Subnormality and Weighted Composition Operators on $L^2$ Spaces

MOHAMMAD REZA AZIMI
Department of Mathematics, Faculty of Sciences, University of Maragheh, Maragheh, Iran
e-mail: mhr.azimi@maragheh.ac.ir

Abstract. Subnormality of bounded weighted composition operators on $L^2(\Sigma)$ of the form $Wf = uf \circ T$, where $T$ is a nonsingular measurable transformation on the underlying space $X$ of a $\sigma$-finite measure space $(X, \Sigma, \mu)$ and $u$ is a weight function on $X$; is studied. The standard moment sequence characterizations of subnormality of weighted composition operators are given. It is shown that weighted composition operators are subnormal if and only if $\{J_n(x)\}_{n=0}^{+\infty}$ is a moment sequence for almost every $x \in X$, where $J_n = h_n E_n(|u|^2) \circ T^{-n}$, $h_n = d \circ T^{-n} / d \mu$ and $E_n$ is the conditional expectation operator with respect to $T^{-n}\Sigma$.

1. Introduction

Weighted composition operators acting on certain function spaces are operators of the form $Wf = uf \circ T$, where $T$ is a measurable transformation on a $\sigma$-finite measure space $(X, \Sigma, \mu)$ and $u$ is a weight function on $X$. These operators include composition operators of the form $Cf = f \circ T$ and multiplication operators with symbol $u$. So far most general properties of composition operators and major classes of known normal, quasinormal, hyponormal and weak hyponormal composition operators have been widely studied by numerous mathematicians (See [1],[2],[13],[15],[17],[19]). Conditions for weighted composition operators to belong to some certain specific classes are found in [8]. However general properties of weighted composition operators have not been studied as well as composition operators. Originally subnormal composition operators have been studied by A. Lambert in [12] based on the standard moment sequence characterizations. Recently, in [18] subnormality of composition operators have been characterized in the Laplace density case. In this paper we shall use the standard moment sequence

Received July 6, 2013; revised October 19, 2013; accepted October 23, 2013.
2010 Mathematics Subject Classification: Primary 47B20; Secondary 47B38.
Key words and phrases: Subnormal, Weighted composition operators, Conditional expectation, Moment sequence.
characterizations of subnormality of weighted composition operators.

2. Preliminaries

Let \((X, \Sigma, \mu)\) be a complete \(\sigma\)-finite measure space and \(T\) be a nonsingular measurable transformation from \(X\) onto \(X\), i.e. \(T^{-1}\Sigma \subseteq \Sigma\) and such that \(\mu \circ T^{-1}\) is absolutely continuous with respect to \(\mu\) which is denoted by \(\mu \circ T^{-1} \ll \mu\). Given a real or complex valued measurable weight function \(u\) on \(X\), the weighted composition operator on \(L^2(\Sigma) := L^2(X, \Sigma, \mu)\) induced by \(T\) and \(u\) is defined by \(Wf = uf \circ T\).

It is worth reminding the reader that the nonsingularity of \(T\) assures that \(W\) is well defined as a mapping of equivalence classes of functions on the support of \(u\). Let \(h = h_1 := d\mu \circ T^{-1}/d\mu\) be the Radon-Nikodym derivative and for each \(n \geq 1\) let \(h_n := d\mu \circ T^{-n}/d\mu\). For \(f \in L^2(\Sigma)\) or \(f \geq 0\) a.e let \(E(f) := E(f|T^{-1}\Sigma)\) be the conditional expectation of \(f\) with respect to \(\sigma\)-subalgebra \(T^{-1}\Sigma\). For each \(n \geq 2\), let \(E_n(f) := E(f|T^{-n}\Sigma)\). Conditional expectation operator \(E\) is defined as an orthogonal projection from \(L^2(\Sigma)\) onto \(L^2(T^{-1}\Sigma)\) by the assignment \(f \rightarrow E(f)\). If \(g\) is a \(T^{-1}\Sigma\)-measurable function then \(E(fg) = gE(f)\). One may find more noteworthy properties of conditional expectation operator in [6, 16, 2] and the references therein.

In [15] it is shown that \(C\) defines a bounded operator on \(L^2(\Sigma)\) if and only if \(h \in L^\infty(\Sigma) := L^\infty(X, \Sigma, \mu)\). Necessary and sufficient condition for \(W\) to be bounded on \(L^2(\Sigma)\) is that \(J := hE(|u|^2) \circ T^{-1} \in L^\infty(\Sigma)\) which is found in [7]. Conditional expectation is an functional projection and we shall use it in this paper frequently. For example observe the recurrent relationship \(h_{n+1} = hE(h_n) \circ T^{-1}\) proved in [11].

A measurable transformation \(T : X \rightarrow X\) is said to be essentially surjective if \(\mu(X - T(X)) = 0\). It is shown in [9] that a bounded weighted composition operator \(W\) is injective if and only if \(T\) is essentially bounded. Throughout this paper it is assumed that \(T\) is essentially bounded.

Recall that a bounded operator \(A\) defined on a Hilbert space is normal if and only if \(A^*A = AA^*\) (where \(A^*\) is an adjoint of \(A\)) and \(A\) is subnormal if \(A\) has an extension to a normal operator on a larger Hilbert space. Let \(A = U|A|\) be the canonical polar decomposition for \(A\) and let \(p \in (0, \infty)\). An operator \(A\) is \(p\)-hyponormal if \((A^*A)^p \geq (AA^*)^p\) and \(A\) is \(p\)-quasihyponormal if \(A^*(A^*A)^pA \geq A^*(AA^*)^pA\). For all unit vectors \(x \in X\), if \(||A|^pU|A|^px|| \geq ||A|^px||^2\), then \(A\) is called a \(p\)-paranormal operator (See [5]). For instance consider the known relationships between these classed as follows

\[
\text{normal} \implies \text{quasinormal} \implies \text{subnormal} \implies \text{hyponormal},
\]
\[
p - \text{hyponormal} \implies p - \text{quasihyponormal} \implies p - \text{paranormal}.
\]

Further information may be found in [1, 8, 2].

3. Subnormal Weighted Composition Operators
The assumption that $T$ is essentially bounded; plays an important role to make use of the following theorem stated in [14].

**Theorem 3.1.** Let $A$ be an operator on a Hilbert space $H$ with kernel $\{A\} = 0$. Then $A$ is subnormal if and only if for every $x$ in $H$ the sequence $\{\|A^n x\|\}_{n=0}^{+\infty}$ is a moment sequence, i.e., there is an interval $I = [0, r]$ such that for each $x$ in $H$ one can find a Borel measure $m_x$ such that for each $n \geq 0$,

$$\|A^n x\|^2 = \int_I t^n dm_x(t).$$

**Remark 3.2.** For each $n \in \mathbb{N}$ and $F \in \Sigma$, define the measure $\lambda_n^u$ by

$$\lambda_n^u(F) = \int_{T^{-n}(F)} |u|^2 d\mu.$$

According to the following chain

$$\lambda_n^u \ll \mu \circ T^{-n} \ll ... \ll \mu \circ T^{-2} \ll \mu \circ T^{-1} \ll \mu,$$

the assumption $\mu \circ T^{-1} \ll \mu$ implies that $\lambda_n^u \ll \mu$. Hence there exists the Radon-Nikodym derivative $J_n := \frac{d\lambda_n^u}{d\mu}$. For $F \in \Sigma$, we have

$$\lambda_n^u(F) = \int_{T^{-n}(F)} |u|^2 d\mu = \int_{T^{-n}(F)} E_n(|u|^2)d\mu = \int_F h_n E_n(|u|^2) \circ T^{-n} d\mu.$$

Then $d\lambda_n^u = h_n[E_n(|u|^2)] \circ T^{-n} d\mu$, since $F$ was chosen arbitrarily. It follows that $J_n = h_n[E_n(|u|^2)] \circ T^{-n}$. Note that $J_1 = J$ and let $J_0 = h_0 = 1$. Moreover, for an arbitrary $F \in \Sigma$,

$$\lambda_n^u(F) = \int_{T^{-1}(F)} |u|^2 d\mu = \int_{T^{-1}(F)} E_n(|u|^2)d\mu = \int_{T^{-1}(F)} h_n E_n(|u|^2) \circ T^{-n} d\mu = \int_F E(J_n) d\mu = \int_F h E(J_n) \circ T^{-1} d\mu.$$
Then $J_{n+1} = hE(J_n) \circ T^{-1}$. If Theorem 3.1 is now applied to a bounded weighted composition operator we have the following.

**Theorem 3.3.** $W$ is subnormal if and only if for each $f \in L^2(\Sigma)$, $\{\int_X |J_n| f^2 \, d\mu\}_{n=0}^{+\infty}$ is a moment sequence.

**Proof.** Let $f \in L^2(\Sigma)$, by calculating the $n$-th iteration of $W$ we will have

$$W^n f = \prod_{i=0}^{n-1} (u \circ T^i)(f \circ T^n).$$

Thus,

$$||W^n f||^2 = \int_X \left| \prod_{i=0}^{n-1} (u \circ T^i)(f \circ T^n) \right|^2 \, d\mu = \int_X \prod_{i=0}^{n-2} |u \circ T^i|^2 |f \circ T^{n-1}|^2 \, d\lambda_1 = \int_X \prod_{i=0}^{n-3} |u \circ T^i|^2 |f \circ T^{n-2}|^2 \, d\lambda_2 \ldots = \int_X |f|^2 \, d\lambda_u = \int_X J_n|f|^2 \, d\mu.$$  \hfill $\Box$

In what follows we are going to determine the subnormality of the weighted composition operators just by the sequence $\{J_n\}_{n=0}^{+\infty}$ based on the following standard moment sequence characterization.

**Theorem 3.4.** Let $\{\lambda_n\}_{n=0}^{+\infty}$ be a sequence of positive real numbers. Then $\{\lambda_n\}_{n=0}^{+\infty}$ is moment sequence if and only if for some interval $I$ the linear functional $\varphi$ defined on $P(I)$ - the set of polynomials over $I$ - by $\varphi(\sum_{n=0}^{+\infty} a_n t^n) = \sum_{n=0}^{+\infty} a_n \lambda_n$ is positive (i.e., $\varphi(p) \geq 0$ whenever $P(t) \geq 0$ on $I$).

**Proof.** See [20]. \hfill $\Box$

The application of above theorem characterizes subnormality of weighted composition operators initially.

**Theorem 3.5.** Let $W \in B(L^2(\Sigma))$ and $I = [0, \|J\|_{\infty}]$. Define the linear transformation $\mathcal{L} : P(I) \to L^\infty(\Sigma)$ by $\mathcal{L}(\sum_{n=0}^{+\infty} a_n t^n) = \sum_{n=0}^{+\infty} a_n J_n$. Then $W$ is subnormal if and only if $\mathcal{L}$ is positive, in a sense that $\mathcal{L}(p) \geq 0$ a.e $[\mu]$ whenever $p \geq 0$ on $I$. 

Proof. Let $W$ be a subnormal and $p$ be an arbitrary nonnegative polynomial on $I$. Let $K \subseteq \{x \in X : \mathcal{L}(p)(x) < 0\}$ with $\mu(K) < +\infty$. By Theorem 3.3 there is a Borel measure $m$ on $I$ such that for every $n \geq 0$,

$$\int_K J_n d\mu = \int_I t^n dm(t).$$

Then

$$\int_K \mathcal{L}(p)d\mu = \sum_{n=0}^{+\infty} a_n \int_K J_n d\mu = \sum_{n=0}^{+\infty} a_n \int_I t^n dm(t) = \int_I p(t) dm(t) \geq 0.$$

It follows that $\mathcal{L}(p) \geq 0$ a.e, hence $\mathcal{L}$ is positive.

Conversely, suppose that $\mathcal{L}$ is positive. Note that $\mathcal{L}$ is continuous with respect to essential sup norm since $\mathcal{L}$ preserves the unit element of both sides and $J_n(x) \in I = [0, \|J\|_{\infty}]$ a.e for each $n \geq 0$. Then $\mathcal{L}$ can be normally extended to a positive linear mapping from $C(I)$ (the space of all continuous functions on $I$) into $L^\infty(\Sigma)$.

Correspond for each $f \in L^2(\Sigma)$ a linear functional $\varphi_f$ on $C(I)$ by

$$\varphi_f(g) = \int_X \mathcal{L}(g)|f|^2 d\mu.$$

Since $\mathcal{L}(g) \in L^\infty(\Sigma)$, $\varphi_f$ is bounded. Actually its norm can not be exceed $\|\mathcal{L}\|\|f\|_2^2$. By the Riesz representation theorem there is a unique Borel measure $m_f$ such that

$$\varphi_f(g) = \int_I g(t) dm_f(t), \quad g \in C(I).$$

Hence $\int_X \mathcal{L}(g)|f|^2 d\mu = \int_I g(t) dm_f(t)$. For each $n \geq 0$, putting $g(t) = t^n$ in the last equation leads that

$$\int_X J_n |f|^2 d\mu = \int_I t^n dm_f(t).$$

It follows from Theorem 3.3 that $W$ is subnormal. \qed

**Corollary 3.6.** $W \in \mathcal{B}(L^2(\Sigma))$ is subnormal if and only if for every $\Sigma$-measurable set $A$ of finite measure the sequence $\{\int_{I-A} |u|^2 d\mu\}_{n=0}^{+\infty}$ is a moment sequence.
Proof. First suppose that \( W \) is subnormal. Then by Theorem 3.3 we have
\[
\int_X J_n d\mu = \int_A h_n [E_n(T^n)] \circ T^{-n} d\mu
\]
\[
= \int_{T^{-n}A} E_n(|u|^2) d\mu
\]
\[
= \int_{T^{-n}A} |u|^2 d\mu.
\]
Conversely, let \( A \) be an arbitrary \( \Sigma \)-measurable set of finite measure on which \( \{\int_{T^{-n}A} |u|^2 d\mu\}^{+\infty}_{n=0} \) is a moment sequence. Let \( p(t) \geq 0 \) on \( I = [0, \|J\|_\infty] \). Then there is a Borel measure \( m \) such that for each \( n \), \( \int_{T^{-n}A} |u|^2 d\mu = \int_I t^n dm(t) \). Then
\[
\int_A \mathcal{L}(p) d\mu = \sum_{n=0}^{+\infty} a_n \int_A J_n d\mu
\]
\[
= \sum_{n=0}^{+\infty} a_n \int_{T^{-n}A} |u|^2 d\mu
\]
\[
= \sum_{n=0}^{+\infty} a_n \int_I t^n dm(t)
\]
\[
\geq 0.
\]
Hence \( \mathcal{L}(p) \geq 0 \) a.e., since \( A \) was chosen arbitrarily. This fact yields that \( W \) is subnormal by Theorem 3.5. \( \square \)

**Proposition 3.7.** \( W \in \mathcal{B}(L^2(\Sigma)) \) is subnormal if and only if \( \{J_n(x)\}^{+\infty}_{n=0} \) is a moment sequence for almost every \( x \in X \).

**Proof.** The proof follows very closely the proof of Corollary 4 in [12]. Suppose first that \( W \) is subnormal. Let \( P^+(I) \) be the set of all nonnegative polynomials on \( I \). Let \( P^+_r(I) \) be the set of all polynomials with rational coefficients which are nonnegative on \( I \). By Theorem 3.5 for each \( p \in P^+_r(I) \), \( \mathcal{L}(p) \geq 0 \) a.e. Let \( X_p := \{x \in X : \mathcal{L}(p) \geq 0\} \). Then \( \mu(X - X_p) = 0 \). Since \( P^+_r(I) \) is a countable dense subset of \( P^+(I) \) we have \( \mu(X - X^*) = 0 \) where \( X^* = \bigcup X_p : p \in P^+_r(I) \). It means that for each \( p \in P^+(I) \), \( \mathcal{L}(p) \geq 0 \) a.e. Therefore by Theorem 3.5 it is inferred that \( \{J_n(x)\}^{+\infty}_{n=0} \) is a moment sequence for every \( x \in X^* \).

Conversely, suppose that \( \{J_n(x)\}^{+\infty}_{n=0} \) is a moment sequence for almost every \( x \in X \). Let \( X_0 \) be a measurable subset of \( X \) with \( \mu(X - X_0) = 0 \) in a way that \( \{J_n(x)\}^{+\infty}_{n=0} \) is a moment sequence on \( X_0 \). If \( p \) is an arbitrary nonnegative polynomial on \( I \), then by Theorem 3.4, \( \mathcal{L}(p) \geq 0 \) exactly on \( X_0 \). Thus the positivity of \( \mathcal{L} \) yields that \( W \) is subnormal by Theorem 3.5. \( \square \)

**Example 3.8.** Consider the case that \( h \circ T = h \) and \( E(|u|^2) \circ T = E(|u|^2) \). This case in turn characterizes \( p \)-quasihyponormal and \( p \)-paranormal weighted composition
operators; see [8]. Now the relationship $J_{n+1} = hE(J_n) \circ T^{-1}$ follows that $J_n = h^{n-1}J$ for each $n \geq 0$. Thus $L(\sum_{n=0}^{+\infty} a_n t^n) = a_0 + J \sum_{n=1}^{+\infty} a_n h^{n-1}$ is positive since $J(x) \in [0, ||J||_\infty]$ for almost every $x \in X$. Therefore $W$ is subnormal by Theorem 3.5.

**Example 3.9.** Let $X$ be the set of nonnegative integers, let $\Sigma$ be the $\sigma$-algebra of all subsets of $X$, and take $\mu$ to be the point mass measure determined by the $\mu(k) := m_k, k \in X$. Define the measurable transformation $T$ by

$$T(k) = \begin{cases} 0 & k = 0, 1, \\ k - 1 & k \geq 2 \end{cases}$$

and the weight function $u$ by

$$u(k) = \begin{cases} 1 & k = 0, 1, \\ k & k \geq 2. \end{cases}$$

By routine computations one may verify that

$$h(k) = \frac{1}{m_k} \sum_{j \in T^{-1}(k)} m_j,$$

$$h_n(k) = \frac{1}{m_k} \sum_{j \in T^{-1}(k)} h_{n-1}(j) m_j,$$

$$J_n(k) = \frac{1}{m_k} \sum_{j \in T^{-n}(k)} (u(j))^2 m_j.$$

So by inserting $T$ and $u$ in the above formulas we obtain

$$J_1(0) = \frac{1}{m_0} (m_0 + m_1);$$

$$J_n(0) = \frac{1}{m_0} (m_0 + m_1 + 2^2 m_2 + 3^2 m_3 + \ldots + n^2 m_n);$$

$$J_n(k) = \frac{m_{k+n}}{m_k}, \quad k \geq 1.$$  

In particular, fix $a > 0$ and let $\epsilon > 0$ be an arbitrary. Let $m_0 = \epsilon$ and for every $k \geq 1$, let

$$m_k = \frac{1}{k^2} \int_a^{a+\epsilon} (t^k - t^{k-1}) dt.$$ 

In Proposition 3.7 it is proved that $W$ is subnormal if and only if for every $k \geq 1$ the sequence $\{J_n(k)\}_{n=0}^{+\infty}$ is a moment sequence. Namely the sequences $\{m_0 + \sum_{i=1}^{n} i^2 m_i\}_{n=0}^{+\infty}$ and $\{m_{k+n} \}_{n=0}^{+\infty}$ are moment sequences. But these assertions are
true since for every \( k \geq 1 \) and \( n \geq 0 \), we have

\[
m_0 + \sum_{i=1}^{n} i^2 m_i = m_0 + \sum_{i=1}^{n} \int_{a}^{a+\epsilon} (t^i - t^{i-1}) dt = \int_{a}^{a+\epsilon} t^n dt
\]

and

\[
\frac{m_{k+n}}{m_k} = \frac{1}{(k+n)^2 m_k} \int_{a}^{a+\epsilon} (t^{k+n} - t^{k+n-1}) dt = \int_{a}^{a+\epsilon} t^n (\frac{t^k - t^{k-1}}{(k+n)^2 m_k}) dt = \int_{a}^{a+\epsilon} t^n d\mu(t),
\]

where \( d\mu = (\frac{t^{k+n-1}}{(k+n)^2 m_k}) dt \) is a nonnegative measure on \([a, a + \epsilon]\). Hence \( W \) is subnormal.

References


