phi_m$, then $(u_{fg}, v_{fg}) \in H$. We can check easily that

$$N_{\alpha\beta\gamma}(u_{fg}, v_{fg}) = (f, g).$$

So $N^{-1} : H \to H$ is well defined, so the lemma is proved. □

Using Lemma 2.1 we can derive the following Lemma 2.2.

Lemma 2.2. *Assume that the conditions* $(B1)$, $(B2)$ *and* $(B3)$ *hold. Then the system*

$$\begin{cases} -\Delta u &= \alpha u + \beta v \quad \text{in } \Omega, \\ \Delta v &= \beta u + \gamma v, \quad \text{in } \Omega, \\ u &= v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.2}$$

has only the trivial solution $(0,0)$.

LEMMA 2.3. *Assume that the conditions* $(B1)$, $(B2)$ *and* $(B3)$ *hold. Then for any* (τ, t) *with* $\tau < 0$ *and* $t < 0$, *the linear system*

$$\begin{cases} -\Delta u &= \alpha u + \beta v + \tau\phi_1 \quad \text{in } \Omega, \\ \Delta v &= \beta b u + \gamma v + t\phi_1 \quad \text{in } \Omega, \\ u &= v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has a unique positive solution $(u_{\tau t}, v_{\tau t}) \in H$, *which is of the form*

$$u_{\tau t} = [\frac{\beta^2\tau + \beta t(\lambda_1 - \alpha)}{(\lambda_1 - \alpha)((\lambda_1 - \alpha)(-\lambda_1 - \gamma) - \beta^2)} + \frac{\tau}{\lambda_1 - \alpha}]\phi_1 < 0,$$

$$v_{\tau t} = [\frac{\beta\tau + t(\lambda_1 - \alpha)}{(\lambda_1 - \alpha)(-\lambda_1 - \gamma) - \beta^2}]\phi_1 < 0.$$

Since for any (τ, t) *with* $\tau < 0$ *and* $t < 0$, $u_{\tau t} < 0$ *and* $v_{\tau t} < 0$, *we can choose* (τ_0, t_0) *such that for any* (τ, t) *with* $\tau < \tau_0$ *and* $t < t_0$, $u_0 = u_{fg} + u_{\tau t} < 0$ *and* $v_0 = v_{fg} + v_{\tau t} < 0$.

Proof. We note that $(u_{\tau t}, v_{\tau t})$ is a solution of system (2.3) with $u_{\tau t} < 0$ and $v_{\tau t} < 0$ for any (τ, t) with $\tau < 0$ and $t < 0$, and the uniqueness is the consequence of Lemma 2.1. □

Combing Lemma 2.1 with Lemma 2.3, we obtain the following lemma.

LEMMA 2.4. *Assume that the conditions* $(B1)$, $(B2)$ *and* $(B3)$ *hold. there exist constants* $\tau_0 < 0$ *and* $t_0 < 0$ *such that for any* (τ, t) *with* $\tau < \tau_0$ *and* $t < t_0$, *the linear system*

$$\begin{cases} -\Delta u &= \alpha u + \beta v + \tau\phi_1 + f \quad \text{in } \Omega, \\ \Delta v &= \beta u + \gamma v + t\phi_1 + g \quad \text{in } \Omega, \\ u &= v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.4}$$

has a unique negative solution $(u_0, v_0) \in H$ *with* $u_0 < 0$ *and* $v_0 < 0$, *which is of the form*

$$\begin{aligned} u_0 &= \sum_m (\frac{(-\lambda_m - \gamma)f_m + \beta g_m}{(\lambda_m - \alpha)(-\lambda_m - \gamma) - \beta^2})\phi_m \\ &\quad + [\frac{\beta^2\tau + \beta t(\lambda_1 - \alpha)}{(\lambda_1 - \alpha)((\lambda_1 - \alpha)(-\lambda_1 - \gamma) - \beta^2)} + \frac{\tau}{\lambda_1 - \alpha}]\phi_1, \end{aligned}$$

$$v_0 = \sum_m (\frac{(\lambda_m - \alpha)g_m + \beta f_m}{(\lambda_m - \alpha)(-\lambda_m - \gamma) - \beta^2})\phi_m + [\frac{\beta\tau + t(\lambda_1 - \alpha)}{(\lambda_1 - \alpha)(-\lambda_1 - \gamma) - \beta^2}]\phi_1,$$

where $f = \sum_m f_m\phi_m$ *with* $\sum_m f_m^2 < +\infty$ *and* $g = \sum_m g_m\phi_m$ *with* $\sum_m g_m^2 < +\infty$.

Proof. Since for any (τ, t) with $\tau < 0$ and $t < 0$, $u_{\tau t} > 0$ and $v_{\tau t} < 0$, we can choose (τ_0, t_0) such that for any (τ, t) with $\tau > \tau_0$ and $t < t_0$, $u_0 = u_{fg} + u_{\tau t} < 0$ and $v_0 = v_{fg} + v_{\tau t} < 0$. □

THEOREM 2.1. *Assume that*

$$(B1)\qquad det\begin{pmatrix} \lambda_1-\alpha & -\beta \\ -\beta & -\lambda_1-\gamma \end{pmatrix} > 0,$$

$$(B2)\qquad det\begin{pmatrix} \lambda_i-\alpha & -\beta \\ -\beta & -\lambda_i-\gamma \end{pmatrix} < 0 \qquad \textit{for}\ \ i \geq 2,$$

$$(B3)\qquad \alpha > 0,\ \beta > 0,\ \gamma < 0,\ \ \lambda_1 - \alpha > 0.$$

Then there exists (τ_0, t_0) *with* $\tau_0 < 0$ *and* $t_0 < 0$ *such that for any* (τ, t) *with* $\tau < \tau_0$ *and* $t < t_0$, *(1.1) has at least one negative solution* (u, v).

Proof. We note that the negative solution (u_0, v_0) with $u_0 < 0$ and $v_0 < 0$ of (2.4) is also a negative solution of (1.1). □

3. Eigenvalue problem of the matrix

Let us set

$$E_{\lambda_i} = \text{span}\{\phi_j |\ \lambda_j = \lambda_i\}.$$

Let us denote by $(\xi^+_{\lambda_i}, \eta^+_{\lambda_i})$ and $(\xi^-_{\lambda_i}, \eta^-_{\lambda_i})$ the eigenvectors of $\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in M_{2\times 2}(R)$ corresponding to $\mu^+_{\lambda_i}$ and $\mu^-_{\lambda_i}$ respectively. Let us also set

$$\begin{aligned}
D_{\lambda_1} &= \{(\alpha,\beta,\gamma) \in R^3 |\ (\lambda_1-\alpha)(-\lambda_1-\gamma) - \beta^2 \geq 0\},\\
D_{\lambda_i} &= \{(\alpha,\beta,\gamma) \in R^3 |\ (\lambda_i-\alpha)(-\lambda_i-\gamma) - \beta^2 \leq 0\} \qquad \text{for}\ \ i \geq 2,\\
D'_{\lambda_i} &= D_{\lambda_i} \cap \{-\gamma \leq \alpha\},\\
D''_{\lambda_i} &= D_{\lambda_i} \cap \{-\gamma \geq \alpha\},\\
H_{\lambda_i} &= \{(\xi\phi, \eta\phi) \in H |\ (\xi,\eta) \in R^2, \phi \in E_{\lambda_i}\},\\
H^+_{\lambda_i} &= \{(\xi^+_{\lambda_i}\phi, \eta^+_{\lambda_i}\phi) \in H |\ \phi \in E_{\lambda_i}\},\\
H^-_{\lambda_i} &= \{(\xi^-_{\lambda_i}\phi, \eta^-_{\lambda_i}\phi) \in H |\ \phi \in E_{\lambda_i}\},
\end{aligned}$$

$$
\begin{aligned}
X^{+}(\alpha,\beta,\gamma) &= (\oplus_{\mu^{+}_{\lambda_i}>0} H^{+}_{\lambda_i}) \oplus (\oplus_{\mu^{-}_{\lambda_i}>0} H^{-}_{\lambda_i}),\\
X^{-}(\alpha,\beta,\gamma) &= (\oplus_{\mu^{+}_{\lambda_i}<0} H^{+}_{\lambda_i}) \oplus (\oplus_{\mu^{-}_{\lambda_i}<0} H^{-}_{\lambda_i}),\\
X^{0}(\alpha,\beta,\gamma) &= (\oplus_{\mu^{+}_{\lambda_i}=0} H^{+}_{\lambda_i}) \oplus (\oplus_{\mu^{-}_{\lambda_i}=0} H^{-}_{\lambda_i}).
\end{aligned}
$$

Then $X^{+}(\alpha,\beta,\gamma)$, $X^{-}(\alpha,\beta,\gamma)$ and $X^{0}(\alpha,\beta,\gamma)$ are the positive, negative and null space relative to the quadratic form $Q_{\alpha,\beta,\gamma}$ in H. Because $(\lambda_i-\alpha)(-\lambda_i-\gamma)-\beta^2 \neq 0$, $X^{0}(\alpha,\beta,\gamma)=\{0\}$.

The following theorem can be obtained by easy computations.

THEOREM 3.1. *Assume that the conditions* $(B1)$, $(B2)$ *and* $(B3)$ *hold. Let* $(\alpha,\beta,\gamma)\in R^3$ *and* $i\in N$. *Then*
(i) $H^{+}_{\lambda_i}$ *and* $H^{-}_{\lambda_i}$ *are the eigenspaces for the operator* $N_{\alpha,\beta,\gamma}$, $N_{\alpha,\beta,\gamma}(u,v) = (-\Delta u-\alpha u-\beta v, \Delta v-\beta u-\gamma v)$ *associated with* $Q_{\alpha,\beta,\gamma}$ *with eigenvalues* $\frac{\mu^{+}_{\lambda i}}{\lambda_i}$ *and* $\frac{\mu^{-}_{\lambda_i}}{\lambda_i}$ *respectively.*
(ii) $H^{+}_{\lambda_i}$ *and* $H^{-}_{\lambda_i}$ *generate* H.
(iii)

$$
\begin{gathered}
\text{If } (\alpha,\beta,\gamma)\in D'_{\lambda_1},\ \mu^{-}_{\lambda_1}<\mu^{+}_{\lambda_1}<0,\\
\text{if } (\alpha,\beta,\gamma)\in D''_{\lambda_1},\ 0<\mu^{-}_{\lambda_1}<\mu^{+}_{\lambda_1},\\
\text{if } i\geq 2,\ \mu^{-}_{\lambda_i}<0<\mu^{+}_{\lambda_i},\\
\lim_{i\to\infty}\frac{\mu^{-}_{\lambda_i}}{\lambda_i}=-1, \qquad \lim_{i\to\infty}\frac{\mu^{+}_{\lambda_i}}{\lambda_i}=1 \qquad \text{for } i\geq 2,\\
\lim_{(\alpha,\beta,\gamma)\to(\alpha_0,\beta_0,\gamma_0)} \mu^{-}_{\lambda_i}(\alpha,\beta,\gamma)=\mu^{-}_{\lambda_i}(\alpha_0,\beta_0,\gamma_0)
\end{gathered}
$$

and

$$
\lim_{(\alpha,\beta,\gamma)\to(\alpha_0,\beta_0,\gamma_0)} \mu^{+}_{\lambda_i}(\alpha,\beta,\gamma)=\mu^{+}_{\lambda_i}(\alpha_0,\beta_0,\gamma_0).
$$

uniformly with respect to $i\in N$.

Assume that the conditions $(B1)$, $(B2)$ and $(B3)$ hold. Let us choose $(\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}$ and U be an any neighborhood of $(\alpha_0,\beta_0,\gamma_0)$. Then $U=(U\cap D'_i)\oplus(U\backslash D'_{\lambda_i})$,

$$
\begin{gathered}
\text{if } (\alpha,\beta,\gamma)\in U\cap D'_{\lambda_1},\ \text{then } \mu^{-}_{\lambda_1}<\mu^{+}_{\lambda_1}\leq 0,\ \text{so } X^{+}(\alpha,\beta,\gamma)=\emptyset.\\
\text{if } (\alpha,\beta,\gamma)\in U\backslash D'_{\lambda_1},\ \text{then } 0\leq\mu^{-}_{\lambda_1}<\mu^{+}_{\lambda_1},\ \text{so } X^{-}(\alpha,\beta,\gamma)=\emptyset,\\
\text{if } i\geq 2 \text{ and } (\alpha,\beta,\gamma)\in U,\ \text{then } \mu^{-}_{\lambda_i}\leq 0\leq\mu^{+}_{\lambda_i},\ \text{so } X^{+}(\alpha,\beta,\gamma)\neq\emptyset,\\
X^{-}(\alpha,\beta,\gamma)\neq\emptyset.
\end{gathered}
$$

Thus for all $i \in N$ and $(\alpha_0, \beta_0, \gamma_0) \in \partial D_{\lambda_i}$, there exist a neighborhood U of $(\alpha_0, \beta_0, \gamma_0)$ such that for all $(\alpha, \beta, \gamma) \in U$, $X^+(\alpha, \beta, \gamma)$ can be split as an orthogonal sum

$$X^+(\alpha, \beta, \gamma) = Y_2(\alpha, \beta, \gamma) \oplus Y_3(\alpha, \beta, \gamma) \qquad \text{with } \dim(Y_2(\alpha, \beta, \gamma) < +\infty.$$

COMPETING INTERESTS

The authors declare that they have no competing interests.

AUTHORS'S CONTRIBUTIONS

All authors contributed equally to the manuscript and read and approved the final manuscript.

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