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On the History of the Birth of Finsler Geometry at Göttingen

괴팅겐에서 핀슬러 기하가 탄생한 역사

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Arrivals of Hilbert and Minkowski at Göttingen put mathematical science there in full flourish. They further extended its strong mathematical tradition of Gauss and Riemann. Though Riemann envisioned Finsler metric and gave an example of it in his inaugural lecture of 1854, Finsler geometry was officially named after Minkowski's academic grandson Finsler. His tool to generalize Riemannian geometry was the calculus of variations of which his advisor Carathéodory was a master. Another Göttingen graduate Busemann regraded Finsler geometry as a special case of geometry of metric spaces. He was a student of Courant who was a student of Hilbert. These figures all at Göttingen created and developed Finsler geometry in its early stages. In this paper, we investigate history of works on Finsler geometry contributed by these frontiers.

Keywords: Finsler metric, Riemannian metric, calculus of variations, convexity, tensor calculus, P. Finsler, H. Busemann, C. Gauss, B. Riemann, H. Minkowski, D. Hilbert, C. Carathéodory, R. Courant.

MSC: 01A55, 01A60 ZDM: A30

1 Introduction

Finsler geometry is long forgotten field of differential geometry mainly because, as Riemann himself remarked in his inaugural lecture, it is hard to do any meaningful calculation with such generalized metric. During mid 20th century, there were only sporadic attempts in this field mainly by the groups of geometers of eastern Europe and Japan. Though many areas in differential geometry had blossomed in this period, Finsler geometry was hardly touched until 1990s.

It was S. S. Chern, a grand master of differential geometry in the 20th century, who revived this dormant field and lured geometers into this field. In [10], he reinterpreted his old theory [9] in the modern language: with the notion of fiber bun-

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dles and the theory of connections on them in his disposal, he can interpret his old Euclidean connections in the contemporary setting. He proposed a torsion free connection which is almost compatible with the metric. With this connection, he deduced second variation of the arc length function. This is the starting point of studying Finsler spaces from the modern point of view.

Upon his new interpretation on the connection, Chern wrote two historical introductions to Finsler geometry, one [11] in the Notices of the American Mathematical Society and another one [12] in a series of Contemporary Mathematics of the American Mathematical Society. He commented that Finsler space is a geometry of simple integral

$$\int_a^b L(\alpha(t), \alpha'(t))dt$$

and is as old as the calculus of variations. We will see why this is so in §3.3 and §4.1. It is hard to believe that the systematic study of the geometry of spaces with such generalized metrics is delayed by about 60 years.

Another historical remark can be found in the preface of the book [20] by H. Rund. Here he recognized earlier works done by G. Bliss, G. Landsberg, W. Blaschke before the inception of the Finsler geometry by P. Finsler. He also noted that Finsler geometry took a curious turn away from the calculus of variations. From 1925 and on, tensor calculus was applied to the Finsler geometry by J. Synge, J. Taylor and L. Berwald to name a few. Finsler geometry took another turn when É. Cartan introduced connection and covariant derivative. But Rund argued that Cartan's method is not desirable from the geometrical point of view.

In 1948, one of the creators of Finsler geometry, H. Busemann gave an invited one-hour lecture on the geometry of Finsler spaces [4] in the meeting of the American Mathematical Society. Here at the beginning, he gave a brief historical sketch of Finsler spaces. It is kind of a summary of his book [3] and reflects his own point of view.

Here is a mathematical genealogy of mathematicians who appear in this paper. F. Klein and F. von Lindemann are included for obvious reasons. Klein who spent the year 1871 as a lecturer at Göttingen and then moved to Erlangen in southern Germany in 1872 interacted with Minkowski and Hilbert. Lindemann was a supervisor of both Minkowski and Hilbert at Königsberg.

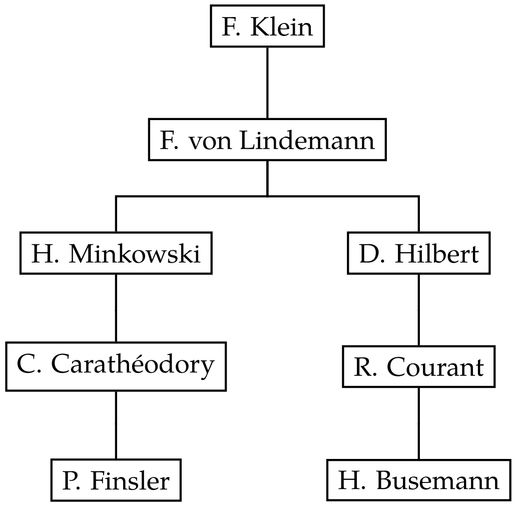

Table 1. Mathematical genealogy according to the doctral advisors

This paper is on the early history of Finsler geometry initiated at Göttingen in its most glorious days. In the sections below, we give biographical sketches of the last six mathematicians with regard to their contributions in Finsler geometry. Only Courant's work on this field is not known to the author. It is fair to begin with Riemann, because Riemann had Finsler metric in mind in his inaugural lecture.

The naissance of Riemannian geometry owes much to the sole work of B. Riemann during 1850's. Though the Riemann's work is regarded as a generalization of his thesis advisor C. Gauss's work[15] on the theory of curved surfaces, his imagination is far beyond the one by Gauss. In his inaugural lecture of 1854, he introduced the concept of n dimensional spaces and the generalized metrics on them. On the other hand, the creation of Finsler geometry is the culmination of the works of mathematicians at Göttingen, even though P. Finsler has his name in Finsler geometry.

In [26], author studied the development of differential geometry by E. Christoffel, G. Ricci-Curbastro and T. Levi-Civita during the late 19th and early 20th century. Author explained how the works of C. Gauss and B. Riemann can be applied to the equivalence problem of the differential 2 forms and ended up with absolute differential calculus or tensor calculus from the historical point of view. This was the only computational tools in differential geometry available in 1930's, though it is now somewhat outdated. So the association of Finsler geometry with highly complicated and time consuming tensor calculus is more or less historical.

Another reason is somewhat philosophical. According to the a priori philosophy of I. Kant who was a contemporary of C. Gauss, we are endowed with an inner product rather than a norm on the space we are living. That means we have a tool of measuring the angles between two vectors, in addition to a tool of measuring the length of vectors. While Riemannian metrics give us both measurements, Finsler metrics only give us measurement of length of vectors. And thus Finsler geometry is a natural way to go beyond Riemannian geometry.

In this paper, we have used widely accepted English expressions for geographical names and figures but we also put them in their native languages or in Latin between parentheses. For the sake of simplicity of appearance of the equations, we adopt the so called Einstein summation convention: repeated indices up and down means summation over all the values of the index. Numbers in parenthesis refer to equation numbers.

The Mathematics Genealogy Project [17] provided by North Dakota State University in association with American Mathematical Society is a good source for the ancestry of mathematicians according to their Ph. D advisors. A source of brief biographical information on the figures appeared in this article is the MacTutor History of Mathematics Archive [23]. We also refer the biographical data to the internet encyclopedia Wikipedia [25].

2 Bernhard Riemann

In order to study the beginning of Finsler geometry, we have to begin with the work of B. Riemann[1)] because it was Riemann who first conceived the idea of generalized metrics including Finsler metrics. His predecessor C. Gauss who laid the foundation of modern differential geometry, considered the metrics on two dimensional surfaces inherited from the Euclidean metric

$$ds^2 = \sum (dx^i)^2$$

of the three dimensional Euclidean space $\mathbb{R}^3$[15].

When he entered the University of Göttingen in 1846, he was supposed to study theology upon his father's will. Then the University of Göttingen was a poor place to learn mathematics. As was usual for his contemporaries, he spent two years at the University of Berlin. C. Jacobi, L. Dirichlet, J. Steiner and M. Eisenstein were among his teachers in Berlin. In particular, influence by Dirichlet was instrumental in mathematical development of Riemann. Returning to Göttingen in 1849, he prepared a thesis under the supervision by Gauss. In 1851, Riemann wrote his doctoral thesis on the theory of complex functions. Its title was *Foundations for a general theory of functions of a complex variable (Grundlagen für eine allgemeine Theorie der Funktionen einer*

1) Georg Friedrich Bernhard Riemann was born on September 17, 1826 in Breselenz, Germany and died July 20, 1866 at the aged of 39 in Selasca, Italy.

veränderlichen complexen Grösse).

In German educational system, Habilitation is the post-doctoral work needed to be a private lecturer (Privatdozent) at the universities. It is somewhat equivalent to the tenure system in the USA. The Habilitation consists of two parts: Habilitationsschrift and Habilitationsvortrag. Habilitationsschrift is the formal thesis that has to be presented to the faculty. Habilitationsvortrag is the informal lecture on the topic chosen by one of the faculty on the list prepared by the student. It is known as an inaugural lecture: a lecture before becoming a lecturer.

For Habilitationsschrift, in 1853, Riemann wrote a thesis on Fourier analysis. The title was *On the representability of a function by a trigonometric series (Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe)*. And finally in 1854, Riemann gave an inaugural lecture [19] entitled *On the hypotheses which underlie geometry (Über die Hypothesen welche der Geometrie zu Grunde liegen)*, which is considered to be the beginning of Riemannian geometry.

Here he introduced the concept of n-fold spaces and generalized metrics.

> $ds =$ the square root of an everywhere positive homogeneous of the second degree in the quantities dx, in which the coefficients are continuous functions of the quantities x.

The generalized metric should be positive homogeneous of the second degree in the quantities dx in order that the length of a curve is defined independent of the choice of parameters.

In a modern terminology, the length of a curve $\alpha(t), a \leq t \leq b$, can be defined by

$$\int_a^b L(\alpha(t), \alpha'(t))dt. \tag{1}$$

By change of variables, the positive homogeneity in the second variable guarantees that the integral is independent of the choice of the parameter t.

In the case of $L(x, y) = \sqrt{\sum g_{ij}(x)y^i y^j}$, the length of a curve $\alpha(t), a \leq t \leq b$, is

$$\int_a^b \sqrt{\sum g_{ij}(\alpha(t))(\alpha^i)'(t)(\alpha^j)'(t)}dt.$$

This is what we now call Riemannian metric and includes $ds = \sqrt{\sum (dx^i)^2}$ as a special case. It is clear that Riemann envisioned beyond Riemannian metric. He noticed that positivity and homogeneity are the conditions imposed on the general metric. He even gave an example of a Finsler metric:

> The line element can be expressed as the fourth root of a differential expression of the fourth degree.

He also commented why he wanted to restrict the generalized metrics as Riemannian ones:

> Investigation of this more general class would actually require no essentially different principles, but it would be rather time consuming and throw proportionally little new light on the study of Space, especially since the results cannot be expressed geometrically; I consequently restrict myself to those manifolds where the line element can be expressed by the square root of a differential expression of the second degree.

This paper was only posthumously in 1867 by his companion Dedekind and became a classic of mathematics. It was published again in 1919 by Weyl after the work of Einstein on the theory of general relativity. There are several versions of the English translation of his inaugural lecture including the one by M. Spivak in [21].

Riemann's posthumous paper had much impact on the development of tensor calculus by E. Christoffel, G. Ricci-Curbastro, T. Levi-Civita. The tensor calculus was instrumental on the advent of Einstein's theory of general relativity [13]. Only after the work of Einstein on the theory of general relativity, tensor calculus began to be widely accepted by mathematicians. The implication of Riemann's work in this direction was studied by author in [26].

Since 1930s, Finsler geometry made a sharp turn on the methodology. Newly developed tensor calculus was applied to the settings of Finsler geometry as Einstein did for his theory of relativity. It hindered the development of Finsler geometry because, as Riemann himself foretold, its calculation was too complicated to extract any meaningful consequences.

3 Forefathers

3.1 Minkowski

Hermann Minkowski was born on June 22, 1864 in Aleksota, Russia and died prematurely due to appendicitis on January 12, 1909 at Göttingen. His family moved to Königsberg[2)] then under the rule of Prussia in his childhood to avoid racial persecution in Russia. He entered the University at Königsberg in 1880. After spending 2 years at Berlin, he came back to Königsberg and first met his life-long friend and colleague D. Hilbert.

He was awarded his doctoral degree in 1885 under the guidance of F. von Lindemann. The title of his thesis was *Studies on the quadratic forms, determining the number*

2) It was a city in the monastic state of the Teutonic Knights, the Duchy of Prussia, the Kingdom of Prussia and Germany until 1946. It was annexed by the Soviet Union after the World War II and renamed Kaliningrad. It was a hometown of I. Kant, C. Goldbach, D. Hilbert to name a few and a birth place of the mathematical branches of topology and graph theory of L. Euler. But only few traces of the old city remain today.

of different forms with a given genus (Untersuchungen über quadratische Formen, Bestimmung der Anzahl verschiedener Formen, welche ein gegebenes Genus enthält). Minkowski did his postdoctoral research also at Königsberg. The title of his Habilitationsschrift was *Spatial visualization and minima of positive definite quadratic forms (Räumliche Anschauung und Minima positiv definiter quadratischer Formen)* which was extended to his monograph *Geometry of Numbers* [18].

In [18], he introduced a way of assigning its length to each vector in $\mathbb{R}^n$. Given a convex set $W \subset \mathbb{R}^n$, we can define a Minkowski norm by

$$\|v\| = \sup_{t \geq 0} \{tv \in W\} .$$

In the calculus of variations, the boundary ∂W of W is called the indicatrix. Note that $\partial W = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$.

From 1894 to 1902, he was teaching at Eidgenössische Polytechnikum Zürich.[3)] A. Einstein took several courses of Minkowski here. They had common interest in the theory of relativity. The new idea of four dimensional Minkowski space or spacetime developed by Minkowski became a mathematical foundation of the theory of special relativity of Einstein. When he came back here as a researcher, Einstein worked with M. Grossmann and H. Weyl. This later work eventually led him to the theory of general relativity.

Minkowski and Hilbert were friends in the days of Königsberg and finished their doctoral degree under the same advisor Lindemann within on a few months apart. They also remained as a collegue at Göttingen until untimely death of Minkowski by complications of appendicitis .

3.2 Hilbert

David Hilbert was born on January 23, 1862 in Königsberg, Prussia and died on February 14, 1943 at Göttingen. Minkowski only two years younger than Hilbert was a cater-cousin from their Königsberg days. They went to the university at Königsberg and both were awarded their doctoral degrees in 1885 under the guidance of F. von Lindemann. Hilbert worked on the invariant properties of certain algebraic forms. The title of his thesis was *Über invariante Eigenschaften specieller binärer Formen, insbesondere der Kugelfunctionen*. He also finished his Habitation there and was pleased with staying in his native town in spite of Klein's offer to come to Göttingen. His Habilitationsschrift was also on the invariant theory.

Klein persuaded mathematicians at Göttingen to give him professorship. He eventually joined the faculty at Göttingen in 1895 and remained there until his death. Dur-

3) It is a federal institute founded by the Swiss Federal Government in 1854 to educate engineers and scientists. It is also known as Eidgenössische Technische Hochschule or ETH in short.

ing his tenure, Göttingen became the worldwide research center of mathematics and was totally collapsed after the 1933 racist law. He commented on this situation of his department[4]:

> The mathematics in Göttingen freed of the Jews had really none any more.

Hilbert is now remembered for the problem list. He collected 23 unsolved problems that mathematicians should do in the 20th century. He presented 10 out of them at the 2nd International Congress of Mathematicians in 1900 at Paris. In his draft, there were 24 problems but the last one did not appear in the printed form.[5] So there appeared only 23 problems in the Proceedings of the International Congress of Mathematicians[6]. These problems had challenged many generations of mathematicians and influenced the future shape of mathematics. Out of these 23 problems, 4th and 23rd problems are related to Finsler geometry:

P4. The straight line as the shortest path joining two points.

P23. The further development of the methods of the calculus of variations.

Hilbert's 4th problem is too vague to be stated settled or not. But this problem still sheds new light in the field of Finsler geometry. A very readable introduction to the Hilbert's 4th problem pertaining to Finsler geometry can be found in the Chapter 4 of S. Kobayashi's book in Japanese translated into Korean by the author [16].

Hilbert's 23rd problem is not stated concretely unlike other problems as he commented on it.

> Nevertheless, I should like to close with a general problem, namely the indication of a branch of mathematics repeatedly mentioned in this lecture - which, in spite of the considerable advancement lately given by it by Weierstrass, does not receive the general appreciation which, in my opinion, is its due - I mean the calculus of variations.

4) In 1930, Hilbert retired. By the 1933, life in Göttingen changed completely when the Nazis came to power. Jewish lecturers were expelled. By the autumn of 1933, most had left or were dismissed.

5) In 2000 German historian Rüdiger Thiele found the forgotten 24th problem in Hilbert's unpublished mathematical notebooks where he wrote down mathematical and philosophical scribbles [24]. The missing problem is on the foundation of mathematics.

6) He delivered the address in German but the manuscript was in French. The problem list first appeared in Göttinger Nachrichten in 1900, before it was published officially in the Proceedings of the 2nd International Congress of Mathematicians. A French version appeared under the title of *Sur les problèmes futurs des mathématiques* in Compte Rendu du Deuxième Congrès International des Mathématiciens in 1902. Another version for the physicists appeared in Archiv der Mathematik und Physik in 1901. An English translation appeared in the Bulletin of the American Mathematical Society in 1902.

In 1899–1901, he lectured on the calculus of variations at Göttingen. He considered the problem of finding a function $y = y(x)$ which assumes a minimum of the integral

$$I = \int_a^b L(t, x(t), x'(t))dt. \tag{2}$$

Many students including Carathéodory at Göttingen were influenced by his lectures and worked on this field. The Lagrangian in (2) includes fundamental function of Finsler geometry as a special case.

3.3 Carathéodory

Constantin Carathéodory[7)] was a Greek mathematician spending most of his mathematical life in Germany. He was awarded his doctoral degree in 1904 under the supervision of H. Minkowski. In his doctoral dissertation entitled *On the discontinuous solutions to the variational problems (Über die diskontinuierlichen Lösungen in der Variationsrechnung)*, he invented a method of finding a minimizers of certain integrals. It was based on the use of Hamilton-Jacobi equations and this method is now called *the royal road to the calculus of variations*.[8)] This work was much influenced by Hilbert and Klein. In 1905, he wrote the Habilitationsschrift entitled *Über die starken Maxima und Minima bei einfachen Integralen*. Though Carathéodory's official advisor was Minkowski, his earlier works on the calculus of variations were influenced by Hilbert.

He also discovered the relationship between calculus of variations and partial differential equations of the first order. This work was recorded in his textbook of the same title [6]. This classic in this field was later expanded and translated into English in two parts [7]. Among many contributions to the wide area of mathematics and physics, it was the techniques of the calculus of variations that was inherited to his student P. Finsler.

As a student of Minkowski, Carathéodory was also interested in the theory of special relativity and corresponded with A. Einstein on this subject. Around this time, Göttingen mathematicians were interested in and working on the theory of relativity. And Einstein himself paid visits to Göttingen and gave lectures on the theory of relativity. He got much help from Göttingen mathematicians: Minkowski, Hilbert and Weyl. The story on this regard is briefly reviewed in author's [26].

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7) He was born on September 13, 1873 in Berlin and died on February 2, 1950 in München. His father was a diplomat of Ottoman Empire but he sided with Greeks.

8) H. Börder used the term *the royal road to the calculus of variations (Königsweg der Variationsrechnung)* in [2]. Börder was not a doctoral student of Carathéodory but he earned his Habilitation in 1934 under Carathéodory.

3.4 Courant

Richard Courant was born on January 8, 1888 in Lublinitz, Germany and died on January 27, 1972 in New Rochelle, New York. He is probably best known for his finite element method that was used in the proof of a version of the Riemann mapping theorem.

After studying at Breslau and Zürich, Courant ended up with studying at Göttingen. He was a student of Hilbert at Göttingen at its height. Courant obtained his doctoral degree from Göttingen in 1910 under D. Hilbert. The title of his thesis was *On the application of Dirichlet's principle to the problems of conformal mappings (Über die Anwendung des Dirichletschen Prinzipes auf die Probleme der konformen Abbildung)*. Though it was unusual to do Habilitation in one's doctoral institute, Hilbert offered him to continue his study at Göttingen for the Habilitation. Courant worked on the Dirichlet principle for his Habilitationsschrift. Since 1912, he taught at Göttingen until his departure from Germany. Busemann was his doctoral student shortly before his departure from Germany.

Courant left Germany in 1933 as many Jews did after the racial law by Nazis passed.[9] In 1936, he settled in New York University and founded an institute for graduate studies in applied mathematics based on the Göttingen model.[10] He was a founding director from 1953 to 1958 of the Institute. It was renamed in 1964 as the Courant Institute of Mathematical Sciences in his honor and is now one of the most prestigious institute in its field.

4 Creators

4.1 Finsler

Paul Finsler was born in Germany in 1894 and spent most of his mathematical life in Switzerland.[11] He joined the faculty of the University of Zurich in 1927 and stayed there until his death. When he entered the University of Göttingen as a graduate student in 1913, Göttingen was the center of mathematical activities. Göttingen then had prestigious mathematics faculty including D. Hilbert, F. Klein, E. Landau, C.

9) Courant was forced to leave Göttingen when the Nazis came to power in 1933. Weyl then director of the Mathematics Institute made every effort to have Courant reinstated but failed. Weyl himself left Germany via Zürich and landed at the Institute for Advanced Study, Princeton. That ended the Göttingen era in mathematics.

10) He previously established one in Göttingen and another one in Münster serving as director from 1928 until his departure from Germany in 1933.

11) Finsler was born on April 11, 1894 in Heilbronn, Germany and died on April 29, 1970 in Zürich, Switzerland.

Runge, M. Born[12] and C. Carathéodory to name a few. In 1919, he was awarded doctoral degree. The thesis entitled *Curves and surfaces in general spaces (Über Kurven und Flächen in allgemeinen Räumen)* was supervised by Carathéodory.

Carathéodory was a master of calculus of variations. This branch was a long tradition in Göttingen: his predecessors Minkowski and Hilbert worked on the variational problems in the calculus of variations. This topic was included in the problem list of Hilbert addressed in 1900 International Congress of Mathematicians held at Paris.

The main tool that Finsler had used to tackle geometric problems of spaces endowed with the Finsler metrics was the techniques of calculus of variations he learnt from his advisor. He immediately noticed that to ensure the existence of minimizer of the integral

$$L[\alpha] = \int_a^b L(\alpha(t), \alpha'(t))dt, \tag{3}$$

a condition should be imposed on the Finsler metric $L(x, y)$. It is the convexity condition that

$$\left[\frac{\partial L^2}{\partial y^i \partial y^j}(x, y)\right]$$

is positive definite.

His doctoral work on the Finsler geometry had lasting impact on the later generation's research. But as usual at his time[13], the printed copy of his thesis was not readily available to the mathematicians in general. Many new results and ideas were found to be in his thesis. It was only 1950 that an unaltered photographic reproduction of this thesis was published with extensive bibliography of books and papers up to 1949 compiled by H. Schubert. His thesis consisted of 3 chapters, 17 sections and 82 subsections:

Ch.1 Fundamental Concepts

- I. Geometric shape
- II. Projection
- III. Length
- IV. Angular metric
- V. Degree of contact

Ch.2 Theory of Curves

12) He was a distinguished mathematician at Göttingen. He worked on combining ideas of Einstein and Minkowski. In 1914, he became a chair of the University of Berlin where he acquainted with M. Planck and A. Einstein. In 1933, he fled Germany to avoid the Nazis persecution and established his research group mainly of European refugees at the University of Edinburgh, England. After his retirement in 1953, returned to Bad Pyrmont near Göttingen. In 1954, he was awarded the Nobel Prize for Physics for his statistical studies of wave functions.

13) Even Riemann's inaugural lecture was only published posthumously.

VI. Osculating space and normal curve

VII. First curvature

VIII. Higher degree curvature

IX. Other definition of higher degree curvature

X. Natural equations for curves

Ch.3 Theory of Surfaces

XI. Introduction

XII. Geodesics, theorem of Meusnier, asymptotics

XIII. Higher degree curvature for spaces

XIV. Dupin indicatrix

XV. Curvature for 2 dimensional spaces

XVI. Curvature and torsion of geodesics

XVII. Curvature and torsion of asymptotics, developable surfaces

It is surprising that the above table of contents of the doctoral thesis of Finsler just looks like the one of nowadays typical undergraduate textbook of differential geometry. About eight decades later, S. S. Chern in [11][14)] set up a dictum:

> Almost all the results of Riemannian geometry can be developed in the Finsler setting.

After his doctoral work, he never did research back on the topics of differential geometry. Both his Habilitationsschrift and Habilitationsvortrag done at the University of Köln in 1922 are on the set theory. He is now rather known for his work on this field, Finsler set theory.

4.2 Busemann

Herbert Busemann was born on May 12, 1905 in Berlin and died on February 3, 1994 in Santa Ynez, California, USA. He studied at Munich, Paris, Rome and Göttingen and was awarded doctoral degree from the University of Göttingen in 1931 under the supervision of R. Courant. The title of his thesis was *Über die Geometrien, in denen die "Kreise mit unendlichem Radius" die kürzesten Linien sind*. He remained there as an assistant until 1933, when he fled Nazi Germany. He eventually settled in USA.

He was known for his synthetic[15)] approach to geometric problems. He had contributed in the area of Finsler geometry and convexity influenced by the work of

14) The title of the article is just *Finsler Geometry is Just Riemannian Geometry without the Quadratic Restriction.*

15) By "synthetic" we mean the deduction of the geometric properties of a metric space via "synthesis" from the axioms of its distance function. In [3, 5], he tried to find a synthetic description of the Finsler's thesis [14].

Minkowski. His point of view was reflected in the book titled *Metric methods in Finsler Spaces and in the Foundation of Geometry* published in the series of Annals of Mathematics Studies of Princeton University Press in 1942. Here the emphasis is on the spaces in which the geodesic joining given two points is unique. And the book begins with the axioms for finite dimensional metric spaces to possess geodesics satisfying usual properties. These spaces include Finsler spaces. Unlike its title, this book is not on the geometry of Finsler spaces but on the geometry of metric spaces with geodesics. Such spaces are called G-spaces, where G stands for geodesic with all the usual properties. In 1955, Busemann published *The Geometry of Geodesics* which is more or less a revised and expanded edition of the previous book. Here geodesics now have local uniqueness properties. This assumption is more plausible to Finsler geometry or Riemannian geometry.

His approach produced many new results and gave a unified way to prove many previously known results, and thereby reveals startling facts: much of Riemannian geometry is not truly Riemannian and much of differential geometry does not require differentiability. His method may have methodological advantages over the one we are accustomed to. But we can hardly replace calculations with syllogism because most of contemporary mathematicians are deeply accustomed to the analytic techniques.

Both books were hard to be read. But his approach was well regarded rather outside of USA. In 1985, he was awarded the Labachevski Prize administered by Soviet Academy of Sciences for his book *The Geometry of Geodesics*. He was the first American mathematician to receive it. His academic grandfather Hilbert was also awarded the prize in 1903.

Busemann also noticed that tensor calculus would be an indispensable tool for Finsler geometry by the time he wrote the first book [3]. But in [5], he did envision a synthetic way of understanding Finsler geometry.

5 Conclusion

A Finsler space is a generalization of a Riemannian space where the length function is defined in terms of Minkowskian norm and Minkowski's geometry holds pointwisely. In 1927, J. Taylor first gave this generalized geometry its name: Finsler geometry [22]. Finsler's name was also used in 1933 by Élie Cartan in his paper [8] to distinguish it from Riemannian space.

A Minkowski space V is a real vector space with a norm on it. The hypersurface

$$I = \{v \in V \mid L(v) = 1\}$$

is called the indicatrix. The tangent space $T_x M$ of a Finsler space at $x \in M$ with a fundamental function $L(x, y)$ is a Minkowski space. The indicatrix I_x in each tangent space T_xM of a Finsler space M is called the indicatrix at $x \in M$. In [14], Finsler called the indicatrix by *Einheitflächen* and Finsler geometry by *Allgemeine Geometrie.*

For a Riemannian space with a fundamental function $L(x,y) = \sqrt{\sum g_{ij}y^iy^j}$, the indicatrices I_x are congruent. For a Finsler space, this phenomenon no longer holds in general. If the tangent spaces are congruent to a single Minkowski space, then the Finsler space is said to be modeled on a Minkowski space.

In the modern terminology, a Finsler metric on the space M is a function $L(x,y)$, called a fundamental function, defined on the tangent bundle TM satisfying certain conditions. In order that a length $L[\alpha]$ of a curve α is defined independent of a choice of parameters, we should require that $L(x,y)$ is positive homogeneous of degree 1 in the variables y. That is

$$L(x, \lambda y) = \lambda L(x,y), \quad \lambda > 0. \tag{F2a}$$

For practical reasons, we also require some of:

$$L(x,y) \geq 0, \quad L(x,y) = 0 \iff y = 0. \tag{F1}$$

$$L(x,-y) = L(x,y). \tag{F2b}$$

$$\left[g_{ij} = \frac{\partial^2 L^2}{\partial y^i \partial y^j}\right] \text{ is positive definite.} \tag{F3}$$

$$L(x,y) \text{ is differentiable away from } y = 0. \tag{F4}$$

Combining (F2a) and (F2b), we would rather have the homogeneity condition:

$$L(x, \lambda y) = \lambda L(x,y), \quad \lambda \in \mathbb{R}. \tag{F2}$$

Riemann who was only interested in defining the length function (1) only required (F1) and (F2b) on the generalized metric. (F1) is plausible because the length should be positive. (F2b) guarantees that the length of a curve does not depend on the choice of its orientation.

On the other hand, Finsler imposed conditions (F1) and (F3) on the generalized metrics. To him, (F3) on the metric is indispensable because he was interested in the variational problems

$$\inf \{L[\alpha]\}$$

of the integral (3). This requirement was already used by his predecessor Carathéodory and Minkowski. Under this assumption (F3) on the fundamental function $L(x,y)$, the variational principle leads to the Euler-Lagrange equation for the integral

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0.$$

This convexity (F3) on the metric L is equivalent to saying that the indicatrices I_x are convex. This simple observation signifies that to have a Finsler metric we are required to have convex sets and vice versa. This enables us to study Finsler geometry in synthetic way. Buseman in §4.2 had worked on Finsler geometry in this direction. As was seen in §3.1, Minkowski worked on the problem of convex bodies. In his monograph [18], Minkowski implicitly introduced a fundamental function $L(x, y)$ by defining a convex set in $\mathbb{R}^n$.

K. Menger[16)] was the first one who studied geometry of geodesics in metric spaces. He ensured the existence of geodesics to complete the axioms of Fréchet. Menger and Busemann both envisioned geometric problems particularly on the geodesics in grand scale: particular study of geometric problem is a special case of geometry of metric spaces. Menger and his school had contributed to the geometry of metric spaces and the calculus of variations. But their approach is quite different from the traditional one and can be found in [1].

In [5], Busemann proposed a way to handle geometric problems of certain metric spaces in a unified way. He extended the five axioms in the first book of Euclid's Elements to the category of metric spaces. Such spaces are called geodesic spaces or simply G-spaces. A Finsler metric induces a distance on the space. The distance defined in terms of the infinitesimal length is always an inner distance and the topology as a metric space is equivalent to the topology of the underlying space. Thus complete Riemannian spaces and complete Finsler spaces are all G-spaces. It seemed to him that Finsler geometry could be better understood with further generalization.

Following the work of Hilbert on calculus of variations, Carathéodory investigated variational problems with convex/non-convex indicatrices. Finsler found a way of interpreting the convexity of the indicatrix analytically and ended up with the condition (F3) of positive definiteness of the Finsler metrics.

Finally, we briefly include two specialists who contributed in Finsler geometry at its very early stages. Their tool was also the calculus of variations.

G. Bliss[17)] is known for his work on the calculus of variations. Bliss spent the academic year 1902–03 at Göttingen and interacted with Klein, Hilbert, Minkowski, and Carathéodory among others. Bliss strengthened the necessary conditions of Euler, Weierstrass, Legendre, and Jacobi into sufficient conditions and had a work on Finsler geometry.

G. Landsberg[18)] should be mentioned. He worked on Finsler geometry in a non-Finsler framework. Now the Landsberg space named after him is still elusive. It was first introduced around 1907. He was a teacher of Courant of his undergraduate days.

Considering the respective works on Finsler geometry, we should replace the math-

16) Karl Menger born on January 13, 1902 and died on October 5, 1985 was an Austrian-American mathematician.

17) Gilbert Ames Bliss born on May 9, 1876 and died May 8, 1951 was an American mathematician.

18) Georg Landsberg born on January 30, 1865 and died September 14, 1912 was a German mathematician.

ematical genealogy on p. 135 by a new mathematical ancestry.

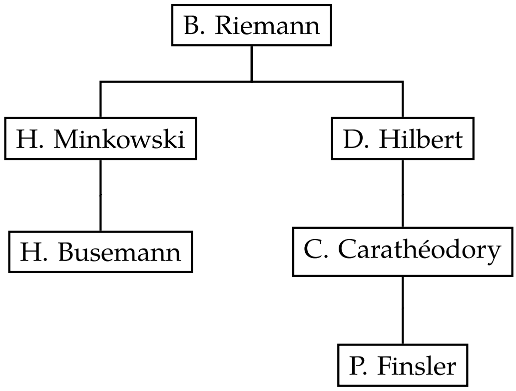

Table 2. Mathematical genealogy according to the works on Finsler geometry

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