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EINSTEIN'S CONNECTION IN 3-DIMENSIONAL ES-MANIFOLD

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Abstract. The manifold $^*g-ESX_n$ is a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by the unified field tensor $^*g^{\lambda\nu}$ through the ES-connection which is both Einstein and semi-symmetric. The purpose of the present paper is to prove a necessary and sufficient condition for a unique Einstein's connection to exist in 3-dimensional $^*g - ESX_3$ and to display a surveyable tnesorial representation of 3-dimensional Einstein's connection in terms of the unified field tensor, employing the powerful recurrence relations in the first class.

1. Preliminaries

This paper is a direct continuation of our previous paper [1], which will be denoted by I in the present paper. All considerations in this paper are based on the results and symbolism of I. Whenever necessary, they will be quoted in the present paper. In this section, we introduce a brief collection of basic concepts, notations, and results of I, which are frequently used in the present paper([2],[3],[4]).

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(a) n-simensional $^*g$-unified field theory

Let X_n be an n-dimensional generalized Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformations $x^\nu \rightarrow x^{\nu'}$ for which

$$det(\frac{\partial x'}{\partial x}) \neq 0 \tag{1.1}$$

In $n-g-UFT$ the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{1.2}$$

where

$$\mathfrak{g} = det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = det(k_{\lambda\mu}) \tag{1.3}$$

In $n-{}^*g-UFT$ the algebraic structure on X_n is imposed by the basic real tensor $^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta^\nu_\mu \tag{1.4}$$

It may be also decomposed into its symmetric part $^*h^{\lambda\nu}$ and skew-symmetric part $^*k^{\lambda\nu}$:

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu} \tag{1.5}$$

Since $det({}^*h^{\lambda\nu}) \neq 0$, we may define a unique tensor $^*h_{\lambda\mu}$ by

$$^*h_{\lambda\mu}{}^*h^{\lambda\nu} = \delta^\nu_\mu \tag{1.6}$$

In $n-{}^*g$-UFT we use both $^*h^{\lambda\nu}$ and $^*h_{\lambda\mu}$ as tensors for raising and/or lowering indices of all tensors in X_n in the usual manner. We then have

$$^*k_{\lambda\mu} = {}^*k^{\rho\sigma}{}^*h_{\lambda\rho}{}^*h_{\mu\sigma}, \qquad {}^*g_{\lambda\mu} = {}^*g^{\rho\sigma}{}^*h_{\lambda\rho}{}^*h_{\mu\sigma} \tag{1.7}$$

so that

$$^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu} \tag{1.8}$$

The differential geometric structure on X_n is imposed by the tensor $^*g^{\lambda\nu}$ by means of a connection $\Gamma_\lambda{}^\nu{}_\mu$ defined by a system of equations

$$D_\omega{}^*g^{\lambda\nu} = -2S_{\omega\alpha}{}^\nu{}^*g^{\lambda\alpha} \tag{1.9}$$

where D_ω denotes the symbol of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$ and $S_{\lambda\mu}{}^\nu$ is the torsion tensor of $\Gamma_\lambda{}^\nu{}_\mu$. Under certain conditions the system (1.9) admits a unique solutions $\Gamma_\lambda{}^\nu{}_\mu$.

It has been shown in [5] that if the system (1.9) admits $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form

$$\Gamma_\lambda{}^\nu{}_\mu = {}^*\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + U^\nu{}_{\lambda\mu} + S_{\lambda\mu}{}^\nu. \tag{1.10}$$

where

$$U_{\nu\lambda\mu} = \overset{100}{S}{}_{(\lambda\mu)\nu} + 2\overset{(10)0}{S}{}_{\nu(\lambda\mu)} \tag{1.11}$$

(b) Some notations and results

The following quantities are frequently used in our further considerations:

$$^*g = det(^*g_{\lambda\mu}), \quad ^*h = det(^*h_{\lambda\mu}), \quad ^*k = det(^*k_{\lambda\mu}) \tag{1.12}$$

$$^*\mathfrak{g} = \frac{^*g}{^*h}, \quad ^*\mathfrak{k} = \frac{^*k}{^*h}. \tag{1.13}$$

$$K_p = {}^*k_{[\alpha_1}{}^{\alpha_1}\, {}^*k_{\alpha_2}{}^{\alpha_2} \cdots {}^*k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots). \tag{1.14}$$

$${}^{(0)*}k_\lambda{}^\nu = \delta_\lambda^\nu, \ {}^{(p)*}k_\lambda{}^\nu = {}^*k_\lambda{}^\alpha \ {}^{(p-1)*}k_\alpha{}^\nu \quad (p = 1, 2, \cdots). \tag{1.15}$$

$$K_{\omega\mu\nu} = \nabla_\nu{}^*k_{\omega\mu} + \nabla_\omega{}^*k_{\nu\mu} + \nabla_\mu{}^*k_{\omega\nu} \tag{1.16}$$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the christoffel symbols ${}^*\left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\}$ defined by ${}^*h_{\lambda\mu}$ in the usual way.

In X_n it was proved in [5] that

$$K_0 = 1, \ K_n = {}^*\mathfrak{k} \text{ if } n \text{ is even, and } \mathrm{K_n = 0 \text{ if } n \text{ is odd.}} \tag{1.17}$$

$$^*\mathfrak{g} = 1 + K_2 + \cdots + K_{n-\sigma}. \tag{1.18}$$

$$\sum_{s=0}^{n-\sigma} K_s \ {}^{(n-s)*}k_\lambda{}^\nu = 0 \quad (p = 0, 1, 2, \cdots). \tag{1.19}$$

We also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$ skew-symmetric in the first two indices by T:

$$\overset{pqr}{T} = \overset{pqr}{T}_{\omega\mu\lambda} = T_{\alpha\beta\gamma}{}^{(p)*}k_\omega{}^{\alpha(q)*}k_\mu{}^{\beta(r)*}k_\lambda{}^\gamma \tag{1.20}$$

and for an arbitrary tensor $T^{\cdots}_{\cdots}$ for $p = 1, 2, 3, \cdots$:

$$^{(p)}T^{\nu\cdots}_{\cdots} =^{(p-1)}{}^*k^\nu{}_\alpha T^{\alpha\cdots}_{\cdots}. \tag{1.21}$$

On the other hand, it has shown in [6] that the tensor $S_{\lambda\mu}{}^\nu$ satisfies

$$S = B - 3\overset{(110)}{S} \tag{1.22}$$

where

$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}{}^*k_{\omega]}{}^{\alpha*}k_\nu{}^\beta \tag{1.23}$$

In our subsequent chapter, we start with the relation (1.22) to solve the system (1.9). Furthermore, for the first class, the nonholonomic solution of (1.22) may be given by

$$\underset{xyz}{M} S_{xyz} = B_{xyz} \tag{1.24}$$

or equivalently

$$\begin{aligned} 4\underset{xyz}{M} S_{xyz} = (2 + \underset{z}{M}\underset{x}{M} + \underset{z}{M}\underset{y}{M})K_{xyz} + \underset{z}{M}(\underset{x}{M} + \underset{z}{M})K_{zxy} \\ + \underset{z}{M}(\underset{y}{M} + \underset{z}{M})K_{yzx} \end{aligned} \tag{1.25}$$

where

$$\underset{xyz}{M} = 1 + \underset{x}{M}\underset{y}{M} + \underset{y}{M}\underset{z}{M} + \underset{z}{M}\underset{x}{M} \tag{1.26}$$

Therefore, in virtue of (1.24), we see that a necessary and sufficient condition for the system (1.9) to have a unique solution in the first class is

$$\underset{xyz}{M} \neq 0 \;\; for\, all\, x, y, z \tag{1.27}$$

(c) n-dimensional ES manifold $n - {}^*g$-UFT

In this subsection, we display an useful representation of the ES connection in $n - {}^*g$-UFT.

DEFINITION 1.1. A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be *semi-symmetric* if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda}X_{\mu]}. \tag{1.28}$$

for an arbitrary non-null vector X_μ.

A connection which is both semi-symmetric and Einstein is called an ES connection. An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by ${}^*g^{\lambda\nu}$ by means of an ES connection, is called an n-dimensional ${}^*g-ES$ manifold. We denote this manifold by ${}^*g - ESX_n$ in our further considerations.

In ${}^*g - ESX_n$, the following theorems were proved in I.

THEOREM 1.2. *The main recurrence relation in the first class is*

$$^{(p+3)*}k_\lambda{}^\nu = -K_2{}^{(p+1)*}k_\lambda{}^\nu, \quad (p = 0, 1, 2, \cdots) \tag{1.29}$$

THEOREM 1.3. *The basic scalars* $\underset{x}{M}$ *satisfy*

$$\underset{x}{M}\underset{y}{M}(\underset{x}{M} + \underset{y}{M}) = 0, \quad (x \neq y) \tag{1.30}$$

$$\underset{x}{M}\underset{y}{M}(\underset{x}{M}\underset{y}{M} - K_2) = 0, \quad (x \neq y) \tag{1.31}$$

THEOREM 1.4. *(Recurrence relations in the first class) If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the first class of* $3 - {}^*g - ESX_3$ *:*

$$\overset{(12)r}{T} = 0, \qquad \overset{22r}{T} = K_2 \overset{11r}{T} \tag{1.32}$$

$$\overset{r(12)}{T}{}_{\nu[\omega\mu]} = 0, \qquad \overset{r22}{T}{}_{\nu[\omega\mu]} = K_2 \overset{r11}{T}{}_{\nu[\omega\mu]} \tag{1.33}$$

2. Einstein's connection $\Gamma_\lambda{}^\nu{}_\mu$ in the first class

In this section, we shall derive surveyable tensorial representations of $S_{\lambda\mu}{}^\nu$ and hence $\Gamma_\lambda{}^\nu{}_\mu$ in terms of ${}^*g^{\lambda\nu}$, employing the recurrence relations.

In the following theorem, we shall prove two relations in X_n. These relations will be used in our subsequent theorem when we are concerned with the solution of (1.9).

THEOREM 2.1. *We have*

$$\overset{(pq)r}{B} = \overset{(pq)r}{S} + \overset{(p'q')r}{S} + \overset{(p'q)r'}{S} + \overset{(pq')r'}{S} \tag{2.1}$$

$$\begin{aligned} 2\overset{(pq)r}{B}{}_{\omega\mu\nu} &= \overset{(pq)r}{K}{}_{\omega\mu\nu} + \overset{r''(pq)}{K}{}_{\nu[\omega\mu]} \\ &+\tfrac{1}{2}(\overset{(pq')r'}{K}{}_{\omega\mu\nu} + \overset{(p'q)r'}{K}{}_{\omega\mu\nu} + \overset{r'p'q}{K}{}_{\nu[\omega\mu]} + \overset{r'q'p}{K}{}_{\nu[\omega\mu]}) \end{aligned} \tag{2.2}$$

where

$$p' = p+1, \quad q' = q+1, \quad r' = r+1, \quad r'' = r+2 \tag{2.3}$$

Proof. In virtue of (1.22) and (1.20), the first relation (2.1) is obtained as in the following way:

$$\begin{aligned} \overset{(pq)r}{B} &= \overset{(pq)r}{B}{}_{\omega\mu\nu} = \tfrac{1}{2}B_{\omega\beta\gamma}({}^{(p)*}k_\omega{}^{\alpha(q)*}k_\mu{}^\beta + {}^{(q)*}k_\omega{}^{\alpha(p)*}k_\mu{}^\beta)^{(r)*}k_\nu{}^\gamma \\ &= \tfrac{1}{2}(S_{\alpha\beta\gamma} + S_{\epsilon\eta\gamma}{}^*k_\alpha{}^{\epsilon*}k_\beta{}^\eta + S_{\epsilon\beta\eta}{}^*k_\alpha{}^{\epsilon*}k_\gamma{}^\eta + S_{\alpha\epsilon\eta}{}^*k_\beta{}^{\epsilon*}k_\gamma{}^\eta) \\ &\qquad \times({}^{(p)*}k_\omega{}^{\alpha(q)*}k_\mu{}^\beta + {}^{(q)*}k_\omega{}^{\alpha(p)*}k_\mu{}^\beta)^{(r)*}k_\nu{}^\gamma \end{aligned} \tag{2.4}$$

After a lengthy calculation, we note that the right-hand side of the above equation is equal to that of (2.1). Similarly, we verify (2.2) using (1.20) and (1.23). □

THEOREM 2.2. *A necessary and sufficient condition for the system* (1.9) *to admit a unique solution* $\Gamma_\lambda{}^\nu{}_\mu$ *is that*

$$1-(K_2)^2 \neq 0 \tag{2.5}$$

Proof. Since $\underset{xyz}{M}$ defined by (1.26), is symmetric in x, y, z and satisfies

$$\underset{33x}{M} = 1, \quad \underset{311}{M} = \underset{322}{M} = 1-K_2, \quad \underset{123}{M} = \underset{112}{M} = \underset{122}{M} = 1+K_2 \tag{2.6}$$

we have the condition (2.5) in virtue of (1.27). □

THEOREM 2.3. *The system of equations* (1.22) *is reduced to a system of the following 5 equations:*

$$\text{(2.7)}\qquad \begin{cases} B = S + 2\overset{(10)1}{S} + \overset{110}{S} \\ \overset{(10)1}{B} = \overset{(10)1}{S} + \overset{(20)2}{S} + \overset{112}{S} \\ \overset{110}{B} = (1+K_2)\overset{110}{S} \\ \overset{(20)2}{B} = (K_2)^2\overset{(10)1}{S} + \overset{(20)2}{S} - K_2\overset{112}{S} \\ \overset{112}{B} = (1+K_2)\overset{112}{S} \end{cases}$$

Proof. This assertion follows from (2.1) using (1.29), (1.32) and (1.33). □

THEOREM 2.4. *The tensor* $\overset{(pq)r}{B}{}_{\omega\mu\nu}$ *are given as linear combinations of* $\overset{(pq)r}{K}{}_{\omega\mu\nu}$, *as follows:*

$$\text{(2.8)}\qquad \begin{cases} 2\overset{(10)1}{B}{}_{\omega\mu\nu} = \overset{(10)1}{K}{}_{\omega\mu\nu} + \frac{1}{2}(\overset{112}{K}{}_{\omega\mu\nu} + \overset{(20)2}{K}{}_{\omega\mu\nu} + \overset{211}{K}{}_{\nu[\omega\mu]} - K_2\overset{101}{K}{}_{\nu[\omega\mu]}) \\ 2\overset{110}{B}{}_{\omega\mu\nu} = \overset{110}{K}{}_{\omega\mu\nu} \\ 2\overset{(20)2}{B}{}_{\omega\mu\nu} = \overset{(20)2}{K}{}_{\omega\mu\nu} + \frac{1}{2}[(K_2)^2\overset{(10)1}{K}{}_{\omega\mu\nu} - K_2\overset{112}{K}{}_{\nu[\omega\mu]} - K_2\overset{202}{K}{}_{\nu[\omega\mu]}] \\ 2\overset{112}{B}{}_{\omega\mu\nu} = \overset{112}{K}{}_{\omega\mu\nu} \end{cases}$$

Proof. These relations are obtained from (2.2) in virtue of (1.29), (1.32) and (1.33). □

THEOREM 2.5. *If the condition* (2.5) *is satisfied, the unique solution of* (1.22) *is given by*

$$\text{(2.9)}\qquad [1-(K_2)^2](S-B) = -2\overset{(10)1}{B} + (K_2-1)\overset{110}{B} + 2\overset{(20)2}{B} + 2\overset{112}{B}$$

or equivalently

$$
\begin{aligned}
&[1-(K_2)^2](2S_{\omega\mu\nu} - K_{\omega\mu\nu} - \overset{110}{K}{}_{\nu[\omega\mu]} - \overset{200}{K}{}_{\nu[\omega\mu]}) = \\
&\quad - \overset{(10)1}{K}{}_{\omega\mu\nu} + \overset{112}{K}{}_{\omega\mu\nu} + \overset{(20)2}{K}{}_{\omega\mu\nu} - \overset{211}{K}{}_{\nu[\omega\mu]} \\
&\quad + (K_2-1)\overset{110}{K}{}_{\omega\mu\nu} + K_2(\overset{101}{K}{}_{\nu[\omega\mu]} - \overset{112}{K}{}_{\nu[\omega\mu]} - \overset{202}{K}{}_{\nu[\omega\mu]})
\end{aligned} \tag{2.10}
$$

Proof. (2.9) is the solution of (2.7), while (2.10) is obtained by substituting (2.8) into (2.9) and making use of recurrence relations. □

THEOREM 2.6. *The tensor $U^{\nu}{}_{\lambda\mu}$ is given by*

$$
\begin{aligned}
&[1-(K_2)^2](U_{\nu\lambda\mu} - \overset{[10]0}{B}{}_{\lambda\mu\nu} - 2\overset{(10)0}{B}{}_{\nu(\lambda\mu)}) = \\
&\quad - K_2(\overset{[10]2}{B}{}_{\lambda\mu\nu} + 2\overset{(10)2}{B}{}_{\nu(\lambda\mu)}) + (K_2-1)\overset{[21]0}{B}{}_{\lambda\mu\nu} \\
&\quad + \overset{[02]1}{B}{}_{\lambda\mu\nu} + \overset{[21]2}{B}{}_{\lambda\mu\nu} - 2\overset{(20)1}{B}{}_{\nu(\lambda\mu)} - 2\overset{111}{B}{}_{\nu(\lambda\mu)}
\end{aligned} \tag{2.11}
$$

or equivalently

$$
\begin{aligned}
&[1-(K_2)^2](2U_{\nu\lambda\mu} + \overset{[01]0}{K}{}_{\lambda\mu\nu} - 2\overset{(10)0}{K}{}_{\nu(\lambda\mu)}) = (K_2-1)\overset{[21]0}{K}{}_{\lambda\mu\nu} \\
&+ \overset{[02]1}{K}{}_{\lambda\mu\nu} + K_2\overset{[01]2}{K}{}_{\lambda\mu\nu} - 2(K_2\overset{(10)2}{K}{}_{\nu(\lambda\mu)} - \overset{(20)1}{K}{}_{\nu(\lambda\mu)} - \overset{111}{K}{}_{\nu(\lambda\mu)})
\end{aligned} \tag{2.12}
$$

Proof. The representations (2.11), (2.12) are direct consequences of substituting (2.9), (2.10) into (1.11). □

Now that we have obtained the tensor $S_{\lambda\mu}{}^{\nu}$ and $U^{\nu}{}_{\lambda\mu}$ in terms of ${}^*g^{\lambda\nu}$, it is possible for us to determine $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ by only substituting for S and U into (1.10).

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