Value Distribution of the Product of a Meromorphic Derivative and a Power of the Function

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Abstract. In the paper we discuss the value distribution of the product of the derivative of a transcendental meromorphic function and a power of the function.

1. Introduction

W. K. Hayman [5] proved the following result.

Theorem A. ([5]) If \( n \geq 3 \) is an integer and \( f \) is a transcendental meromorphic function, then \( f^n f' \) assumes all finite values, except possibly zero, infinitely often.

Hayman [7] also conjectured that Theorem A might be valid for \( n = 1 \) and \( n = 2 \). E. Mues [12] settled the conjecture for \( n = 2 \) and the case \( n = 1 \) was settled by W. Bergweiler and A. Eremenko [1] and by H. H. Chen and M. L. Fang [3].

In 1999 X. C. Pang and L. Zalcman [13] considered the general order derivative of an entire function. They proved the following result.

Theorem B. ([13]) Let \( f \) be a transcendental entire function, all of whose zeros have multiplicity at least \( k \) and let \( n \) be a positive integer. Then \( f^n f^{(k)} \) assume every nonzero finite value infinitely often.

Recently J. P. Wang [16] considered the meromorphic case and proved the following theorem.

Theorem C. ([16]) Let \( f \) be a transcendental meromorphic function all of whose zeros have multiplicity at least \( t \). Then for any positive integer \( k \geq 2 \), \( ff^{(k)} \) assumes...
every nonzero finite value infinitely often provided that $t = k + 1$ for $2 \leq k \leq 4$, $t = 5$ for $k = 5$ and $t = 6$ for $k \geq 6$.

N. Steinmetz [15] proved that if $f$ is a transcendental meromorphic function, then $f^n f^{(k)}$ assume every nonzero finite value infinitely often, where $n(\geq 2)$ and $k$ are positive integers.

In 1994 Yik Man Chiang asked the question of value distribution of $ff' - a$, where $a = a(z)(\not\equiv 0, \infty)$ is a small function of $f$ i.e., $T(r,a) = S(r,f)$. In response to this question W. Bergweiler [2] proved the following theorem.

**Theorem D.** ([2]) Let $f$ be a transcendental meromorphic function of finite order and $a = a(z)(\not\equiv 0)$ be a polynomial. Then $ff' - a$ has infinitely many zeros.

In 2005 I. Lahiri and S. Dewan [10] observed that if $a = bz^n$, $n$ is a nonnegative integer and $b$ is a nonzero constant, then the order restriction on $f$ can be withdrawn. Their result is as follows.

**Theorem E.** ([10]) Let $f$ be a transcendental meromorphic function. Then $f^n f' - bz^n$ has infinitely many zeros, where $b(\not\equiv 0)$ is a constant and $n(\geq 0)$, $p(\geq 1)$ are integers.

If one considers a small function, then following two results are worth mentioning, which follow from two inequalities proved by Q. D. Zhang [18].

**Theorem F.** Let $f$ be a transcendental meromorphic function with $\delta(\infty; f) > \frac{7}{9}$. Then $ff' - a$ has infinitely many zeros, where $a = a(z)(\not\equiv 0, \infty)$ is a small function of $f$.

**Theorem G.** Let $f$ be a transcendental meromorphic function with $\delta(\infty; f) + 2\delta(0; f) > 1$. Then $ff' - a$ has infinitely many zeros, where $a = a(z)(\not\equiv 0, \infty)$ is a small function of $f$.

K. W. Yu [17] treated the small function case without imposing any restriction on $f$. However he had to consider a small function and its negative as a pair of targets. He proved the following theorem.

**Theorem H.** ([17]) If $a = a(z)(\not\equiv 0, \infty)$ is a small function of a transcendental meromorphic function $f$, then at least one of $ff' + a$ and $ff' - a$ has infinitely many zeros.

In 2003 I. Lahiri and S. Dewan [9] considered the general order derivative and proved the following result.

**Theorem I.** (cf. Corollary 1 [9]) Let $f$ be a transcendental meromorphic function and $k$ be a positive integer. Suppose that $F_1 = ff^{(k)} - a$ and $F_2 = ff^{(k)} + a$, where $a = a(z)(\not\equiv 0, \infty)$ is a small function of $f$. Then $\Theta(0; F_1) + \Theta(0; F_2) \leq 2 - \frac{2}{2k+1}$. 

336 Indrajit Lahiri and Rajib Mukherjee
The problem of value distribution of $f^p f^{(k)} - a$ remains open, where $f$ is a transcendental meromorphic function, $a = a(z)(\neq 0, \infty)$ is a small function of $f$ and $p, k$ are positive integers. In the paper we deal with this problem.

We respectively denote by $N_k(r, f)$ and $\overline{N}_k(r, f)$ the counting function and the reduced counting function of those zeros of $f$ which have multiplicities less than or equal to $k$, where $k$ is a positive integer.

For standard definitions and notations of the value distribution theory we refer the reader to [6].

We now state the main result of the paper.

**Theorem 1.** Let $f$ be a transcendental meromorphic function and $\alpha = \alpha(z)(\neq 0, \infty)$ be a small function of $f$ such that the zero-pole sets of $f$ and $\alpha$ are disjoint. Suppose that the zeros of $f$ have multiplicity at least $t$, where $t = \lceil \frac{k+1}{p} \rceil + 1$ if $p \geq 2$, $t = k + 1$ if $p = 1$ and $1 \leq k \leq 4$ and $t = \min\{k, 6\}$ if $p = 1$ and $k \geq 5$. If $F = f^p f^{(k)} - \alpha$, then one of the following holds:

1. $\Theta(0; F) \leq 1 - \frac{p}{k + p + 1}(1 - \frac{k + 1}{pt})$ if $p \geq 2$;
2. $\Theta(0; F) \leq 1 - \frac{k}{4t(k+2)}$ if $p = 1$ and $1 \leq k \leq 4$;
3. $\Theta(0; F) \leq 1 - \frac{(t-4)(k+1) - t}{4t(k+1)(k+2)}$ if $p = 1$ and $k \geq 5$.

**2. Lemmas**

In this section we state two necessary lemmas. Let $f$ be a transcendental meromorphic function and $n, p$ be positive integers. A differential polynomial $P$ of $f$ is defined by $P(z) = \sum_{k=1}^{n} \phi_k(z)$, where $\phi_k(z) = \alpha_k(z) \prod_{j=0}^{p} (f^{(j)}(z)^{S_{kj}}$, $\alpha_k(z) \neq 0$, $S_{kj}$ are nonnegative integers and $T(r, \alpha_k) = S(r, f)$.

If we suppose only $m(r, \alpha_k) = S(r, f)$, then $P(z)$ is called a quasi-differential polynomial.

The quantities $\overline{d}(P) = \max_{1 \leq k \leq n} \{ \sum_{j=0}^{p} S_{kj} \}$ and $\underline{d}(P) = \min_{1 \leq k \leq n} \{ \sum_{j=0}^{p} S_{kj} \}$ are respectively called the degree and lower degree of $P(z)$. If, in particular, $\overline{d}(P) = \underline{d}(P)$, then $P(z)$ is called homogeneous.

**Lemma 1.** ([8]) Let $f$ be a transcendental meromorphic function and $P = P(z)$ be a nonconstant differential polynomial in $f$ with $\underline{d}(P) > 1$. Suppose that $Q = \max_{1 \leq k \leq n} \{ \sum_{j=1}^{p} jS_{kj} \}$. Then

$$T(r, f) \leq \frac{Q + 1}{\overline{d}(P) - 1} N(r, 0; f) + \frac{1}{\overline{d}(P) - 1} N(r, 1; P) + S(r, f).$$
Lemma 2. (p.39 [11]) Let $f$ be a nonconstant meromorphic function and $Q_1, Q_2$ be quasi-differential polynomials in $f$ with $Q_2 \neq 0$. Let $n$ be a positive integer and $f^n Q_1 = Q_2$. If $\overline{d}(Q_2) \leq n$, then $m(r, Q_1) = S(r, f)$.

3. Proof of Theorem 1

First we suppose that $p \geq 2$. We put $P = \frac{1}{\alpha} f^p f^{(k)}$. Then $d(P) = \overline{d}(P) = p + 1$ and $Q = k$. So by Lemma 1 we get

$$T(r, f) \leq \frac{k + 1}{p} N(r, 0; f) + \frac{1}{p} N(r, 1; P) + S(r, f)$$

and so

$$(3.1) \quad p(1 - \frac{k + 1}{pt}) T(r, f) \leq N(r, 1; P) + S(r, f).$$

We note that \{cf. [4], [14]\}

$$(3.2) \quad T(r, f) + S(r, f) \leq CT(r, F) + S(r, F)$$

and

$$(3.3) \quad T(r, F) \leq (k + p + 1) T(r, f) + S(r, f),$$

where $C$ is a nonzero constant.

From (3.2) and (3.3) we see that $S(r, f)$ and $S(r, F)$ are mutually interchangeable. So from (3.1) and (3.3) we get

$$\frac{p}{k + p + 1} (1 - \frac{k + 1}{pt}) T(r, F) \leq N(r, 0; F) + S(r, F).$$

This implies $\Theta(0; F) \leq 1 - \frac{p}{k + p + 1} (1 - \frac{k + 1}{pt})$.

Now we suppose that $p = 1$. Let us put

$$(3.4) \quad F = f f^{(k)} - \alpha$$

and

$$(3.5) \quad a = \frac{f' f^{(k)}}{f^2} f^{(k+1)} - f^{(k+1)} \frac{F'}{F}.$$ 

Then

$$(3.6) \quad f a = \alpha \frac{f'}{\alpha} - \frac{F'}{F}.$$
Let \( \frac{\alpha'}{\alpha} - \frac{f'}{f} \equiv 0 \). Then on integration we get \( F = c\alpha \), where \( c(\neq 0) \) is a constant. Hence we get from (3.4)

\[
(3.7) \quad ff^{(k)} = (1 + c)\alpha.
\]

Since \( ff^{(k)} \neq 0 \), we have \( 1 + c \neq 0 \). From (3.7) we get

\[
(3.8) \quad N(r, 0; f) \leq N(r, 0; \alpha) = S(r, f).
\]

Also from (3.7) we obtain \( \frac{1}{f^2} = \frac{1}{(1 + c)\alpha} \frac{f^{(k)}}{f} \) and so \( m(r, \frac{1}{f^2}) = S(r, f) \). This implies

\[
(3.9) \quad m(r, 0; f) = S(r, f).
\]

From (3.8), (3.9) and the first fundamental theorem we get \( T(r, f) = S(r, f) \), a contradiction. Therefore \( \frac{\alpha'}{\alpha} - \frac{f'}{f} \neq 0 \). So from (3.6) we get by Lemma 2

\[
(3.10) \quad m(r, a) = S(r, f).
\]

Let \( z_1 \) be a pole of \( f \) with multiplicity \( q(\geq 2) \). Then \( z_1 \) is a simple pole of \( \alpha(\frac{\alpha'}{\alpha} - \frac{f'}{f}) \) as \( \alpha(z_1) \neq 0, \infty \) and so \( z_1 \) is a zero of \( \alpha \) with multiplicity \( q - 1 \). Hence

\[
(3.11) \quad N_{(2)}(r, \infty; f) \leq 2N(r, 0; a),
\]

where \( N_{(2)}(r, \infty; f) \) denotes the counting function of multiple poles of \( f \).

Let \( z_2 \) be a zero of \( f \) with multiplicity \( q(\geq k + 1) \). Then \( z_2 \) is a zero of \( F' + \alpha' = f'f^{(k)} + f f^{(k+1)} \) with multiplicity at least \( 2q - (k + 1) \).

Since \( F + \alpha = ff^{(k)} \), we see that \( z_2 \) is a zero of \( F + \alpha \) with multiplicity \( 2q - k \).

From (3.6) we get \( fa = (F' + \alpha') - \frac{F'(F + \alpha)}{f} \). So \( z_2 \) is a zero of \( fa \) with multiplicity at least \( 2q - (k + 1) \). Therefore \( z_2 \) is not a pole of \( a \).

Also from (3.6) we see that a simple pole of \( f \) is not a pole of \( a \). Hence the poles of \( a \) are contributed by the zeros of \( F \) and by zeros of \( f \) with multiplicities less than or equal to \( k \) and the poles of \( \alpha \). Therefore

\[
(3.12) \quad N(r, \infty; a) = \overline{N}(r, \infty; a) \leq \overline{N}_{kj}(r, 0; f) + \overline{N}(r, 0; F) + S(r, f).
\]

By (3.10) and (3.12) we get

\[
(3.13) \quad T(r, a) \leq \overline{N}_{kj}(r, 0; f) + \overline{N}(r, 0; F) + S(r, f).
\]

From (3.6) we obtain

\[
(3.14) \quad m(r, f) \leq m(r, 0; a) + S(r, f)
\]

\[
= T(r, a) - N(r, 0; a) + S(r, f).
\]
Let $z_0$ be a simple pole of $f$. Then from (3.6) we see that $a(z_0) \neq 0, \infty$. Now in some neighbourhood of $z_0$ we get

$$f(z) = \frac{c_1}{z - z_0} + c_0 + O(z - z_0), \quad (3.15)$$

$$a(z) = a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2, \quad (3.16)$$

and

$$\alpha(z) = \alpha(z_0) + \alpha'(z_0)(z - z_0) + O(z - z_0)^2, \quad (3.17)$$

where $c_1 \neq 0$ and $\alpha(z_0) \neq 0, \infty$.

Differentiating (3.15) we get

$$f^{(j)}(z) = \frac{(-1)^j j! c_1}{(z - z_0)^{j+1}} + O(1) \quad \text{for } j = 1, 2, 3, \ldots \quad (3.18)$$

From (3.5) and (3.6) we have

$$af = \alpha f f^{(k)} + a f^{(k+1)} + f^2 f^{(k)} a - \alpha' f f^{(k)}. \quad (3.19)$$

Now by (3.15) – (3.19) we get

$$\begin{align*}
\{a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2\} \{\frac{c_1}{z - z_0} + c_0 + O(z - z_0)\} \\
\{a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2\}
\end{align*}$$

$$= \{a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2\} \left\{\frac{-c_1}{(z - z_0)^2} + O(1)\right\}$$

$$= \left\{\frac{(-1)^k k! c_1}{(z - z_0)^{k+1}} + O(1)\right\} + \{a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2\}$$

$$= \left\{\frac{c_1}{z - z_0} + c_0 + O(z - z_0)\right\} \left\{\frac{(-1)^{k+1} c_1 k + 1)!}{(z - z_0)^{k+2}} + O(1)\right\}$$

$$+ \left\{\frac{c_1}{z - z_0} + c_0 + O(z - z_0)^2\right\} \left\{\frac{(-1)^k k! c_1}{(z - z_0)^{k+1}} + O(1)\right\}$$

$$\{a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2\} - \{a'(z_0) + O(z - z_0)\}$$

$$= \left\{\frac{c_1}{z - z_0} + c_0 + O(z - z_0)\right\} \left\{\frac{(-1)^k k! c_1}{(z - z_0)^{k+1}} + O(1)\right\}. \quad (3.20)$$

Equating the coefficients of $\frac{1}{(z - z_0)^{k+3}}$ and $\frac{1}{(z - z_0)^{k+4}}$ of both sides of (3.20) we respectively get

$$c_1 a(z_0) = \alpha(z_0)(k + 2). \quad (3.21)$$
and

\[ \frac{c_0}{c_1} = \frac{\alpha'(z_0)}{\alpha(z_0)} - \frac{(k + 2)a'(z_0)}{(k + 3)a(z_0)}. \]

From (3.15) we have

\[ \frac{f'}{f} = \frac{-1}{z - z_0} + \frac{c_0}{c_1} + O(z - z_0). \]

Also from (3.6) we obtain

\[ \alpha \left( \frac{\alpha'}{\alpha} - \frac{F'}{F} \right) = fa \]

\[ = \left\{ \frac{c_1}{z - z_0} + c_0 + O(z - z_0) \right\} \left\{ a(z_0) + a'(z_0)(z - z_0) + O(z - z_0)^2 \right\} \]

\[ = c_1a(z_0) \left\{ \frac{1}{z - z_0} + \frac{c_0}{c_1} + \frac{a'(z_0)}{a(z_0)} \right\} + O(z - z_0). \]

From (3.21) – (3.24) we get in some neighbourhood of \( z_0 \)

\[ \alpha \left( \frac{\alpha'}{\alpha} - \frac{F'}{F} \right) = \alpha(z_0)(k + 2) \left\{ \frac{2a'(z_0)}{\alpha(z_0)} - \frac{f'}{f} - \frac{(k + 1)a'(z_0)}{(k + 3)a(z_0)} \right\} + O(z - z_0). \]

Let

\[ h = (k + 2)\frac{f'}{f} - \frac{F'}{F} - \frac{(2k + 3)\alpha'}{\alpha} + \frac{(k + 1)(k + 2)a'}{(k + 3)a}. \]

First we suppose that \( h(z) \equiv 0 \). Then on integration we get

\[ f^{(k+2)(k+3)}a^{(k+1)(k+2)} = A\alpha^{(2k+3)(k+3)}F^{k+3}, \]

where \( A(\neq 0) \) is a constant.

Let \( z_3 \) be a zero of \( f \) with multiplicity \( p(\geq 1) \). Then from (3.27) we see that \( z_3 \) is a pole of \( a \) with multiplicity \( q \) such that \( (k + 2)(k + 3)p = (k + 1)(k + 2)q \) and so \( q = \frac{k + 3}{k + 1}p > 1 \). This is impossible because from (3.5) we see that a zero of \( f \) is at most a simple pole of \( a \). Hence \( f \) has no zero. Since a zero of \( F \) is a possible pole of \( a \) and \( f \) has no zero, we have from (3.27) \( \overline{N}(r, 0; F) = S(r, f) \). Therefore from (3.13) we get

\[ T(r, a) = S(r, f). \]
Now by the first fundamental theorem we get

\[ N(r, 0; f^{(k)}) = N(r, 0; \frac{f^{(k)}}{f}) \]

\[ \leq T(r, \frac{f^{(k)}}{f}) + O(1) \]

\[ = N(r, \frac{f^{(k)}}{f}) + S(r, f) \]

\[ = kN(r, \infty; f) + S(r, f) \]

\[ = N(r, \infty; f^{(k)}) - N(r, \infty; f) + S(r, f). \]

(3.29)

From (3.5) we obtain

\[ \frac{1}{f^{(k)}} = \frac{1}{a} \left( \frac{f'}{f} + \frac{f^{(k+1)}}{f} - \frac{F'}{F} \right) \]

and so in view of (3.28) we get \( m(r, 0; f^{(k)}) = S(r, f) \). Hence by the first fundamental theorem we get

\[ T(r, f^{(k)}) = N(r, 0; f^{(k)}) + S(r, f). \]

(3.30)

From (3.29) and (3.30) we see that

\[ N(r, \infty; f) = S(r, f). \]

(3.31)

So from (3.14), (3.28) and (3.31) we get

\[ T(r, f) \leq T(r, a) - N(r, 0; a) + S(r, f) = S(r, f), \]

a contradiction.

Let \( h(z) \neq 0 \). Then from (3.25) we see that \( h(z_0) = 0 \). Hence

\[ N_1(r, \infty; f) \leq N(r, 0; h) \leq T(r, h) + O(1) = N(r, h) + m(r, h) + O(1) = N(r, h) + S(r, f). \]

(3.32)

The possible poles of \( h \) are (i) zeros and poles of \( \alpha \), (ii) zeros and poles of \( a \), (iii) zeros and poles of \( f \) and (iv) zeros and poles of \( F \). We further note that

(I) A pole of \( F \) is either a pole of \( f \) or a pole of \( \alpha \) or of both;

(II) A zero of \( f \) with multiplicity \( \geq k + 2 \) is a zero of \( a \). Also a zero of \( f \) with multiplicity \( k + 1 \) is a nonzero regular point of \( a \);

(III) A simple pole of \( f \) is a zero of \( h \) and a multiple pole of \( f \) is a zero of \( a \);

(IV) A zero of \( F \), which is not a zero or a pole of \( \alpha \), is a pole of \( a \).
Let $z_4$ be a zero of $f$ with multiplicity $k$. Then $F(z_4) \neq 0, \infty$ and $f^{(k)}(z_4) \neq 0, \infty$. Then from (3.5) we see that $z_4$ is a simple pole of $a$. Therefore we have

\begin{equation}
N(r,h) \leq N(r,0; a) + N(r,\infty; a) + N_{k-1}(r,0; f) + N_{k+1}(r,0; f) + S(r,f),
\end{equation}

where $N_{k+1}(r,0; f)$ denotes the reduced counting function of those zeros of $f$ which have multiplicity exactly equal to $k+1$.

Now considering (3.11), (3.13), (3.14), (3.32) and (3.33) we get

\begin{align*}
T(r,f) &= m(r,f) + N_1(r,\infty; f) + N_2(r,\infty; f) \\
&\leq T(r,a) - N(r,0; a) + 2N(r,0; a) + N(r,0; a) \\
&\quad + N(r,\infty; a) + N_{k-1}(r,0; f) + N_{k+1}(r,0; f) + S(r,f) \\
&\leq 4T(r,a) + N_{k-1}(r,0; f) + N_{k+1}(r,0; f) + S(r,f) \\
&\leq 4N_k(r,0; f) + N_{k-1}(r,0; f) + N_{k+1}(r,0; f) + 4N(r,0; F) + S(r,f).
\end{align*}

Let $1 \leq k \leq 4$. then from (3.34) and (3.3), for $p = 1$, we get by the hypothesis

\begin{align*}
T(r,f) &\leq \frac{1}{k+1}T(r,f) + 4N(r,0; F) + S(r,F) \quad \text{and so} \quad \frac{1}{k+1}(1 - \frac{1}{k+1})T(r,F) \leq 4N(r,0; F) + S(r,F). \quad \text{This implies} \quad \Theta(0; F) \leq 1 - \frac{k}{4(k+1)(k+2)}.
\end{align*}

Next let $k \geq 5$. Then from (3.34) and (3.3), for $p = 1$, we get by the hypothesis

\begin{align*}
T(r,f) &\leq \frac{4}{t}T(r,f) + \frac{1}{k+1}T(r,f) + 4N(r,0; F) + S(r,F) \quad \text{and so} \quad \frac{1}{k+1}(1 - \frac{4}{t})T(r,F) \leq 4N(r,0; F) + S(r,F). \quad \text{This proves the theorem.}
\end{align*}

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References


