SUFFICIENT CONDITIONS FOR STARLIKENESS

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Abstract. We obtain the conditions on \( \beta \) so that \( 1 + \beta z p'(z) \lessdot \frac{1 + 4z}{3 + 2z^2/3} \) implies \( p(z) \lessdot \frac{2 + z}{2 - z} \), \( 1 + (1 - \alpha)z \), \( (1 + (1 - 2\alpha)z)/(1 - z) \), \( (0 \leq \alpha < 1) \), \( \exp(z) \) or \( \sqrt{1 + z} \). Similar results are obtained by considering the expressions \( 1 + \beta z p'(z)/p(z) \), \( 1 + \beta z p'(z)/p(z)^2 \) and \( p(z) + \beta z p'(z)/p(z) \).

These results are applied to obtain sufficient conditions for normalized analytic function \( f \) to belong to various subclasses of starlike functions, or to satisfy the condition \( |\log(zf'(z)/f(z))| < 1 \) or \( |(zf'(z)/f(z))^2 - 1| < 1 \) or \( zf'(z)/f(z) \) lying in the region bounded by the cardioid \((9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0\).

1. Introduction

Let \( A \) denote the class of analytic functions in the unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) of the form \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \). An analytic function \( p(z) = 1 + cz + \cdots \) is a function with a positive real part if \( \text{Re} p(z) > 0 \). The class of all such functions is denoted by \( P \). For two functions \( f \) and \( g \) analytic in \( \mathbb{D} \), \( f \) is subordinate to \( g \), denoted by \( f \lessdot g \), if there is an analytic function \( w \) in \( \mathbb{D} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \). In particular, if the function \( g \) is univalent in \( \mathbb{D} \), then \( f \lessdot g \) is equivalent to \( f(0) = g(0) \) and \( f(\mathbb{D}) \subset g(\mathbb{D}) \). Noticing that several subclasses of univalent functions are characterized by the quantities \( zf'(z)/f(z) \) or \( 1 + zf''(z)/f'(z) \) lying in a region in the right-half plane, Ma and Minda [6] gave a unified presentation of various subclasses of convex and starlike functions. They considered analytic functions \( \varphi \) with positive real part in \( \mathbb{D} \) that map the unit disc \( \mathbb{D} \) onto regions starlike with respect to 1, symmetric with respect to the real axis and normalized by the conditions \( \varphi(0) = 1 \) and \( \varphi'(0) > 0 \). Ma and Minda [6] introduced the following classes:

\[
S^*(\varphi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \lessdot \varphi(z) \right\}
\]
and

$$C(\phi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \right\}.$$  

For special choices of $\phi$, $S^*(\phi)$ reduces to well-known subclasses of starlike functions. For example, when $-1 \leq B < A \leq 1$, $S^*[A, B] := S^*(1 + Az)/(1 + Bz)$ is the class of Janowski starlike function $[4, 10]$ and $S^*[1 - 2\alpha, -1]$ is the class $S^*(\alpha)$ of starlike functions of order $\alpha$, introduced by Robertson $[12]$ and $S^* := S^*(0)$ is the class of starlike functions. Similarly, $S^*_L := S^*(\sqrt{1 + z})$ is the subclass of $S^*$ introduced by Sokol and Stankiewicz $[18]$, consisting of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. More results regarding these classes can be found in $[1, 3, 5, 11, 13, 16, 17]$. Recently, Sharma et al. $[14]$ introduced and studied the properties of the class

$$S^*(1 + (4/3)z + (2/3)z^2) = S^*_C.$$  

Precisely, $f \in S^*_C$ provided $zf'(z)/f(z)$ lies in the region bounded by the cardioid $(9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0$. The class $S^*_L := S^*(e^z)$, introduced recently by Mendariratta et al. $[7]$, consists of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1$.

Let $p$ be an analytic function defined on $D$ with $p(0) = 1$. Recently Ali et al. $[2]$ determined the condition on $\beta$ for $p(z) \prec \sqrt{1 + z}$ when $1 + \beta zf'(z)/p^2(z)$ with $n = 0, 1, 2$ or $(1 - \beta)p(z) + \beta p^2(z) + \beta zp'(z)$ is subordinated to $\sqrt{1 + z}$. Motivated by the works in $[1, 3, 9, 15, 17]$, in Section 2, we determine the sharp conditions on $\beta$ so that $p(z) \prec (2 + z)/(2 - z)$ or $1 + (1 - \beta)z/(1 - z)$, $(0 \leq \alpha < 1)$ when $1 + \beta zp'(z) \prec 1 + 4z/3 + 2z^2/3$. Conditions on $\beta$ so that $1 + \beta zp'(z)/p(z) \prec 1 + 4z/3 + 2z^2/3$ implies $p(z) \prec (1 + z)/(1 - z)$ or $1 + z$ are also discussed. Conditions on $\beta$ are derived so that the subordination $1 + \beta zp'(z)/p^2(z) \prec 1 + 4z/3 + 2z^2/3$ implies $p(z) \prec (1 + z)/(1 - z)$ or $(2 + z)/(2 - z)$ or $1 + z$. We also determine the conditions on $\beta$ so that $p(z) \prec (1 + z)/(1 - z)$ or $1 + 4z/3 + 2z^2/3$, when $p(z) + \beta zp'(z)/p(z) \prec 1 + 4z/3 + 2z^2/3$. Section 3 of the paper investigates the sharp conditions on $\beta$ so that $1 + \beta zp'(z)/p^2(z) \prec 1 + 4z/3 + 2z^2/3$ $(n = 0, 1, 2)$ implies $p(z) \prec e^z$. Similarly, in Section 4, we consider differential implications with the superordinate function $e^z$ replaced by the superordinate function $\sqrt{1 + z}$. In addition to this, condition on $\beta$ is determined so that $p(z) \prec \sqrt{1 + z}$ when $p(z) + \beta zp'(z)/p(z) \prec 1 + 4z/3 + 2z^2/3$. In Section 5, we give applications of our results which will yield sufficient conditions for $f \in \mathcal{A}$ to belong to the various subclasses of starlike functions.

The following results will be required in our investigation.

**Lemma 1.1** ([8, Corollary 3.4h, p. 135]). Let $q$ be univalent in $D$, and let $\phi$ be analytic in a domain $D$ containing $q(D)$. Let $qz'(z)\phi(q(z))$ be starlike. If $p$ is analytic in $D$, $p(0) = q(0)$ and satisfies $zp'(z)\phi(p(z)) \prec qz'(z)\phi(q(z))$, then $p \prec q$ and $q$ is the best dominant.

The following is a more general version of the above lemma.
Lemma 1.2 ([8, Theorem 3.4, p. 134]). Let \( q \) be univalent in \( \mathbb{D} \) and let \( \varphi \) and \( \nu \) be analytic in a domain \( D \) containing \( q(\mathbb{D}) \) with \( \varphi(w) \neq 0 \) when \( w \in q(\mathbb{D}) \). Set \( Q(z) := zq'(z)\varphi(q(z)) \), \( h(z) := \nu(q(z)) + Q(z) \). Suppose that (i) either \( h \) is convex or \( Q(z) \) is starlike univalent in \( \mathbb{D} \) and (ii) \( \text{Re}(zh'(z)/Q(z)) > 0 \) for \( z \in \mathbb{D} \). If \( p \) is analytic in \( \mathbb{D} \), \( p(0) = q(0) \) and satisfies
\[
(1) \quad \nu(p(z)) + zp'(z)\varphi(p(z)) < \nu(q(z)) + zq'(z)\varphi(q(z)),
\]
then \( p < q \) and \( q \) is the best dominant.

Lemma 1.3 ([8, Corollary 3.4a, p. 120]). Let \( q \) be analytic in \( \mathbb{D} \) and \( \phi \) be analytic in a domain \( D \) containing \( q(\mathbb{D}) \) and suppose (i) \( \text{Re}(\phi(q(z))) > 0 \) and either (ii) \( q \) is convex, or (iii) \( Q(z) = zq'(z)\phi(q(z)) \) is starlike. If \( p \) is analytic in \( \mathbb{D} \), \( p(0) = q(0) \), \( p(\mathbb{D}) \subset D \) and \( p(z) + zp'(z)\phi(p(z)) < q(z) \), then \( p < q \).

2. Results associated with starlikeness

Let \( p \) be an analytic function in \( \mathbb{D} \) with \( p(0) = 1 \). In the first result, conditions on \( \beta \) are obtained so that the subordination
\[
1 + \beta z p'(z) < 1 + \frac{4z}{3} + \frac{2z^2}{3}
\]
implies \( p(z) < (2 + z)/(2 - z) \) or \( 1 + (1 - \alpha)z \) or \( 1 + (1 + 2\alpha)z)/(1 - z) \), \((0 \leq \alpha < 1)\).

Theorem 2.1. Let \( \beta_0 \approx 1.90987 \) be the root of the equation \( 9 + 47\beta + 90\beta^2 - 216\beta^3 + 81\beta^4 = 0 \). Let \( p \) be an analytic function defined on \( \mathbb{D} \) with \( p(0) = 1 \) satisfying
\[
1 + \beta z p'(z) < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then the following sharp results hold:
\[
\begin{align*}
(\text{a}) & \quad \text{If } \beta \leq -4.5 \text{ or } \beta \geq \beta_0, \text{ then } p(z) < (2 + z)/(2 - z). \\
(\text{b}) & \quad \text{If } |\beta| \geq 2/(1 - \alpha), (0 \leq \alpha < 1), \text{ then } p(z) < 1 + (1 - \alpha)z. \\
(\text{c}) & \quad \text{If } \beta \leq -4/(1 - \alpha) \text{ or } \beta \geq 4/(3(1 - \alpha)), (0 \leq \alpha < 1), \text{ then } p(z) < (1 + (1 - 2\alpha)z)/(1 - z).
\end{align*}
\]

Proof. Define the function \( q : \mathbb{D} \to \mathbb{C} \) by \( q(z) = (1 + Az)/(1 + Bz) \), \((-1 \leq B < A \leq 1)\) with \( q(0) = 1 \). Let us define \( \varphi(w) = \beta \) and \( Q(z) = zq'(z)\varphi(q(z)) \).

Since \( q \) is the convex univalent function, \( Q \) is starlike in \( \mathbb{D} \). It follows from Lemma 1.1, that the subordination
\[
1 + \beta z p'(z) < 1 + \beta zq'(z)
\]
implies \( p(z) \prec q(z) \). The theorem is proved by computing \( \beta \) so that
\[
(2) \quad 1 + \frac{4z}{3} + \frac{2z^2}{3} < 1 + \beta z q'(z) = 1 + \frac{\beta(A - B)z}{(1 + Bz)^2} := h(z).
\]

Set \( \psi(z) = 1 + 4z/3 + 2z^2/3 \). Clearly, \( \psi(\mathbb{D}) = \left\{ w \in \mathbb{C} : | -2 + \sqrt{6w - 2}| < 2 \right\} \).

The subordination \( \psi(z) \prec h(z) \) holds if \( \partial h(\mathbb{D}) \subset \mathbb{C} \setminus \psi(\mathbb{D}) \). Thus, by using
A calculation shows that if
\[ x = \frac{6\beta(A - B)e^{it}}{(1 + Be^{it})^2} - 2 \]
we have
\[ \left| \sqrt{4 + \frac{6\beta(A - B)e^{it}}{(1 + Be^{it})^2} - 2} \right| \geq 2. \]

Set
\[ w = u + iv = 4 + (6\beta(A - B)e^{it})/(1 + Be^{it})^2. \]

Then, condition (3) holds if \(|\sqrt{w} - 2| \geq 2\) which is same as \(|w| \geq 4\text{Re}(\sqrt{w})\).

On further simplification, we get
\[ (u^2 + v^2 - 8u + 2) \geq 0. \]

(a) Take \(A = 1/2, B = -1/2\) in (4). Then
\[ u = 4 + \frac{24\beta(5\cos t - 4)}{(5 - 4\cos t)^2}, \quad v = \frac{72\beta\sin t}{(5 - 4\cos t)^2}. \]

So, (5) reduces to
\[ \frac{-768}{(5 - 4\cos t)^4}(1921 - 3712\beta + 2376\beta^2 - 432\beta^4 - 80(37 - 69\beta + 36\beta^2)\cos t \]
\[ + 16(83 - 132\beta + 36\beta^2)\cos 3t - 320\cos 3t + 320\beta\cos 3t + 32\cos 4t \geq 0. \]

We need to find the values of \(\beta\) for which \(f(x) \geq 0\) in the interval \(-1 \leq x \leq 1\), where \(x = \cos t\) and
\[ f(x) = -(1921 - 3712\beta + 2376\beta^2 - 432\beta^4 - 80(37 - 69\beta + 36\beta^2)x \]
\[ + 16(83 - 132\beta + 36\beta^2)(2x^2 - 1) - 320(4x^3 - 3x) \]
\[ + 320\beta(4x^3 - 3x) + 32(8x^4 - 8x^2 + 1)). \]

A calculation shows that
\[ f'(x) = -16(-5 + 4x)(25 + 16x^2 - 57\beta + 36\beta^2 + 20x(-2 + 3\beta)) = 0 \]
if \(x = x_1 = 5/4\) or \(x = x_2 = (10 - 15\beta - 3\sqrt{-8\beta + 9\beta^2})/8\) or \(x = x_3 = (10 - 15\beta + 3\sqrt{-8\beta + 9\beta^2})/8\). Note that \(-1 \leq x_2, x_3 \leq 1\) if and only if \(\beta > 8/9\). These observations lead to two cases:

Case 1: \(\beta > 8/9\). In this case, \(f''(x_2) < 0\) and \(f''(x_3) > 0\). Thus \(f(x)\) attains its minimum value at \(x = x_3\), it follows that \(f(x) \geq 0\) for \(-1 \leq x \leq 1\) if and only if
\[ f(x_3) = \frac{27\beta^2}{2} \left( 24 + 153\beta^2 + 40\sqrt{-8\beta + 9\beta^2} - 3\beta(68 + 15\sqrt{-8\beta + 9\beta^2}) \right) \geq 0, \]
which is possible if \(\beta \geq \beta_0\). Hence \(p(z) < q(z)\) if \(\beta \geq \beta_0 \approx 1.90987\).

Case 2: \(\beta \leq 8/9\). In this case, \(f'(1) \geq 0\), \(f'(-1) \geq 0\) and \(f'(x)\) has no zero in \([-1, 1]\). Hence by Intermediate Value Theorem, \(f'(x) \geq 0\) for \(-1 \leq x \leq 1\). Thus, \(f(x) \geq 0\) for \(-1 \leq x \leq 1\) if and only if
\[ f(-1) = 27(-3 + 2\beta)^3(9 + 2\beta) \geq 0, \]
which is possible if $\beta \leq -4.5$. Hence $p(z) \prec q(z)$ if $\beta \leq -4.5$. This completes the proof for part (a).

(b) Take $A = 1 - \alpha$, $B = 0$, $(0 \leq \alpha < 1)$ in (4). Then

$$u = 4 + 6\beta(1 - \alpha)\cos t, \quad v = 6\beta(1 - \alpha)\sin t.$$ 

So, (5) takes the following form

$$g(t) := 48(27\beta^4(1 - \alpha)^4 - 72\beta^2(1 - \alpha)^2 - 16 - 64\beta(1 - \alpha)\cos t) \geq 0.$$ 

We need to find all possible values of $\beta$ for which $g(t)$ is non negative for $t \in [-\pi, \pi]$. Clearly, $g(t)$ attains its minimum value at $t = 0$ if $\beta > 0$ and $t = \pm\pi$ if $\beta < 0$. If $\beta > 0$, then $g(t) \geq 0$ if and only if

$$g(0) = 48(-2 + \beta(1 - \alpha))(2 + 3\beta(1 - \alpha))^3 \geq 0$$

which is true if $\beta \geq 2/(1 - \alpha)$. Next if $\beta < 0$, then $g(t) \geq 0$ if and only if

$$g(\pi) = 48(2 + \beta(1 - \alpha))(-2 + 3\beta(1 - \alpha))^3 \geq 0$$

which is possible if $\beta \leq -2/(1 - \alpha)$. Hence $p(z) \prec q(z)$ if $|\beta| \geq 2/(1 - \alpha)$.

(c) Take $A = 1 - 2\alpha$, $B = -1$, $(0 \leq \alpha < 1)$ in (4). Then, we get

$$u = 4 - 3\beta(1 - \alpha)\sin^2 t/2, \quad v = 0.$$ 

So, (5) reduces to

$$(u^2 - 8u)^2 - 64u^2 \geq 0,$$

which on further simplification becomes $u(u - 16) \geq 0$ which implies that

$$(-4\sin^2 t/2 + 3\beta(1 - \alpha))(\beta(1 - \alpha) + 4\sin^2 t/2) \geq 0$$

which is possible if $\beta \geq 4/3(1 - \alpha)$ or $\beta \leq -4/(1 - \alpha)$. This completes the proof for (c). □

Next result depicts the conditions on $\beta$ so that the subordination

$$1 + \beta z p'(z) / p(z) \prec 1 + z + 2z^2$$

implies $p(z) \prec (1 + z)/(1 - z)$ or $1 + z$ where $p$ is an analytic function in $\mathbb{D}$ with $p(0) = 1$.

**Theorem 2.2.** Let $p$ be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying

$$1 + \beta z p'(z) / p(z) \prec 1 + z + 2z^2,$$

then the following sharp results hold:

(a) If $|\beta| \geq \sqrt{(4\sqrt{3} + 8)/(3\sqrt{3})} \approx 1.6947$, then $p(z) \prec (1 + z)/(1 - z)$.

(b) If $\beta \geq 4$ or $\beta \leq -2$, then $p(z) \prec 1 + z$. 

Proof. Let the function \( q : \mathbb{D} \to \mathbb{C} \) be defined by \( q(z) = (1 + Az)/(1 + Bz) \), \((-1 \leq B < A \leq 1) \) with \( q(0) = 1 \). Let us define \( \varphi(w) = \beta/w \) and \( Q(z) = z\varphi'(z)\varphi(q(z)) = \beta(A - B)z/(1 + Az)(1 + Bz) \). A computation shows that

\[
\frac{zQ'(z)}{Q(z)} = \frac{1 - ABz^2}{(1 + Az)(1 + Bz)}.
\]

Thus with \( z = re^{it}, r \in (0, 1), t \in [-\pi, \pi], \) yields

\[
\Re\left( \frac{1 - ABz^2}{(1 + Az)(1 + Bz)} \right) = \frac{(1 - ABr^2)(1 + (A + B)r \cos t + ABr^2)}{|1 + Are^{it}|^2|1 + Bre^{it}|^2}.
\]

Since \( 1 + ABr^2 + (A + B)r \cos t \geq (1 - Ar)(1 - Br) > 0 \) for \( A + B \geq 0 \) and similarly, \( 1 + ABr^2 + (A + B)r \cos t \geq (1 + Ar)(1 + Br) > 0 \) for \( A + B \leq 0 \), it follows that \( Q(z) \) is starlike in \( \mathbb{D} \). An application of Lemma 1.1 reveals that the subordination

\[
1 + \beta \frac{z\varphi'(z)}{p(z)} < 1 + \beta \frac{z\varphi(q(z))}{q(z)}
\]

implies \( p(z) \prec q(z) \). Now our result is established if we prove

\[
1 + \frac{4z^2}{3} + \frac{2z^2}{3} < 1 + \beta \frac{z\varphi'(z)}{q(z)} = 1 + \frac{\beta(A - B)z}{(1 + Az)(1 + Bz)} := h(z).
\]

Let \( \psi(z) = 1 + 4z/3 + 2z^2/3 \). Then \( \psi(\mathbb{D}) = \{w \in \mathbb{C} : | -2 + \sqrt{6w - 2} | < 2\} \). The subordination \( \psi(z) \prec h(z) \) holds if \( \partial h(\mathbb{D}) \subset \mathbb{C} \setminus \psi(\mathbb{D}) \). Thus, by using the definition of \( h \) as given in (6), the subordination \( \psi(z) \prec h(z) \) holds if for \( t \in [-\pi, \pi], \) we have

\[
\left| \left( \sqrt{4 + \frac{6\beta(A - B)e^{it}}{(1 + Ae^{it})(1 + Be^{it})}} - 2 \right) \right| \geq 2.
\]

Set

\[
w = u + iv = 4 + (6\beta(A - B)e^{it})/(1 + Ae^{it})(1 + Be^{it})
\]

Then, proceeding as in Theorem 2.1, we have to deduce (5).

(a) Take \( A = 1, B = -1 \) in (7). Then \( u = 4 \) and \( v = 6\beta/\sin t \). Substituting \( u \) and \( v \) in (5), we get

\[
\left( \frac{36\beta^2}{\sin^2 t} - 16 \right)^2 - 64 \left( 16 + \frac{36\beta^2}{\sin^2 t} \right) \geq 0.
\]

Our problem is now to find all possible values of \( \beta \) for which \( p(x) \geq 0 \) for \( x \in [-1, 1], \) where \( x = \sin t \) and \( p(x) = -16x^4 - 72x^2\beta^2 + 27\beta^4 \). Clearly, \( p(x) \geq -16 - 72\beta^2 + 27\beta^4 \geq 0 \) if \( |\beta| \geq \sqrt{(4/\sqrt{3} + 8)/(3\sqrt{2})} \approx 1.6947 \).

(b) Take \( A = 1, B = 0 \) in (7). Then, \( u = 4 + 3\beta \) and \( v = 3\beta \tan t/2 \). So, (5) becomes

\[
-3 \cot^2 t (3(32 + 64\beta + 48\beta^2 - 9\beta^4)) + 16(8 + 16\beta + 9\beta^2) \cos t + 32(1 + 2\beta) \cos 2t \geq 0.
\]
Now our problem is to find all values of $\beta$ for which $g(x)$ is non negative in the whole interval $-1 \leq x \leq 1$ where $x = \cos t$ and $g(x) = -3(3(32+64\beta+48\beta^2-9\beta^4)+16(8+16\beta+9\beta^2)x+32(1+2\beta)(2x^2-1))$. A calculation shows that $g'(x) = 0$ if $x = x_0 = (-8 - 16\beta - 9\beta^2)/(8(1 + 2\beta))$ and $g''(x) = -384(1 + 2\beta)$. Let us first assume that $\beta < -1/2$. In this case, $g''(x_0) > 0$. Thus, $\min g(x) = g(x_0) = 162\beta^4(2 + \beta)/(1 + 2\beta)$. Hence, $g(x)$ is non negative if and only if $g(x_0)$ is non negative which is possible only if $\beta \leq -2$. Let us next assume that $\beta \geq -1/2$. In this case, we get $g''(x) \leq 0$ so that $g'(x) \leq g'(-1) = -432\beta^2 \leq 0$ and hence $g(x)$ is decreasing function. Therefore, $g(x) \geq 0$ if and only if $g(1) = 3(-4 + \beta)(4 + 3\beta)^3 \geq 0$ which can happen only when $\beta \geq 4$. Hence we get our required result.

In the next result, the conditions on $\beta$ are derived so that the subordination

$$1 + \beta z p'(z) / p^2(z) < 1 + 4z / 3 + 2z^2 / 3$$

implies $p(z) \prec (1 + z)/(1 - z) or (2 + z)/(2 - z)$ or $1 + z$ where $p$ is an analytic function in $D$ with $p(0) = 1$.

**Theorem 2.3.** Let $\beta_0 \approx -1.90987$ be the smallest real root of $9 - 47\beta + 90\beta^2 + 216\beta^3 + 81\beta^4 = 0$. Let $p$ be an analytic function defined on $D$ with $p(0) = 1$ satisfying

$$1 + \beta z p'(z) / p^2(z) < 1 + 4z / 3 + 2z^2 / 3,$$

then the following sharp results hold:

(a) If $\beta \geq 4$ or $\beta \leq -4/3$, then $p(z) \prec (1 + z)/(1 - z)$.
(b) If $\beta \geq 9/2$ or $\beta \leq \beta_0$, then $p(z) \prec (2 + z)/(2 - z)$.
(c) If $\beta \geq 8$ or $\beta \leq -8/3$, then $p(z) \prec 1 + z$.

**Proof.** Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = (1 + Az)/(1 + Bz)$, $(-1 \leq B < A \leq 1)$ and consider the function $Q(z) = \beta z q'(z) / q^2(z) = \beta (A - B) z / (1 + Az)^2$. Consider

$$z Q'(z) / Q(z) = 1 - Az / 1 + Az.$$ 

Let $z = re^{it}$, $-\pi \leq t \leq \pi$, $0 < r < 1$. Then

$$\text{Re} \left( 1 - Az / (1 + Az) \right) = 1 - A^2 r^2 / (1 + Ar e^{it})^2 > 0.$$ 

Hence, $Q$ is starlike in $D$. Now it is easy to see that the subordination

$$1 + \beta z p'(z) / p^2(z) < 1 + \beta z q'(z) / q^2(z)$$

implies $p(z) \prec q(z)$ by Lemma 1.1. So our result will be proved if we can prove

$$\psi(z) := 1 + 4z / 3 + 2z^2 / 3 \prec 1 + \beta z q'(z) / q^2(z) = 1 + \beta (A - B) z / (1 + Az)^2 := h(z).$$
So, we only need to show that for $t \in [-\pi, \pi]$, the following condition holds
\[
\left| \left( \sqrt{4 + \frac{6\beta(A-B)e^{it}}{(1 + A e^{it})^2}} - 2 \right) \right| \geq 2.
\]
Let
\[
w = u + iv = 4 + \frac{6\beta(A-B)e^{it}}{(1 + A e^{it})^2}.
\]
Then, proceeding as in Theorem 2.1, we have to get (5).

(a) Take $A = 1, B = -1$ in (9). Then, $u = 4 + 3\beta \sec^2 t/2$ and $v = 0$. So, (5) reduces to $u(u - 16) \geq 0$. Now, it is easy to see that our target is to find conditions on $\beta$ such that $f(x) \geq 0$ for $-1 \leq x \leq 1$, where
\[
x = \cos \frac{t}{2}, \quad f(x) = (4x^2 + 3\beta)(\beta - 4x^2).
\]
Clearly, $f(x) \geq 0$ if $\beta \leq -4/3$ or $\beta \geq 4$.

(b) Take $A = 1/2, B = -1/2$ in (9). Then,
\[
u = 72\sin t \left( \frac{5 + 4 \cos t}{(5 + 4 \cos t)^2} \right), \quad v = \frac{72\sin t}{(5 + 4 \cos t)^2}.
\]

So, (5) reduces to
\[
\frac{76}{(5 + 4 \cos t)^2} \left( -1921 + 8\beta(-464 - 297\beta + 54\beta^3) - 80(37 + 69\beta + 36\beta^2) \cos t - 16(83 + 12\beta(11 + 3\beta)) \cos 2t - 320(1 + \beta) \cos 3t - 32 \cos 4t \right) \geq 0.
\]

We need to find the values of $\beta$ for which $g(x) \geq 0$ in the interval $-1 \leq x \leq 1$, where $x = \cos t$ and
\[
g(x) = -(5 + 4x)^4 - 16(5 + 4x)^2(4 + 5x)\beta - 72(5 + 4x)^2\beta^2 + 432\beta^4.
\]
A calculation shows that
\[
g'(x) = -16(5 + 4x)(5 + 4x)^2 + 3(19 + 20x)\beta + 36\beta^2 = 0
\]
if $x = x_1 = -5/4$ or $x = x_2 = (-10 - 15\beta - 3\sqrt{8\beta + 9\beta^2})/8$ or $x = x_3 = (-10 - 15\beta + 3\sqrt{8\beta + 9\beta^2})/8$. Note that $x_2, x_3$ are real numbers if and only if $\beta > 0$ or $\beta < -8/9$. These observations lead to three cases:

Case 1: $\beta < -8/9$. In this case, $g''(x_2) > 0$ and $g''(x_3) < 0$. Thus, $g(x)$ attains its minimum value at $x = x_2$, it follows that $g(x) \geq 0$ for $-1 \leq x \leq 1$ if and only if
\[
g(x_2) = \frac{27\beta^2}{2} \left( 24 + 40\sqrt{8\beta + 9\beta^2} + 3\beta(68 + 51\beta + 15\sqrt{8\beta + 9\beta^2}) \right) \geq 0,
\]
which is possible if $\beta \leq -1.90987$.

Case 2: $\beta \geq 0$. In this case, we get $g''(x) \leq 0$ so that $g'(x) \leq g'(1) = -16(1 - 3\beta + 36\beta^2) \leq 0$ and hence $g(x)$ is a decreasing function. Therefore, $g(x) \geq 0$ if and only if $g(1) = 27(-9 + 2\beta)(3 + 2\beta)^3 \geq 0$ which can happen only when $\beta \geq 9/2$. 
Case 3: $-8/9 < \beta < 0$. In this case, $f'(1) < 0$, $f'(-1) < 0$ and $f'(x)$ has no zero in $]-1, 1[$. Hence by Intermediate Value Theorem, $f'(x) < 0$ for $-1 \leq x \leq 1$. Thus $f(x) \geq 0$ for $-1 \leq x \leq 1$ if and only if

$$f(1) = 27(3 + 2\beta)^3(-9 + 2\beta) \geq 0,$$

which is possible if $\beta \leq -3/2$ or $\beta \geq 9/2$. But this is not possible as $-8/9 < \beta < 0$. Hence, $p(z) \prec q(z)$ if $\beta \geq 9/2$ or $\beta \leq -1.99087$.

(c) Take $A = 1, B = 0$ in (9). Then,

$$u = 4 + \frac{3\beta}{2\cos^2 t/2}, \quad v = 0.$$ 

So, (5) reduces to $p(x) \geq 0$, $x \in [-1, 1]$, where

$$x = \cos t, \quad p(x) = (-4 + \beta - 4x)(4 + 3\beta + 4x)^3.$$

Clearly, $p'(x) < 0$. So, $p(x) \geq 0$ if and only if $p(1) = (-8 + \beta)(8 + 3\beta)^3 \geq 0$ which is true if $\beta \geq 8$ or $\beta \leq -8/3$. Hence proved. □

In the following theorem, we find the conditions on $\beta$ so that $p(z) \prec 1 + 4z/3 + 2z^2/3$, whenever $p(z) + \beta z p'(z) p(z) \prec 1 + 4z/3 + 2z^2/3$.

**Theorem 2.4.** Let $p$ be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying

$$p(z) + \beta z p'(z) p(z) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}, \quad \beta > 0.$$ 

Then $p(z) \prec 1 + 4z/3 + 2z^2/3$.

**Proof.** Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = 1 + 4z/3 + 2z^2/3$ with $q(0) = 1$. Let us define $\phi(w) = \beta/w$ ($\beta > 0$). Consider

$$\text{Re} \phi(q(z)) = \beta \text{Re} \left( \frac{1}{q(z)} \right) > 0.$$ 

Next, define the function $Q$ as

$$Q(z) := zq'(z)\phi(q(z)) = \frac{\beta z q'(z)}{q(z)} = \frac{4\beta z(1 + z)}{3 + 4z + 2z^2}.$$ 

From definition of $Q$, we have

$$\frac{zQ'(z)}{Q(z)} = \frac{3 + 6z + 2z^2}{3 + 7z + 6z^2 + 2z^3} =: K(z).$$

For $t \in [-\pi, \pi]$, we have

$$\text{Re}(K(e^{it})) = \frac{1}{2} + \frac{5 + 4 \cos t}{29 + 40 \cos t + 12 \cos 2t}.$$
Now, we will find minimum value of \( f(x) \) for \(-1 \leq x \leq 1\), where 
\[
x = \cos t, \quad f(x) = \frac{5 + 4x}{29 + 40x + 12(2x^2 - 1)}.
\]
A calculation shows that \( f'(x) = 0 \) if \( x = x_1 = -(5 + \sqrt{3})/4 \) or \( x = x_2 = (-5 + \sqrt{3})/4 \). Note that \( x_1 < -1 \) and \( f''(x_2) < 0 \). Also note that \( f(-1) = 1 \) and \( f(1) = 1/9 \). So, \( f(x) \), \(-1 \leq x \leq 1\) attains its minimum value at \( x = 1 \).

Hence, \( \text{Re}(K(e^{it})) \geq 11/18 > 0, \) this shows that \( Q \) is starlike in \( \mathbb{D} \). The result now follows from Lemma 1.3.

We close this section by obtaining the conditions on \( \beta \) so that \( p(z) \prec (1 + z)/(1 - z) \), whenever
\[
p(z) + \beta \frac{zp'(z)}{p(z)} < 1 + \frac{4z}{3} + \frac{2z^2}{3}.
\]

**Theorem 2.5.** Let \( p \) be an analytic function defined on \( \mathbb{D} \) with \( p(0) = 1 \) satisfying
\[
p(z) + \beta \frac{zp'(z)}{p(z)} < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad \text{for} \quad \beta \geq 0.
\]

Then \( p(z) \prec (1 + z)/(1 - z) \).

**Proof.** For \( \beta = 0 \), result hold obviously. Let us assume that \( \beta > 0 \). Define the function \( q : \mathbb{D} \to \mathbb{C} \) by \( q(z) = (1 + z)/(1 - z) \). Also define \( \nu(w) = w \) and \( \varphi(w) = \beta / w \). Clearly, the functions \( \nu \) and \( \varphi \) are analytic in \( \mathbb{C} \) and \( \varphi(w) \neq 0 \).

Consider the functions \( Q \) and \( h \) defined as follows:
\[
Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta q'(z)q(z)}{q(z)} = \frac{2\beta z}{1 - z^2} \quad \text{and} \quad h(z) := \nu(q(z)) + Q(z) = q(z) + Q(z).
\]

Since the mapping \( z/(1 - z^2) \) maps \( \mathbb{D} \) onto the entire plane minus the two half lines \( 1/2 \leq y < \infty \) and \(-\infty < y \leq -1/2 \), \( Q(z) \) is starlike univalent in \( \mathbb{D} \). A computation shows that
\[
zh'(z) = q(z) + Q(1 + z)/(1 - z).
\]

Since the mapping \( zh'(z)/Q(z) \) maps \( \mathbb{D} \) onto the plane \( \text{Re} w > 0 \), all the conditions of Lemma 1.2 are fulfilled and hence it follows that \( p(z) \prec q(z) \). In order to complete the proof, we need to show that
\[
\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} < q(z) + \beta \frac{q'(z)q(z)}{q(z)} = 1 + z + \frac{2\beta z}{1 - z}.
\]

So, we only need to show that for \(-\pi \leq t \leq \pi \), the following condition holds
\[
\left| \left( -\frac{2 + \frac{12\beta e^{it}}{1 - e^{2it}} + \frac{6(1 + e^{it})}{1 - e^{it}} - 2}{1 - e^{2it}} \right) \right| \geq 2.
\]
Set
\[ w = u + iv = -2 + \frac{12\beta e^{it}}{1 - e^{2it}} + \frac{6(1 + e^{it})}{1 - e^{it}} \]
so that
\[ u = -2 \quad \text{and} \quad v = \frac{6(1 + \beta + \cos t)}{\sin t}. \]
Then, substituting the values of \( u \) and \( v \) in (5), we get
\[ \frac{144}{(\sin t)^4} \left( 4 + 3\beta (2 + \beta) + 6(1 + \beta) \cos t + 2 \cos 2t \right)^2 \geq 0 \]
which is possible for any \( \beta \). Hence, \( p(z) \prec q(z) \) if \( \beta \geq 0 \). □

3. Results associated with the function \( e^z \)

In this section, we compute the sharp conditions on \( \beta \) so that \( p(z) \prec e^z \), whenever
\[ 1 + \beta z p'(z) \quad \text{or} \quad 1 + \beta \frac{z p'(z)}{p(z)} \quad \text{or} \quad 1 + \beta \frac{z p'(z)}{p^2(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}, \]
where \( p \) is an analytic function defined on \( \mathbb{D} \) with \( p(0) = 1 \).

**Theorem 3.1.** Let \( p \) be an analytic function defined on \( \mathbb{D} \) and \( p(0) = 1 \). Let \( \beta \geq 2e/3 \) or \( \beta \leq -2e \). If the function \( p \) satisfies the subordination
\[ 1 + \beta z p'(z) \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}, \]
then \( p \) also satisfies the subordination \( p(z) \prec e^z \). The result is sharp.

**Proof.** Let \( q \) be the convex univalent function defined by \( q(z) = e^z \). Then clearly, \( \beta z q'(z) \) is starlike in \( \mathbb{D} \). If the subordination
\[ 1 + \beta z p'(z) \prec 1 + \beta z q'(z) \]
is satisfied, then \( p(z) \prec q(z) \) by Lemma 1.1. It suffices to show that
\[ 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta z q'(z) = 1 + \beta z e^z := h(z). \]
Set \( \psi(z) = 1 + 4z/3 + 2z^2/3. \) Clearly, \( \psi(\mathbb{D}) = \{ w \in \mathbb{C} : |w - 2 + \sqrt{6w - 2}| < 2 \}. \)
The subordination \( \psi(z) \prec h(z) \) holds if \( \partial h(\mathbb{D}) \subset \mathbb{C} \setminus \psi(\mathbb{D}) \). Thus, by using the definition of \( h \) as given in (10), the subordination \( \psi(z) \prec h(z) \) holds if for \( t \in [-\pi, \pi] \), we have
\[ |\sqrt{4 + 6\beta e^{it}e^{zit}} - 2| \geq 2. \]
Set \( w = u + iv = 4 + 6\beta e^{it}e^{zit} \). Then, we only need to show that \( |\sqrt{w} - 2| \geq 2 \) which is same as \( |w| \geq 4 \text{Re}(\sqrt{w}) \). On further simplification, we get
\[ (u^2 + v^2 - 8u)^2 - 64(u^2 + v^2) \geq 0. \]
Clearly, \( u = 4 + 6\beta e^{\cos t} \cos(t + \sin t) \) and \( v = 6\beta e^{\cos t} \sin(t + \sin t) \). Our problem is now to find all possible values of \( \beta \) for which \( f(t) \geq 0 \) for \( t \in [-\pi, \pi] \), where

\[
f(t) = -16 - 72\beta^2 e^{2\cos t} + 27\beta^4 e^{4 \cos t} - 64\beta e^{\cos t} \cos(t + \sin t).
\]

Since \( f(t) \) is an even function of \( t \). It suffices to find the condition on \( \beta \) for which \( f(t) \geq 0 \) for \( t \in [0, \pi] \). Note that

\[
f(0) = (-2 + e\beta)(2 + 3e\beta)^3 \quad \text{and} \quad f(\pi) = \frac{-(2e - 3\beta)^3(2e + \beta)}{e^4}.
\]

So, \( f(0) \geq 0 \) and \( f(\pi) \geq 0 \) if \( -2e \leq \beta \leq 2e/3 \). If \( -2e \leq \beta \leq 2e/3 \), then \( f \) is a decreasing function of \( t \) and since \( f(\pi) \geq 0 \), we conclude that \( f(t) \geq 0 \) for \( t \in [0, \pi] \) if \( \beta \leq -2e \) or \( \beta \geq 2e/3 \). \( \square \)

**Theorem 3.2.** If \( p \) is an analytic function defined on \( D \) with \( p(0) = 1 \) satisfying the subordination

\[
1 + \beta \frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad \text{for} \quad |\beta| \geq 2
\]

then \( p \) also satisfies the subordination \( p(z) \prec e^z \). The result is sharp.

**Proof.** Let the function \( q : \mathbb{D} \to \mathbb{C} \) be defined by \( q(z) = e^z \). Let us define \( \varphi(w) = \beta/w \) and \( Q(z) = zq'(z)\varphi(q(z)) = \beta z \). Clearly, \( Q(z) \) is starlike in \( \mathbb{D} \). An application of Lemma 1.1 reveals that the subordination

\[
1 + \beta \frac{zp'(z)}{p(z)} \prec 1 + \beta \frac{zq'(z)}{q(z)}
\]

implies \( p(z) \prec q(z) \). Now, our result is established if we prove

\[
\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta \frac{zq'(z)}{q(z)} = 1 + \beta z := h(z).
\]

Since the subordination \( \psi(z) \prec h(z) \) holds if \( \partial h(\mathbb{D}) \subset \mathbb{C} \setminus \overline{\psi(\mathbb{D})} \), we only need to show that for \( t \in [-\pi, \pi] \),

\[
\left| \sqrt{4 + 6\beta e^t} - 2 \right| \geq 2.
\]

Set \( w = u + iv = 4 + 6\beta e^t \) so that \( u = 4 + 6\beta \cos t \) and \( v = 6\beta \sin t \). Then, proceeding as in Theorem 3.1, we need to show that (12) holds. After substituting the values of \( u \) and \( v \) in (12), we need to find the values of \( \beta \) for which \( g(t) \geq 0 \) for \( t \in [-\pi, \pi] \), where

\[
g(t) = -16 - 72\beta^2 + 27\beta^4 - 64\beta \cos t.
\]

Note that \( g(t) \) is an even function of \( t \). So, we only need to consider \( g(t) \) for \( t \in [0, \pi] \). Also note that \( g'(t) = 64\beta \sin t \). Let us first assume that \( \beta > 0 \). In this case, \( g(t) \) is an increasing function. Therefore, \( g(t) \geq 0 \) if and only if \( g(0) = (-2 + \beta)(2 + 3\beta)^3 \geq 0 \) which can happen only when \( \beta \geq 2 \). Let us next assume that \( \beta < 0 \). In this case, \( g(t) \) being decreasing function, is non negative
if and only if $g(\pi) = (2 + \beta)(-2 + 3\beta)^3$ is non negative which is possible if $\beta \leq -2$. Hence, $p(z) < q(z)$ if $|\beta| \geq 2$. \hfill \Box

**Theorem 3.3.** Let $p$ be an analytic function defined on $D$ and $p(0) = 1$. Let $\beta \geq 2e$ or $\beta \leq -2e/3$. If the function $p$ satisfies the subordination

$$1 + \beta \frac{zp'(z)}{p'(z)} < 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

then $p(z) \prec e^z$. The result is sharp.

**Proof.** Define the function $q : D \to \mathbb{C}$ by $q(z) = e^z$ and consider the function $Q(z) = \beta z q'(z)/q^2(z) = \beta ze^{-z}$. For $z = x + iy \in D$, we have

$$\text{Re} \left( \frac{zQ'(z)}{Q(z)} \right) = \text{Re}(1-z) = 1 - x > 0.$$ 

Hence, $Q$ is starlike in $D$. Now, it is easy to see that by Lemma 1.1, the subordination

$$1 + \beta \frac{zp'(z)}{p'(z)} < 1 + \beta \frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. So, our result will be proved if we can prove

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} + 1 + \beta \frac{zq'(z)}{q^2(z)} = 1 + \beta ze^{-z} := h(z).$$

Thus, we only need to show that $\partial h(D) \subset \mathbb{C} \setminus \{0\}$ which is equivalent to show that for $t \in [-\pi, \pi]$,

$$\left| \frac{\sqrt{4 + 6\beta e^{it}e^{-e^{-t}}}}{(1 + \beta e^{it}e^{-e^{-t}})} - 2 \right| \geq 2.$$ 

Set $w = u + iv = 4 + 6\beta e^{it}e^{e^{-t}}$. Then, proceeding as in Theorem 3.1, we need to prove (12). Clearly, $u = 4 + 6\beta e^{-\cos t} \cos(t - \sin t)$ and $v = 6\beta e^{-\cos t} \sin(t - \sin t)$. Our problem reduces to find all possible values of $\beta$ for which $k(t)$ is non negative in $[-\pi, \pi]$, where

$$k(t) = -16 - 72\beta^2 e^{-2 \cos t} + 27\beta^4 e^{-4 \cos t} - 64\beta e^{-\cos t} \cos(t - \sin t).$$

Observe that $k(-t) = k(t)$ for $t \in [-\pi, \pi]$. Thus, it is sufficient to find the values of $\beta$ for which $k(t)$ is non negative in $[0, \pi]$. Note that

$$k(0) = \frac{(2e + \beta)(2e + 3\beta)^3}{e^4} \quad \text{and} \quad k(\pi) = (2 + e\beta)(-2 + 3e\beta)^3.$$ 

Clearly, $k(0)$ and $k(\pi)$ both are non negative if $\beta \leq -2e/3$ or $\beta \geq 2e$. Also, if $\beta \leq -2e/3$ or $\beta \geq 2e$, then $k$ is an increasing function of $t$ and $k(0)$ is non negative. Hence, $k(t) \geq 0$ for $t \in [0, \pi]$ if $\beta \leq -2e/3$ or $\beta \geq 2e$. \hfill \Box
4. Results associated with the lemniscate of Bernoulli

In this section, we compute the conditions on $\beta$ so that $p(z) \prec \sqrt{1 + z}$, whenever

$$1 + \beta \frac{zp'(z)}{p^k(z)} \quad (k = 0, 1, 2) \quad \text{or} \quad p(z) + \beta \frac{zp'(z)}{p(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3},$$

where $p$ is an analytic function defined on $\mathbb{D}$ with $p(0) = 1$.

**Theorem 4.1.** Let $\beta \geq 4\sqrt{2}$. Let $p$ be an analytic function defined on $\mathbb{D}$ with $p(0) = 1$ satisfying

$$1 + \beta zp'(z) \prec 1 + 4z^3 + 2z^2,$$

then $p(z) \prec \sqrt{1 + z}$. The result obtained is sharp.

**Proof.** Define the function $q : \mathbb{D} \to \mathbb{C}$ by $q(z) = \sqrt{1 + z}$ with $q(0) = 1$. Since $q(\mathbb{D}) = \{w : |w^2 - 1| < 1\}$ is the right half of the lemniscate of Bernoulli, $q(\mathbb{D})$ is a convex set and hence $q$ is convex and $zq'(z)$ is starlike in $\mathbb{D}$. It follows from Lemma 1.1, that the subordination

$$1 + \beta zp'(z) \prec 1 + \beta zq'(z)$$

implies $p(z) \prec q(z)$. Now, our result is established if we prove the following:

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta zq'(z) = 1 + \frac{\beta z}{2\sqrt{1 + z}} := h(z).$$

Now, proceeding as in earlier sections, it is enough to show that $\partial h(\mathbb{D}) \subset \mathbb{C} \setminus \psi(\mathbb{D})$ which is equivalent to show that for $t \in [-\pi, \pi]$,

$$\left| \sqrt[4]{4 + \frac{3\beta e^{it}}{\sqrt{1 + e^{it}}} - 2} \right| \geq 2.$$

Taking $w = u + iv = 4 + 3\beta e^{it}/(\sqrt{1 + e^{it}})$. Then, we only need to show that

$$(13) \quad (u^2 + v^2 - 8u)^2 - 64(u^2 + v^2) \geq 0.$$

A calculation shows that

$$u = 4 + \frac{3\beta \cos(3t/4)}{\sqrt{2} \cos t/2} \quad \text{and} \quad v = \frac{3\beta \sin(3t/4)}{\sqrt{2} \cos t/2}.$$  

Using these values in (13), our problem reduces to find all possible values of $\beta$ for which $f(t) \geq 0$ for $t \in [-\pi, \pi]$, where

$$f(t) = -\frac{3}{4} \left( 512 - 27\beta^4 + 512 \cos t + 64\beta(9\beta \cos(t/2) + 16\sqrt{2} \cos^{3/2}(t/2) \cos(3t/4)) \right).$$
Note that \( f(t) = f(-t) \) for any \( t \), so it is sufficient to consider the interval \( 0 \leq t \leq \pi \). Also note that \( f''(t) \geq 0 \) for \( \beta > 0 \), so \( f(t) \) attains minimum value at \( t = 0 \). Clearly,
\[
    f(0) = \frac{-3}{4}(1024 + 1024\sqrt{2}\beta + 576\beta^2 - 27\beta^4) \geq 0 \quad \text{for} \quad \beta \geq 4\sqrt{2}.
\]

Thus, \( f(t) \geq 0 \) if \( \beta \geq 4\sqrt{2} \). This completes the proof. \( \square \)

**Theorem 4.2.** Let \( \beta \leq -4 \) or \( \beta \geq 8 \). Let \( p \) be an analytic function defined on \( \mathbb{D} \) with \( p(0) = 1 \) satisfying
\[
    1 + \beta z p'(z) \prec 1 + 4z^3 + 2z^2/3,
\]
then \( p(z) \prec \sqrt{1 + z} \). The result obtained is sharp.

**Proof.** Let the function \( q : \mathbb{D} \to \mathbb{C} \) be defined by \( q(z) = \sqrt{1 + z} \) with \( q(0) = 1 \).
Let us define \( \psi(w) = \beta/w \) and \( Q(z) = zq'(z)q(z) = \beta z/(1 + z) \) which maps \( \mathbb{D} \) onto \( \text{Re } w < \beta/4 \). So, \( Q(z) \) is starlike in \( \mathbb{D} \). An application of Lemma 1.1 reveals that the subordination
\[
    1 + \beta z p'(z) \prec 1 + \beta z q'(z)\quad \text{implies } p(z) \prec q(z).
\]
Now, our result is established if we prove
\[
    (14) \quad \psi(z) := 1 + 4z/3 + 2z^2/3 < 1 + \beta z q'(z)/q(z) = 1 + \beta z/2(1 + z) := h(z).
\]

Hence, we only need to show that \( \partial h(\mathbb{D}) \subset \mathbb{C} \setminus \psi(\mathbb{D}) \) which is same as to show that for \( t \in [-\pi, \pi] \),
\[
    \sqrt{4 + 3\beta e^{it}}/1 + e^{it} - 2 \geq 2.
\]
Set \( w = u + iv = 4 + 3\beta e^{it}/(1 + e^{it}) \). Then, proceeding as in Theorem 4.1, our target is to prove (13). Clearly,
\[
    u = 4 + \frac{3\beta}{2} \quad \text{and} \quad v = \frac{3\beta}{2} \tan \frac{t}{2}.
\]

On substituting \( u \) and \( v \) in (13), we get
\[
    \frac{1}{16} \left( -64 + 9\beta^2 + 9\beta^2 \left( \frac{1 - x^2}{x^2} \right) \right)^2 - 16 \left( 8 + 3\beta \right)^2 + 9\beta^2 \left( \frac{1 - x^2}{x^2} \right) \geq 0,
\]
where \( x = \cos t/2 \). So, our problem reduces to find the values of \( \beta \) for which \( G(x) \geq 0 \) for \( x \in [0, 1] \), where
\[
    G(x) = -12288(1 + \beta)x^4 - 3456\beta^2x^2 + 81\beta^4.
\]
A calculation shows that
\[
    G'(x) = -768(9x\beta^2 + 64x^3(1 + \beta))
\]
and hence $G'(0) = G'(-3\beta/(8\sqrt{1-\beta})) = 0$. Let us first assume that $\beta \geq -1$. Then, $G(x)$ is a decreasing function of $x \in [0, 1]$. Consequently, we have $G(x) \geq 0$ for $x \in [0, 1]$ provided $G(1) = 3(-8 + \beta)(8 + 3\beta)^3 \geq 0$, which is equivalent to $\beta \geq 8$. Next, assume that $\beta < -1$. In this case, $G''(-3\beta/(8\sqrt{1-\beta})) = 13824/\beta^2 > 0$. Thus $G(x)$ attains its minimum value at $x = -3\beta/(8\sqrt{1-\beta})$, it follows that $G(x) \geq 0$ for $0 \leq x \leq 1$ if and only if

$$G(-3\beta/(8\sqrt{1-\beta})) = \frac{81\beta^2(4+\beta)}{1+\beta} \geq 0,$$

provided $\beta \leq -4$. Hence, $p(z) \prec q(z)$ for $\beta \leq -4$ or $\beta \geq 8$. □

**Theorem 4.3.** Let $p$ be an analytic function defined on $D$ and $p(0) = 1$. If the function $p$ satisfies the subordination

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \frac{4z}{3} + \frac{2z^2}{3}, \text{ for } \beta \geq 8\sqrt{2}$$

then $p(z) \prec \sqrt{1+z}$. The result is sharp.

**Proof.** Define the function $q : D \to \mathbb{C}$ by $q(z) = \sqrt{1+z}$ and consider the function $Q(z) = \beta zq'(z)/q^2(z) = \beta z/2(1+z)^{3/2}$. Clearly,

$$\frac{zQ'(z)}{Q(z)} = 1 - \frac{3z}{2(1+z)},$$

which maps $D$ onto plane $\text{Re } w > 1/4$. Hence, $Q$ is starlike in $D$. An application of Lemma 1.1 reveals that the subordination

$$1 + \beta \frac{zp'(z)}{p^2(z)} \prec 1 + \beta \frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. So, our result will be proved if we can prove

$$\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec 1 + \beta \frac{zq'(z)}{q^2(z)} = 1 + \beta \frac{z}{2(1+z)^{3/2}} := h(z).$$

So, we only need to show that $\partial h(D) \subset \mathbb{C} \setminus \psi(D)$ which is equivalent to show that for $t \in [-\pi, \pi]$,

$$\left| \sqrt{4 + \frac{3\beta e^t}{(1+e^t)^{3/2}}} - 2 \right| \geq 2.$$  

Set $w = u + iv = 4 + (3\beta e^t)/(1+e^t)^{3/2}$. Then, proceeding as in Theorem 4.1, we have to find $\beta$ so that (13) holds. Clearly,

$$u = 4 + 3\beta \frac{\cos t/4}{(2 \cos t/2)^{3/2}}, \quad v = 3\beta \frac{\sin t/4}{(2 \cos t/2)^{3/2}}.$$  

Our problem reduces to find all possible values of $\beta$ for which $k(t)$ is non negative in $[-\pi, \pi]$, where

$$k(t) = 3 \left\{ -16384 - 8192\sqrt{2}\beta \frac{\cos^4 t}{4 \sec^3 t} - 2304\beta^2 \sec^3 \frac{t}{2} + 27\beta^4 \sec^6 \frac{t}{2} \right\}.$$
Observe that \( k(-t) = k(t) \) for \( t \in [-\pi, \pi] \). Thus, it is sufficient to find the values of \( \beta \) for which \( k(t) \) is non negative in \([0, \pi]\). For \( \beta \geq 8\sqrt{2} \), \( k \) is an increasing function of \( t \) and \( k(0) = -768 - 384\sqrt{2}\beta - 108\beta^2 + 81\beta^4/64 \) is non negative. Hence, \( k(t) \geq 0, \ t \in [0, \pi] \) for \( \beta \geq 8\sqrt{2} \). □

**Theorem 4.4.** Let \( p \) be an analytic function defined on \( \mathbb{D} \) with \( p(0) = 1 \) satisfying

\[
p(z) + \beta \frac{zp'(z)}{p(z)} < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad \text{for} \quad \beta \geq 12
\]

then \( p(z) \prec \sqrt{1 + z} \).

**Proof.** Define the function \( q : \mathbb{D} \to \mathbb{C} \) by \( q(z) = \sqrt{1 + z} \). Consider the subordination

\[
p(z) + \beta \frac{zp'(z)}{p(z)} \prec q(z) + \beta \frac{zq'(z)}{q(z)}.
\]

Thus, in view of Lemma 1.2, the above subordination can be written as (1) by defining the functions \( \nu \) and \( \varphi \) as

\[
\nu(w) = w \quad \text{and} \quad \varphi(w) = \frac{\beta}{w}, \ (\beta \neq 0).
\]

Clearly, the functions \( \nu \) and \( \varphi \) are analytic in \( \mathbb{C} \) and \( \varphi(w) \neq 0 \). Let the functions \( Q(z) \) and \( h(z) \) be defined as follows:

\[
Q(z) := zq'(z)\varphi(q(z)) = \frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1 + z)} \quad \text{and} \quad h(z) := \nu(q(z)) + Q(z) = \sqrt{1 + z} + \frac{\beta z}{2(1 + z)} .
\]

Since the mapping \( Q(z) \) maps \( \mathbb{D} \) onto the plane \( \text{Re } w < \beta/4 \), \( Q(z) \) is starlike univalent in \( \mathbb{D} \). A computation shows that

\[
\frac{zh'(z)}{Q(z)} = \frac{\sqrt{1 + z}}{\beta} + \frac{1}{1 + z}.
\]

Now, the mapping \( 1/(1 + z) \) maps \( \mathbb{D} \) onto plane \( \text{Re } w > 1/2 \) and \( \text{Re}(\sqrt{1 + z}) > 0, \ z \in \mathbb{D} \). Therefore, \( \text{Re}(zh'/Q(z)) > 0, \ z \in \mathbb{D} \) if \( \beta > 0 \). Thus, all the conditions of Lemma 1.2 are satisfied and hence, it follows that \( p(z) \prec q(z) \). In order to complete the proof, we need to prove that

\[
\psi(z) := 1 + \frac{4z}{3} + \frac{2z^2}{3} \prec q(z) + \beta \frac{zq'(z)}{q(z)} = \sqrt{1 + z} + \frac{\beta z}{2(1 + z)} = h(z).
\]

So, we only need to show that \( \partial h(\mathbb{D}) \subset \mathbb{C} \setminus \psi(\mathbb{D}) \) which is equivalent to show that for \( t \in [-\pi, \pi] \),

\[
\sqrt{-2 + 6\sqrt{1 + e^t} + \frac{3\beta e^t}{1 + e^t} - 2} \geq 2.
\]
Thus, we have to show that
\[
\left| -2 + 6\sqrt{1 + e^{it}} + \frac{3\beta e^{it}}{1 + e^{it}} \right| \geq 16.
\]
Now,
\[
\left| -2 + 6\sqrt{1 + e^{it}} + \frac{3\beta e^{it}}{1 + e^{it}} \right| = \left| 6e^{it/4}\sqrt{2\cos\frac{t}{2}} + \frac{3\beta e^{it/2}}{2\cos\frac{t}{2}} - 2 \right|
\geq \Re \left( 6e^{it/4}\sqrt{2\cos\frac{t}{2}} + \frac{3\beta e^{it/2}}{2\cos\frac{t}{2}} - 2 \right)
= 6\cos\frac{t}{4}\sqrt{2\cos\frac{t}{2}} + \frac{3\beta}{2} - 2
\geq \frac{3\beta}{2} - 2 \geq 16 \text{ for } \beta \geq 12.
\]
Hence, \( p(z) < q(z) \) and this completes the proof.

5. Applications

In this section we give sufficient conditions for functions \( f \in A \) to belong to the various subclasses of starlike functions.

**Theorem 5.1.** Let \( f \in A \) and \( \beta_0 = \sqrt{(4\sqrt{3} + 8)/(3\sqrt{3})} \simeq 1.6947 \). Then following are the sufficient conditions for \( f \in S^* \).

1. The function \( f \) satisfies the subordination
\[
1 + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (|\beta| \geq \beta_0).
\]

2. The function \( f \) satisfies the subordination
\[
1 - \beta + \beta \frac{1}{f'(z)} < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \leq -4/3 \text{ or } \beta \geq 4).
\]

3. The function \( f \) satisfies the subordination
\[
\frac{zf'(z)}{f(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{z}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \geq 0).
\]

**Proof.** Let the function \( p : \mathbb{D} \to \mathbb{C} \) be defined by \( p(z) = zf'(z)/f(z) \). Then \( p \) is analytic in \( \mathbb{D} \) with \( p(0) = 1 \). A calculation shows that
\[
\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}.
\]
The results follow respectively from Theorems 2.2(a), 2.3(a) and 2.5.

**Theorem 5.2.** Let \( f \in A \) and \( \beta_0 = \sqrt{(4\sqrt{3} + 8)/(3\sqrt{3})} \simeq 1.6947 \). Then following are the sufficient conditions for \( z^2f'(z)/f''(z) \in P \).
(1) The function \( f \) satisfies the subordination
\[
1 + \beta \left( \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (|\beta| \geq \beta_0).
\]

(2) The function \( f \) satisfies the subordination
\[
\frac{z^2f'(z)}{f^2(z)} + \beta \left( \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \geq 0).
\]

**Proof.** The two parts of the theorem follows by taking \( p(z) = z^2f'(z)/f^2(z) \) in Theorems 2.2(a) and 2.5 respectively. \( \square \)

**Theorem 5.3.** Let \( f \in A \) and \( 0 \leq \alpha < 1 \).

1. Let \( \beta \leq -4/(1-\alpha) \) or \( \beta \geq 4/(1-\alpha) \). If the function \( f \) satisfies the subordination
\[
1 + \beta \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then \( f \in S^*(\alpha) \).

2. Let \( \beta \leq -9/2 \) or \( \beta \geq \beta_0 \), where \( \beta_0 \) is given by Theorem 2.1. If the function \( f \) satisfies the subordination
\[
1 + \beta \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then \( f \in S^*[1/2,-1/2] \).

3. Let \( \beta \leq \beta_0 \) or \( \beta \geq 9/2 \), where \( \beta_0 \) is given by Theorem 2.3. If the function \( f \) satisfies the subordination
\[
1 - \beta + \beta \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then \( f \in S^*[1/2,-1/2] \).

4. Let \( |\beta| \geq 2/(1-\alpha) \). If the function \( f \) satisfies the subordination
\[
1 + \beta \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then \( f \in S^*[1-\alpha,0] \).

5. Let \( \beta \leq -2 \) or \( \beta \geq 4 \). If the function \( f \) satisfies the subordination
\[
1 + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then \( f \in S^*[1,0] \).

6. Let \( \beta \leq -8/3 \) or \( \beta \geq 8 \). If the function \( f \) satisfies the subordination
\[
1 - \beta + \beta \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then \( f \in S^*[1,0] \).
Proof. The parts of the theorem are obtained by taking \( p(z) = zf'(z)/f(z) \) in Theorems 2.1(c), 2.1(a), 2.3(b), 2.1(b), 2.2(b) and 2.3(c) respectively. \( \square \)

**Theorem 5.4.** Let \( f \in A \) and \( 0 \leq \alpha < 1 \).

1. If \( f \) satisfies \( 1 + \beta zf''(z) < 1 + 4z/3 + 2z^2/3 \) \( (\beta \leq -4/(1 - \alpha) \) or \( \beta \geq 4/(1 - \alpha) ) \), then \( f' < (1 + (1 - 2\alpha)z)/(1 - z) \).
2. If \( f \) satisfies \( 1 + \beta zf''(z) < 1 + 4z/3 + 2z^2/3 \) \( (\beta \leq -9/2 \) or \( \beta \geq \beta_0 \), where \( \beta_0 \) is given by Theorem 2.1 \), then \( f' < (2 + z)/(2 - z) \).
3. If \( f \) satisfies \( 1 + \beta zf''(z) < 1 + 4z/3 + 2z^2/3 \) \((|\beta| \geq 2/(1 - \alpha)) \), then \( f' < 1 + (1 - \alpha)z \).
4. If \( f \) satisfies
\[
1 + \beta \left( \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \leq -2 \) or \( \beta \geq 4),
\]
then \( z^2 f'(z)/f^2(z) < 1 + z \).

Proof. The first three parts follows from Theorems 2.1(c), 2.1(a) and 2.1(b) respectively by taking \( p(z) = f'(z) \). Next, applying Theorem 2.2(b) to the function \( p(z) = z^2 f'(z)/f^2(z) \) yields the last part of the theorem. \( \square \)

Next theorem is an application of Theorem 2.4.

**Theorem 5.5.** Let \( f \in A \) and \( \beta > 0 \).

1. If \( f \) satisfies the subordination
\[
\frac{zf'(z)}{f(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then \( f \in S^*_C \).
2. If \( f \) satisfies
\[
\frac{z^2 f'(z)}{f^2(z)} + \beta \left( \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3},
\]
then
\[
\frac{z^2 f'(z)}{f^2(z)} < 1 + \frac{4z}{3} + \frac{2z^2}{3}.
\]

The three parts of the next theorem are application of Theorems 3.1, 3.2 and 3.3 respectively.

**Theorem 5.6.** Let \( f \in A \). Then following are the sufficient conditions for \( f \in S^*_C \).

1. Let \( \beta \leq -2e \) or \( \beta \geq 2e/3 \). The function \( f \) satisfies the subordination
\[
1 + \beta \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3}.
\]
(2) Let $|\beta| \geq 2$. The function $f$ satisfies the subordination
\[
1 + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3}.
\]

(3) Let $\beta \leq -\frac{2e}{3}$ or $\beta \geq \frac{2e}{3}$. The function $f$ satisfies the subordination
\[
1 - \beta + \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} < 1 + \frac{4z}{3} + \frac{2z^2}{3}.
\]

The two parts of the next theorem are application of Theorems 3.1 and 3.2 respectively.

**Theorem 5.7.** Let $f \in A$.

1. If $f$ satisfies $1 + \beta zf''(z) < 1 + 4z/3 + 2z^2/3$ ($\beta \leq -2e$ or $\beta \geq 2e/3$), then $f' \prec e^z$.
2. If $f$ satisfies
\[
1 + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (|\beta| \geq 2),
\]
then
\[
\frac{z^2 f'(z)}{f^2(z)} \prec e^z.
\]

The remaining results are application of Section 4.

**Theorem 5.8.** Let $f \in A$. Then following are the sufficient conditions for $f \in S_L^*$.  

1. The function $f$ satisfies the subordination
\[
1 + \beta \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \geq 4\sqrt{2}).
\]

2. The function $f$ satisfies the subordination
\[
1 + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \leq -4 \text{ or } \beta \geq 8).
\]

3. The function $f$ satisfies the subordination
\[
1 - \beta + \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \geq 8\sqrt{2}).
\]

4. The function $f$ satisfies the subordination
\[
\frac{zf'(z)}{f(z)} + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \geq 12).
\]

**Theorem 5.9.** Let $f \in A$.

1. If the function $f$ satisfies $1 + \beta zf''(z) < 1 + 4z/3 + 2z^2/3$, $\beta \geq 4\sqrt{2}$, then $f' \prec \sqrt{1 + z}$. 

(2) If the function \( f \) satisfies
\[
1 + \beta \left( \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \leq -4 \text{ or } \beta \geq 8),
\]
then
\[
\frac{z^2f'(z)}{f^2(z)} < \sqrt{1+z}.
\]

(3) If the function \( f \) satisfies
\[
\frac{z^2f'(z)}{f^2(z)} + \beta \left( \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) < 1 + \frac{4z}{3} + \frac{2z^2}{3} \quad (\beta \geq 12),
\]
then
\[
\frac{z^2f'(z)}{f^2(z)} < \sqrt{1+z}.
\]

References


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