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BOUNDEDNESS IN PERTURBED FUNCTIONAL DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

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Abstract. In this paper, we investigate bounds for solutions of perturbed functional differential systems using the notion of t_∞-similarity.

1. Introduction and preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed functional differential systems of (1.1)

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds + r(t, y(t), Ty(t)),\ y(t_0) = y_0, \tag{1.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $r \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t, 0) = 0$, $r(t, 0, 0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator.

For $x \in \mathbb{R}^n$, let $|x| = (\sum_{j=1}^{n} x_j^2)^{1/2}$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x| \le 1} |Ax|$.

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Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then, we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \ \ v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \ \ z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We recall some notions of h-stability [14].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 1.2. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called

(hS)*h-stable* if there exists $\delta > 0$ such that (1.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

The notion of h-stability (hS) was introduced by Pinto [13, 14] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Choi and Koo [2] and Choi et al. [3,4] investigated h-stability and bounds of solutions for the perturbed differential systems. Also, Goo [6,7,8] and Goo et al. [9] studied the boundedness of solutions for the perturbed differential systems.

The main conclusion to be drawn from this paper is that the use of inequalities provides a powerful tool for obtaining bounds for solutions.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [5].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is t_∞-*similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [5, 10].

In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of t_∞-similarity.

We give some related properties that we need in the sequal.

Lemma 1.4. [14] *The linear system*

$$x' = A(t)x,\ x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 1.5. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 1.6. [3] *If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.*

Theorem 1.7. [4] *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS, then the solution $z = 0$ of (1.4) is hS.*

Lemma 1.8. (*Bihari − type inequality*) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big], \ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 1.9. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$ and $0 \leq t_0 \leq t$,*

(1.9)

$$\begin{aligned} u(t) \leq & c + \int_{t_0}^{t} \lambda_1(s) w(u(s)) ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) u(\tau) \\ & + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) w(u(r)) dr) d\tau ds + \int_{t_0}^{t} \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau) w(u(\tau)) d\tau ds. \end{aligned}$$

Then

(1.10)

$$\begin{aligned} u(t) \leq W^{-1}\Big[& W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r) dr) d\tau \\ & + \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau) d\tau) ds\Big], \end{aligned}$$

$t_0 \le t < b_1$, *where* W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau + \lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Define a function $z(t)$ by the right member of (1.9). Then, we have $z(t_0) = c$ and

$$\begin{aligned} z'(t) &= \lambda_1(t)w(u(t)) + \lambda_2(t)\int_{t_0}^{t}(\lambda_3(s)u(s) + \lambda_4(s)\int_{t_0}^{s}\lambda_5(\tau)w(u(\tau))d\tau)ds \\ &\quad +\lambda_6(t)\int_{t_0}^{t}\lambda_7(s)w(u(s))ds \\ &\le (\lambda_1(t) + \lambda_2(t)\int_{t_0}^{t}(\lambda_3(s) + \lambda_4(s)\int_{t_0}^{s}\lambda_5(\tau)d\tau)ds \\ &\quad +\lambda_6(t)\int_{t_0}^{t}\lambda_7(s)ds)w(z(t)), \end{aligned}$$

$t \ge t_0$, since $z(t)$ and $w(u)$ are nondecreasing, $u \le w(u)$, and $u(t) \le z(t)$. Therefore, by integrating on $[t_0, t]$, the function z satisfies

$$\begin{aligned} z(t) \le c + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau \\ +\lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)d\tau)w(z(s)))ds. \end{aligned} \tag{1.11}$$

It follows from Lemma 1.8 that (1.11) yields the estimate (1.10). □

We obtain the following two corollaries from Lemma 1.9.

Corollary 1.10. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t}\lambda_1(s)\int_{t_0}^{s}(\lambda_2(\tau)w(u(\tau)) + \lambda_3(\tau)\int_{t_0}^{\tau}\lambda_4(r)w(u(r))dr)d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s)\int_{t_0}^{s}(\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^{\tau}\lambda_4(r)dr)d\tau)ds\Big],$$

$t_0 \le t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)dr)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Corollary 1.11. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} \lambda_2(\tau)u(\tau)d\tau + \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) \int_{t_0}^{s} \lambda_2(\tau)d\tau + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau)ds\Big],$$

$t_0 \le t < b_1$, where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) \int_{t_0}^{s} \lambda_2(\tau)d\tau + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

2. Main Results

In this section, we investigate boundedness for solutions of the nonlinear perturbed differential systems via t_∞-similarity.

To obtain the bounded property, the following assumptions are needed:
(H1) $w(u)$ is nondecreasing in u such that $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$.
(H2) $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.
(H3) The solution $x = 0$ of (1.1) is hS with the increasing function h.

Theorem 2.1. *Let $a, b, c, k, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$|g(t, y(t))| \le a(t)w(|y(t)|) + b(t) \int_{t_0}^{t} k(s)w(|y(s)|)ds \tag{2.1}$$

and

$$|r(t,y(t),Ty(t))| \le \int_{t_0}^{t} c(s)w(|y(s)|)ds, \tag{2.2}$$

where $a,b,c,k \in L_1(\mathbb{R}^+)$. *Then, any solution* $y(t) = y(t,t_0,y_0)$ *of (1.2) is bounded on* $[t_0,\infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\int_{t_0}^{s}(a(\tau)+c(\tau)+b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau ds\Big],$$

where W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}\int_{t_0}^{s}(a(\tau)+c(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Using the nonlinear variation of constants formula of Alekseev [1], any solution $y(t) = y(t,t_0,y_0)$ of (1.2) passing through (t_0,y_0) is given by

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^{t}\Phi(t,s,y(s)) (\int_{t_0}^{s}g(\tau,y(\tau))d\tau + r(s,y(s),Ty(s)))ds. \tag{2.3}$$

By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Therefore, by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula (2.3), Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.1), and (2.2), we have

$$\begin{aligned}|y(t)| &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)h(s)^{-1}(\int_{t_0}^{s}(a(\tau)w(|y(\tau)|)\\ &\quad +b(\tau)\int_{t_0}^{\tau}k(r)w(|y(r)|)dr)d\tau + \int_{t_0}^{s}c(\tau)w(|y(\tau)|)d\tau)ds\\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)\Big(\int_{t_0}^{s}((a(\tau)+c(\tau))w(\frac{|y(\tau)|}{h(\tau)})d\tau\\ &\quad +b(\tau)\int_{t_0}^{\tau}k(r)w(\frac{|y(r)|}{h(r)})dr)d\tau\Big)ds.\end{aligned}$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Corollary 1.10, we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} \int_{t_0}^{s} (a(\tau) + c(\tau) + b(\tau) \int_{t_0}^{\tau} k(r)dr)d\tau ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the proof is complete. □

REMARK 2.2. Letting $c(s) = 0$ in Theorem 2.1, we obtain the similar result as that of Theorem 3.6 in [8].

We need the following lemma for the proof of Theorem 2.4.

LEMMA 2.3. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c \geq 0$, we have*

(2.4)
$$\begin{aligned} u(t) \leq & c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s)\Big(\int_{t_0}^{s} (\lambda_3(\tau)w(u(\tau)) \\ & + \lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(s)w(u(r))dr)d\tau + \lambda_6(s)\int_{t_0}^{s} \lambda_7(\tau)w(u(\tau))d\tau\Big)ds,\ t \geq t_0. \end{aligned}$$

Then

(2.5)
$$\begin{aligned} u(t) \leq & W^{-1}\Big[W(c) + \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s)\Big(\int_{t_0}^{s} (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(r)dr)d\tau \\ & + \lambda_6(s)\int_{t_0}^{s} \lambda_7(\tau)d\tau\Big)]ds\Big],\ t \geq t_0, \end{aligned}$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$\begin{aligned} b_1 = \sup\Big\{ t \geq t_0 : W(c) + \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s)\Big(\int_{t_0}^{s} (\lambda_3(\tau) \\ + \lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(r)dr)d\tau + \lambda_6(s)\int_{t_0}^{s} \lambda_7(\tau)d\tau\Big)]ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

Proof. Define a function $v(t)$ by the right member of (2.4). Then, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= \lambda_1(t)w(u(t)) + \lambda_2(t)\Big(\int_{t_0}^t (\lambda_3(s)w(u(s)) \\ &\quad + \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)w(u(\tau))d\tau)ds + \lambda_6(t)\int_{t_0}^t \lambda_7(s)w(u(s))ds\Big) \\ &\le \Big[\lambda_1(t) + \lambda_2(t)\Big(\int_{t_0}^t (\lambda_3(s) + \lambda_4(s)\int_{t_0}^s \lambda_5(\tau)d\tau)ds \\ &\quad + \lambda_6(t)\int_{t_0}^t \lambda_7(s)ds\Big)\Big]w(v(t)), \end{aligned}$$

$t \ge t_0$, since $v(t)$ is nondecreasing and $u(t) \le v(t)$. Now, by integrating the above inequality on $[t_0, t]$ and $v(t_0) = c$, we have

$$\begin{aligned} v(t) \le c + \int_{t_0}^t \Big(\lambda_1(s) + \lambda_2(s)\int_{t_0}^s (\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^\tau \lambda_5(r)dr)d\tau \\ + \lambda_6(s)\int_{t_0}^s \lambda_7(\tau)d\tau\Big)w(v(s))ds. \end{aligned} \tag{2.6}$$

Thus, by Lemma 1.8, (2.6) yields the estimate (2.5). □

THEOREM 2.4. *Let $a, b, c, k, q, u, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$|g(t, y(t))| \le a(t)w(|y(t)|) + b(t)\int_{t_0}^t k(s)w(|y(s)|)ds \tag{2.7}$$

and

$$\begin{aligned} |r(t, y(t), Ty(t))| \le c(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \\ \le \int_{t_0}^t q(s)w(|y(s)|)ds, \quad t \ge t_0 \ge 0, \end{aligned} \tag{2.8}$$

where $a, b, c, k, q \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and

$$\begin{aligned} |y(t)| \ \le h(t)W^{-1}\Big[W(c) + c_2\textstyle\int_{t_0}^t (c(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau \\ + c(s)\textstyle\int_{t_0}^s q(\tau)d\tau)ds\Big], \end{aligned}$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \quad c_2 \int_{t_0}^{t}(c(s) + \int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Applying Lemma 1.4, the hS condition of $x = 0$ of (1.1), (2.3), (2.7), (2.8), and the given conditions, we have

$$|y(t)| \leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)\Big(c(s)w(\frac{|y(s)|}{h(s)}) + \int_{t_0}^{s}(a(\tau)w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^{\tau} k(r)w(\frac{|y(r)|}{h(r)})dr)d\tau + c(s)\int_{t_0}^{s} q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau)\Big)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, it follows from Lemma 2.3 that we have

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(c(s) + \int_{t_0}^{s}(a(\tau) + b(\tau)\int_{t_0}^{\tau} k(r)dr)d\tau + c(s)\int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. □

Remark 2.5. Letting $c(t) = 0$ in Theorem 2.4, we obtain the similar result as that of Theorem 3.6 in [8].

We obtain the following corollary from Lemma 2.3 to prove Theorem 2.7.

Corollary 2.6. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c \geq 0$, we have*

$$u(t) \leq c + \int_{t_0}^{t}\lambda_1(s)w(u(s))ds + \int_{t_0}^{t}\lambda_2(s)\Big(\int_{t_0}^{s}\lambda_3(\tau)w(u(\tau))d\tau + \lambda_4(s)\int_{t_0}^{s}\lambda_5(\tau)w(u(\tau))d\tau\Big)ds,\ t \geq t_0.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s)\Big(\int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s)\int_{t_0}^{s} \lambda_5(\tau)d\tau\Big)]ds\Big],$$

$t \ge t_0$, *where* W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} [\lambda_1(s) + \lambda_2(s)\Big(\int_{t_0}^{s} \lambda_3(\tau)d\tau + \lambda_4(s)\int_{t_0}^{s} \lambda_5(\tau)d\tau\Big)]ds \in \mathrm{domW}^{-1}\Big\}.$$

THEOREM 2.7. *Let* $a, b, c, k, u, w \in C(\mathbb{R}^+)$. *Suppose that (H1), (H2), (H3), and* g *in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \le a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)w(|y(s)|)ds,\ t \ge t_0 \ge 0, \tag{2.9}$$

and

$$|r(t, y(t), Ty(t))| \le \int_{t_0}^{t} c(s)w(|y(s)|)ds, \tag{2.10}$$

where $a, b, c, k \in L_1(\mathbb{R}^+)$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (1.2) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t} (a(s) + \int_{t_0}^{s} c(\tau)d\tau + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big], \quad t_0 \le t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 1.8 and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t} (a(s) + \int_{t_0}^{s} c(\tau)d\tau + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. It is well known that the solution of (1.2) is represented by the integral equation (2.3). By the same argument as the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula (2.3), the hS condition of $x = 0$ of (1.1), (2.9), and

(2.10), we have

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) a(s) w(\frac{|y(s)|}{h(s)})ds$$
$$+ \int_{t_0}^{t} c_2 h(t)(\int_{t_0}^{s} c(\tau) w(\frac{|y(\tau)|}{h(\tau)})d\tau + b(s)\int_{t_0}^{s} k(\tau) w(\frac{|y(\tau)|}{h(\tau)})d\tau)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, an application of Corollary 2.6 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + \int_{t_0}^{s} c(\tau)d\tau + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

Remark 2.8. Letting $c(s) = 0$ in Theorem 2.7, we obtain the same result as that of Theorem 3.2 in [6].

Theorem 2.9. *Let $a, b, c, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that (H2), (H3), and g in (1.2) satisfies*

(2.11)
$$|g(t, y(t))| \le a(t)w(|y(t)|), |r(t, y(t), Ty(t))| \le b(t)\int_{t_0}^{t} c(s)|y(s)|ds,$$

where $a, b, c \in L_1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(b(s)\int_{t_0}^{s} c(\tau)d\tau + \int_{t_0}^{s} a(\tau)d\tau)ds\Big]$$

where W, W^{-1} are the same functions as in Lemma 1.8 and

$$b_1 = \sup\Big\{t \ge t_0 : W(c)+c_2\int_{t_0}^{t}(b(s)\int_{t_0}^{s} c(\tau)d\tau+\int_{t_0}^{s} a(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. By the hS condition of $x = 0$ of

(1.1), (2.3), and (2.11), it follows that

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)(b(s)\int_{t_0}^{s} c(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau$$
$$+\int_{t_0}^{s} a(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau)ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, an application of Corollary 1.11 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(b(s)\int_{t_0}^{s} c(\tau)d\tau + \int_{t_0}^{s} a(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. This completes the proof. □

REMARK 2.10. Letting $b(t) = 0$ in Theorem 2.9, we obtain the similar result as that of Theorem 3.5 in [9].

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