NON-ABELIAN TENSOR ANALOGUES OF 2-AUTO ENGEL GROUPS

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Abstract. The concept of tensor analogues of right 2-Engel elements in groups were defined and studied by Biddle and Kappe [1] and Moravec [9]. Using the automorphisms of a given group \( G \), we introduce the notion of tensor analogue of 2-auto Engel elements in \( G \) and investigate their properties. Also the concept of \( 2 \otimes \)-auto Engel groups is introduced and we prove that if \( G \) is a \( 2 \otimes \)-auto Engel group, then \( G \otimes \text{Aut}(G) \) is abelian.

Finally, we construct a non-abelian 2-auto-Engel group \( G \) so that its non-abelian tensor product by \( \text{Aut}(G) \) is abelian.

1. Introduction

Let \( G \) and \( H \) be groups equipped with the actions of \( G \) on \( H \) and \( H \) on \( G \) (both from the right), written as \( h^g \) and \( g^h \) for all \( g \in G \) and \( h \in H \), respectively. It is always understood that a group acts on itself by conjugation. As in [2, 3], all these actions must be compatible in the sense that

\[
g^{(h^g)} = ((g^{h^{-1}})^g)^g, \quad h^{g^h} = ((h^{g^{-1}})^h)^h,
\]

for all \( g, g' \in G \) and \( h, h' \in H \).

Considering the above compatibilities of groups actions, the non-abelian tensor product \( G \otimes H \) is the group generated by the symbols \( g \otimes h \), satisfying the following relations:

\[
(gg' \otimes h) = (g^{g'} \otimes h^{g'}) (g' \otimes h),
\]

\[
g \otimes hh' = (g \otimes h') (g^{h'} \otimes h^{h'}),
\]

for all \( g, g' \in G \) and \( h, h' \in H \). The concept of non-abelian tensor product of groups was introduced by Brown and Loday in [3]. Brown, Johnson and Robertson in [2] started the investigation of non-abelian tensor product as a group theoretical object. As a special case, \( G \otimes G \) is said to be the tensor square of a group \( G \).

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Recall that the set of all right $n$-Engel elements of a group $G$ is defined by $\mathcal{R}_n(G) = \{g \in G \mid [g, n h] = 1, \forall h \in G\}$. Here $[g, h] = g^{-1}h^{-1}gh$, $[g_1, \ldots, g_n, g_{n+1}] = [[g_1, \ldots, g_n], g_{n+1}]$ and $[g, n h] = [g, h, \ldots, h]$. It is shown in [6] that $\mathcal{R}_2(G)$ is a subgroup of $G$.

Using the non-abelian tensor square, the set of all right $n\otimes$-Engel elements of a group $G$ is defined as follows:

$$\mathcal{R}_n^\otimes(G) = \{g \in G \mid [g, n \otimes h] = 1 \otimes, \forall h \in G\},$$

see also [1, 9]. The set of right $n\otimes$-Engel elements has been studied by several authors (see [1, 9, 11, 12]). Biddle and Kappe [1] proved that $\mathcal{R}_2^\otimes(G)$ is always a characteristic subgroup of $G$ contained in $\mathcal{R}_2(G)$. In [9], Moravec determined some further information on $\mathcal{R}_2^\otimes(G)$ and defined the concept of $2\otimes$-Engel groups, which reads as $G$ is a $2\otimes$-Engel group when $[g, h] = 1$ for all $g, h \in G$. He also showed that if $G$ is a $2\otimes$-Engel group, then the non-abelian tensor square $G \otimes G$ is abelian.

The automorphisms group of a given group $G$ is denoted by $\text{Aut}(G)$ and inner automorphisms by $\text{Inn}(G)$. In the present article, using the non-abelian tensor product $G \otimes \text{Aut}(G)$, we introduce the concept of tensor analogue of $2$-auto Engel elements in a group $G$ and denote it by $A\mathcal{R}_2^\otimes(G)$.

**Definition.** Let $G$ be any group. Then

$$A\mathcal{R}_2^\otimes(G) = \{g \in G \mid [g, \alpha] \otimes = 1 \otimes, \forall \alpha \in \text{Aut}(G)\}$$

is the set of $2\otimes$-auto Engel elements of $G$.

In Section 3, we show that $A\mathcal{R}_2^\otimes(G)$ is a characteristic subgroup of $G$ contained in $\mathcal{R}_2^\otimes(G)$ and give some of its properties. Also we introduce the notion of $2\otimes$-auto Engel groups and prove that if $G$ is a $2\otimes$-auto Engel group, then $G \otimes \text{Aut}(G)$ is abelian.

2. Preliminary results

In this section we summarize some of the basic facts, which are needed for the proofs of our main results.

The following propositions of [2] are needed:

**Proposition 2.1** ([2, Proposition 2]). Let $G$ and $H$ be groups equipped with compatible actions on each other. Then

(i) the groups $G$ and $H$ act on $G \otimes H$ so that

$$(g' \otimes h)^g = g'^g \otimes h^g, \quad (g \otimes h')^h = g^h \otimes h'^h,$$

for all $g, g' \in G$, $h, h' \in H$.

(ii) There are group homomorphisms $\lambda : G \otimes H \to G$ and $\lambda' : G \otimes H \to H$ such that

$$(g \otimes h)\lambda = g^{-1}g'h, \quad (g \otimes h)\lambda' = h^{-1}h,$$
for all $g \in G$ and $h \in H$.

**Proposition 2.2** ([2, Proposition 3]). Let $G$ and $H$ be groups equipped with compatible actions on each other. Then for all $g, g' \in G$ and $h, h' \in H$, the following identities are satisfied:

(i) $(g^{-1} \otimes h)g = (g \otimes h)^{-1} = (g \otimes h)^h$;
(ii) $(g' \otimes h')g \otimes h = (g' \otimes h')[g, h]$;
(iii) $g^{-1}g^h \otimes h' = (g \otimes h)^{-1}(g \otimes h)^{h'}$;
(iv) $g^h \otimes h^{-1}h = (g \otimes h)^{-y}(g \otimes h)$;
(v) $g^{-1}g^h \otimes h^{-1}h' = [g \otimes h, g' \otimes h']$.

For a given group $G$, we define the action of $G$ on Aut$(G)$ given by $\alpha^g = \alpha \varphi_g = \varphi_g^{-1} \circ \alpha \circ \varphi_g$ and the action of Aut$(G)$ on $G$ given by $g^\alpha = (g)\alpha$ for all $g \in G, \alpha \in$ Aut$(G)$ and $\varphi_g \in$ Inn$(G)$.

In the following lemma, we show that the above actions are compatible.

**Lemma 2.3.** The above actions are well-defined and compatible.

**Proof.** Clearly the actions of $G$ on Aut$(G)$ and Aut$(G)$ on $G$ are well-defined. Also for all $\alpha, \beta \in$ Aut$(G)$, $g, h \in G$ and $\varphi_g \in$ Inn$(G)$, we have

$$(h)\beta(\alpha^g) = (h)\beta\varphi_g = (h)(\varphi_g^{-1})\circ \beta \circ \varphi_g = (g^{-1})\alpha((g)\alpha h (g^{-1})\alpha)\beta(g)\alpha = (g^{-1}g)\alpha \beta^{-1}(h)\beta\alpha^{-1}(g^{-1})\alpha \beta^{-1}g(\alpha) \alpha = ((g)\alpha \beta(\alpha)(g^{-1})\alpha \beta^{-1}) \circ \varphi_g \circ \alpha = (h)(\varphi_g^{-1}) \circ (\alpha \circ \beta \circ \alpha^{-1}) \circ \varphi_g \circ \alpha = (h)((\beta^{-1})\varphi_g^\alpha).$$

Thus $\beta(\alpha^g) = ((\beta^{-1})\varphi_g^\alpha)$. Also,

$h(\alpha^h) = h(\varphi^{-1}_g \circ \alpha \circ \varphi_g) = (g^{-1})\alpha(h)\alpha(g^{-1})\alpha g = ((ghg^{-1})^\alpha)^g = ((h^g)^{-1} \varphi_g \circ \alpha = (h)((\beta^{-1})\varphi_g^\alpha).$

This completes the proof. $\square$

**Remark 1.** One notes that we may use all the identities for the non-abelian tensor product $G \otimes \text{Aut}(G)$.

**Lemma 2.4** ([7]). Let $g$ and $h$ be elements of a group $G$ and $\alpha, \beta \in$ Aut$(G)$. Then the followings are held:

(i) $[gh, \alpha] = [g, \alpha]h[h, \alpha]$;
(ii) $[g, \alpha^{-1}] = ([g, \alpha]^{-1})^\alpha^{-1};$
(iii) \([g^{-1}, \alpha] = ([g, \alpha]^{-1})g^{-1}\);
(iv) \([g, \alpha \beta] = [g, \beta][g, \alpha]^{\beta}\);
(v) \([g, \alpha]^{\beta} = [g^{\beta}, \alpha^{\beta}]\).

Following [7], the set of all right 2-auto Engel elements in a given group \(G\) is defined as follows:
\[
AR_2(G) = \{ g \in G \mid [g, \alpha, \alpha] = 1, \forall \alpha \in \text{Aut}(G) \}.
\]

Proposition 2.5 ([7, Lemma 3.2]). Let \(g\) be a right 2-auto Engel element and \(\alpha, \beta\) and \(\gamma\) be arbitrary automorphisms of a group \(G\). Then
(i) \(g^{\alpha \text{Aut}(G)} = \langle g^{\alpha} \mid \alpha \in \text{Aut}(G) \rangle\) is abelian and its elements are right 2-auto Engel elements;
(ii) \([g, \alpha, \beta] = [g, \beta, \alpha]^{-1}\);
(iii) \([g, \alpha, \beta] = [g, \alpha, \beta]^2\);
(iv) \([g, \alpha, \beta, \gamma]^2 = 1\);
(v) \([g, \alpha, \beta, \gamma] = 1\).

The following corollary is an immediate consequence of Proposition 2.5(i).

Corollary 2.6. Let \(G\) be a group, \(g \in AR_2(G)\) and \(\alpha, \beta \in \text{Aut}(G)\). Then
\([[g, \alpha], [g, \beta]] = 1\).

Theorem 2.7 ([7, Theorem 3.2]). The set of all right 2-auto Engel elements of a given group forms a characteristic subgroup.

3. Main results

The main goal of this section is to study the tensor analogues of right 2-auto Engel elements of a given group \(G\). So we define the set of all 2⊗-auto Engel elements as follows:
\[
AR_2^{\otimes}(G) = \{ g \in G \mid [g, \alpha] \otimes \alpha = 1_\otimes, \forall \alpha \in \text{Aut}(G) \}.
\]

We remind that \([g, \alpha] = g^{-1}(g)\alpha = g^{-1}g^{\alpha}\) is the autocommutator elements of \(g \in G\) and \(\alpha \in \text{Aut}(G)\), see also [5]. The subgroups \(K(G) = \{ [g, \alpha] \mid g \in G, \alpha \in \text{Aut}(G) \}\) and \(L(G) = \{ g \in G \mid [g, \alpha] = 1, \forall \alpha \in \text{Aut}(G) \}\) are the autocommutator subgroup and the absolute centre of \(G\), respectively. (see [5, 8, 10], for more information.)

The following properties of \(AR_2^{\otimes}(G)\) are useful for our further investigations.

Lemma 3.1. The set of all 2⊗-auto Engel elements of a given group \(G\) contains \(L(G)\) and is contained in the set of 2-auto Engel elements of \(G\).

Proof. It is obvious that \(L(G)\) is a subset of \(AR_2^{\otimes}(G)\). Now, by Proposition 2.1(ii), there exists a homomorphism \(\theta : G \otimes \text{Aut}(G) \to K(G)\) given by \((g \otimes \alpha)\theta = [g, \alpha]\), which gives the claim. \(\square\)

Lemma 3.2. Let \(G\) be a group. Then
(i) \((g \otimes \alpha)^{-1} = g \otimes \alpha^{-1}\);
Proof.
(i) Proposition 2(ii) implies that $1_\otimes = [g, \alpha] \otimes \alpha = (g \otimes \alpha)^{-1}(g \otimes \alpha)\alpha$.
Thus $(g \otimes \alpha)^{-1} = g \otimes \alpha^{-1}$, by Proposition 2(i).
(ii) Using part (i), we have $g \otimes \beta^{-1} \alpha = (g \otimes (\beta^{-1} \alpha)^{-1})^{-1}$. By a simple calculation we obtain

$\left((g \otimes \beta)^{-1}(g \otimes \beta^{-1})^{-\alpha}\right)^{-1} = (g \otimes \alpha)^{-1}(g \otimes \alpha^{-1})^{-\beta}$.

Hence, by Proposition 2(ii), the above equation reduces to $1_\otimes = [g, \alpha] \otimes \beta$.

(iii) Clearly $1_\otimes = [g, \alpha] \otimes \beta$.

Lemma 3.3. Let $g \in AR_2^\otimes(G)$ and $\alpha, \beta \in \text{Aut}(G)$. Then $[g, \alpha] \otimes \alpha = 1_\otimes$.

Proof. Using Lemma 2.4(iv), we have

$1_\otimes = ([g, \beta] \otimes \alpha \beta)^{[g, \alpha] \otimes \beta}$,

or

$1_\otimes = ([g, \beta] \otimes \alpha)^{[g, \alpha] \otimes \beta}([g, \alpha] \otimes \beta)^{[g, \alpha] \otimes \beta}$.

As $g \in AR_2^\otimes(G)$, we observe that $[g, \alpha]$ acts trivially on $\alpha$. Also using Lemma 3.1 and Corollary 2.6, one notes that $[g, \alpha]$ acts trivially on $[g, \beta]$. Hence we obtain

$1_\otimes = ([g, \beta] \otimes \alpha)^{[g, \alpha] \otimes \beta}([g, \alpha] \otimes \beta)^{[g, \alpha] \otimes \beta}$.

Finally, by Lemma 3.2(ii), the above equation reduces to $1_\otimes = [g, \alpha] \otimes \alpha$, which proves the claim.

Biddle and Kappe in [1] proved that $R_2^\otimes(G)$ is a characteristic subgroup of $G$. Now we are in a position to show that $AR_2^\otimes(G)$ is also a characteristic subgroup of the group $G$.

Theorem 3.4. For a given group $G$, the set of all 2_\otimes-auto Engel elements is a characteristic subgroup of $G$.

Proof. Clearly $AR_2^\otimes(G)$ is a characteristic set. We show that $AR_2^\otimes(G)$ is closed under the inverse and product. Assume $g$ is any element of $AR_2^\otimes(G)$, which is also in $AR_2(G)$. Hence $1 = [g, \phi, \alpha, \phi, \beta]$, for all $\alpha \in \text{Aut}(G)$ and $\phi \in \text{Inn}(G)$.

This implies that $[g, \alpha, \phi, \beta] = 1$, using Lemma 2.4(iv). Now, by Lemma 2.4(iii) and using the previous identity, we obtain $[g^{-1}, \alpha] = [g, \alpha]^{-1}$. So by Proposition 2.2(i), we have

$[g^{-1}, \alpha] \otimes \alpha = [g, \alpha]^{-1} \otimes \alpha = ([g, \alpha] \otimes \alpha)^{-1}[g, \alpha]^{-1} = 1_\otimes$,

which implies that $g^{-1} \in AR_2^\otimes(G)$. By Lemma 2.4(i) and using the rules of non-abelian tensor product, we obtain

$[gh, \alpha] \otimes \alpha = ([g, \alpha] \otimes \alpha)^{[h, \alpha]}$,

for all $g, h \in AR_2^\otimes(G)$ and $\alpha \in \text{Aut}(G)$. Hence, Lemma 3.3 gives the proof.
If $A$ is a subset of $\text{Aut}(G)$, we may define the auto-tensor centralizer of $A$ in $G$ as follows:

$$C^\otimes_G(A) = \{g \in G : g \otimes \alpha = 1_\otimes, \ \forall \alpha \in A\}.$$ 

It is easy to check that $C^\otimes_G(A)$ is a subgroup of $G$. The following proposition gives some useful properties of $AR^\otimes_2(G)$, which are needed in proving Theorem 3.8.

**Proposition 3.5.** Let $G$ be a group. Then for all $\alpha, \beta, \gamma \in \text{Aut}(G)$, $g \in AR^\otimes_2(G)$ and $n \in \mathbb{Z}$,

(i) $[g, \alpha] \in C^\otimes_G(\alpha^{\text{Aut}(G)})$;

(ii) $g^{-1} \otimes \alpha = (g \otimes \alpha)^{-1}$;

(iii) $[g, \alpha]^n \otimes \beta = ([g, \alpha] \otimes \beta)^n$;

(iv) $g \otimes \alpha^n = (g \otimes \alpha)^n$;

(v) $[g, \alpha] \otimes \beta, \gamma = 1_\otimes$;

(vi) $g \otimes [\alpha, \beta] = ([g, \alpha] \otimes \beta)^2$.

**Proof.** (i) It is obvious, by using Lemma 3.3.

(ii) By Proposition 2.2(i, iii) and Lemma 3.2(i, iii), we have $g^{-1} \otimes \alpha = (g \otimes \alpha^{-1})^\gamma$, and hence we conclude that $g^{-1} \otimes \alpha = (g \otimes \alpha)^{-1}$.

(iii) It suffices to assume that $n > 0$. By Lemma 3.2(ii) we observe that $[g, \alpha]^n \otimes \beta = ([g, \alpha] \otimes \beta)([g, \alpha]^n \otimes \beta)$. Hence the claim follows by induction on $n$.

(iv) Using Proposition 2.2(iii), we have $g \otimes \alpha^n = (g \otimes \beta)(g \otimes \alpha)([g, \alpha] \otimes \beta)$.

Now, $g \otimes \alpha^n = (g \otimes \alpha)^n$ is obtained by induction on $n$.

(v) Note that Lemma 3.2(ii) implies that the elements of the form $\alpha \otimes \alpha$ commute, for all $\alpha \in \text{Aut}^\otimes(G)$ and $\alpha \in \text{Aut}(G)$. By considering the identity $[g, \alpha] \otimes \beta \gamma = (g, \beta \gamma) \otimes \alpha^{-1}$, we obtain $([g, \alpha] \otimes \gamma)([g, \alpha] \otimes \beta) = ([g, \alpha] \otimes \beta)([g, \alpha \gamma] \otimes \gamma^{-1}, \alpha^{-1})^{-1}$.

Finally, Proposition 2.2(ii, v) imply that $[g, \alpha] \otimes \beta, \gamma = 1_\otimes$.

(vi) Using parts (i) and (ii), Proposition 2.2(iii), Lemma 3.2(ii) and the identity $g \otimes \alpha \beta = (g \otimes \beta)(g \otimes \alpha)([g, \alpha] \otimes \beta)$ give the claim as follows:

$$g \otimes [\alpha, \beta] = (g \otimes \alpha \beta)(g \otimes \alpha^{-1} \beta^{-1}) = ([g, \alpha] \otimes \beta)([g, \alpha^{-1}] \otimes \beta^{-1}) = ([g, \alpha] \otimes \beta)^2. \ □$$

An immediate consequence of Proposition 3.5 is the following characterization of $AR^\otimes_2(G)$.

**Corollary 3.6.** For any group $G$,

$$AR^\otimes_2(G) = \{g \in G : [g, \alpha] \in C^\otimes_G(\alpha^{\text{Aut}(G)}), \ \forall \alpha \in \text{Aut}(G)\}. $$
It is known that \( g^G \) is abelian, for each element \( g \in R_2(G) \). Also Moravec in [9], proves that if \( g \in R_2^\otimes(G) \), then the normal closure \( (g \otimes h)^{G \otimes G} \) is an abelian group for all \( h \in G \).

The following corollary gives a similar result for \( 2^\otimes \)-auto Engel elements.

**Corollary 3.7.** Let \( g \in AR_2^\otimes(G) \). Then the normal closure \( (g \otimes \alpha)^{G \otimes \text{Aut}(G)} \) is an abelian group for every \( \alpha \in \text{Aut}(G) \).

*Proof.* Let \( g \in AR_2^\otimes(G) \). We know that there exists a homomorphism \( \varphi : G \otimes \text{Aut}(G) \rightarrow K(G) \) given by \( (g' \otimes \alpha') \mapsto [g', \alpha'] \). Hence by Propositions 2.2(ii, v) and 3.5(v),

\[
\begin{align*}
[(g \otimes \alpha), (g \otimes \alpha)(g' \otimes \alpha')] &= [(g \otimes \alpha), (g \otimes \alpha)\varphi(g' \otimes \alpha')] \\
&= [(g \otimes \alpha), (g[\beta] \otimes \alpha[g[\beta]])] = 1_{\otimes}.
\end{align*}
\]

This proves the result. \( \square \)

Moravec in [9] shows that if \( G \) is a \( 2^\otimes \)-Engel group, then the non-abelian tensor product \( G \otimes G \) is abelian and \( C^\otimes_G(g) \) is a normal subgroup of \( G \), for each \( g \in G \). We say a group \( G \) is \( 2^\otimes \)-auto Engel group if \( [g, \alpha] \otimes \alpha = 1_{\otimes} \), for all \( g \in G \) and \( \alpha \in \text{Aut}(G) \).

Using [7], we conclude that the cyclic groups of orders 2 and 4 are the only non-trivial abelian 2-auto-Engel groups. Hence one can easily calculate that they are the only abelian \( 2^\otimes \)-auto Engel groups.

Now we are in a position to prove the following:

**Theorem 3.8.** Let \( G \) be a \( 2^\otimes \)-auto Engel group. Then

(i) \( G \otimes \text{Aut}(G) \) is abelian;

(ii) \( C^\otimes_G(\alpha) \) is a characteristic subgroup of \( G \).

*Proof.* (i) Using Propositions 2.2(v) and 3.5(v), we have

\[
[g \otimes \alpha, h \otimes \beta] = [g, \alpha] \otimes [\varphi_h, \beta] = 1_{\otimes},
\]

for all \( g, h \in G \) and \( \alpha, \beta \in \text{Aut}(G) \).

(ii) It is clear that \( C^\otimes_G(\alpha) \) is a subgroup of \( G \). Then by Lemma 3.2(ii), for each \( g \in C^\otimes_G(\alpha) \) and \( \alpha, \beta \in \text{Aut}(G) \),

\[
g^\beta \otimes \alpha = g[g, \beta] \otimes \alpha = (g \otimes \alpha)^{[g, \beta]}(g, \beta \otimes \alpha) = ([g, \alpha] \otimes \beta)^{-1} = 1_{\otimes}.
\]

Hence \( C^\otimes_G(\alpha) \) is a characteristic subgroup of \( G \). \( \square \)

In the following example we give a non-abelian \( 2^\otimes \)-auto-Engel group \( G \), for which \( G \otimes \text{Aut}(G) \) is abelian.
Example. Let $G = (\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$ be the semidirect products of cyclic group of order 4, which is the group $(64, 68)$ in the GAP small groups library [4]. Using GAP, one can check that the centre and the group of automorphisms of $G$ are both elementary abelian 2-groups. Hence, by Proposition 3.1 in [7], the group $G$ is a purely non-abelian 2-auto-Engel group. Now by Corollary 3.7 in [7], $K(G) \leq Z(G)$ and $\text{Aut}(G) = \text{Aut}_c(G)$, where $\text{Aut}_c(G)$ is the group of all central automorphisms of $G$. These automorphisms fix the central factor group of $G$ element-wise, equivalently they commute with inner automorphisms of $G$.

Hence, by Propositions 2.1(i) and 2.2(ii),

$$(g \otimes \alpha)^{-1}(g' \otimes \alpha')(g \otimes \alpha) = (g' \otimes \alpha')^{[g, \alpha]}$$

for all $g, g' \in G$ and $\alpha, \alpha' \in \text{Aut}(G)$. Thus the non-abelian tensor product $G \otimes \text{Aut}(G)$ is abelian.

Finally, using GAP one can check that $G = (\mathbb{Z}_4 \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_4$ and $H = (\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ are the smallest non-abelian 2-auto-Engel groups of order 64. For more descriptions of these groups see [7]. We remark that by Lemma 3.1, $AR^2_2(G)$ is contained in $AR_2(G)$ and using Theorem 3.8 if the group $G$ is a $2_\otimes$-auto Engel, then $G \otimes \text{Aut}(G)$ is abelian. Hence we strongly believe that $G$ and $H$ are both $2_\otimes$-auto Engel groups.

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