SAMELSON PRODUCTS IN FUNCTION SPACES

JEAN-BAPTISTE GATINZI AND RUGARE KWASHIRA

Abstract. We study Samelson products on models of function spaces. Given a map \( f : X \to Y \) between 1-connected spaces and its Quillen model \( L(f) : L(V) \to L(W) \), there is an isomorphism of graded vector spaces \( \Theta : H^*(\text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(W))) \to H^*(L(W) \oplus \text{Der}(L(V), L(W))). \) We define a Samelson product on \( H^*(\text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(W))). \)

1. Introduction

Throughout this paper, spaces are assumed to be 1-connected finite CW-complexes. Given a map \( f : X \to Y \) between 1-connected spaces and its Quillen model \( L(f) : L(V) \to L(W) \), let \( \text{map}(X, Y; f) \) and \( \text{map}_*(X, Y; f) \) denote the path component containing \( f \) in the space of base point-free and base point-preserving functions respectively. Lupton-Smith [3] extended the notion of derivation of a differential graded Lie algebra to a derivation with respect to a map of differential graded Lie algebras. They proved the following vector space isomorphisms:

\[
\pi_n(\text{map}(X, Y; f)) \otimes Q \cong H_n(sL(W) \oplus \text{Der}(L(V), L(W)); f),
\]

and

\[
\pi_n(\text{map}_*(X, Y; f)) \otimes Q \cong H_n(\text{Der}(L(V), L(W)); f).
\]

The authors in their paper [2] established an isomorphism

\[
\pi_{n-1}(\text{map}(X, Y; f)) \otimes Q \cong H_n(\text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(W))).
\]

Lupton-Smith [4] have expressed the Whitehead product on

\[
H_n(sL(W) \oplus \text{Der}(L(V), L(W))).
\]

We study the Samelson product in the homology of \( \text{Hom}(P, L(W)) \) where \( P = (TV \otimes (Q \oplus sV), D) \) is the acyclic TV-differential module with a differential defined as follows:

\[
Dv = dv \otimes 1, \quad Dsv = v \otimes 1 - S(dv \otimes 1)
\]

Received September 2, 2014.
2010 Mathematics Subject Classification. 55P62, 55Q15.
Key words and phrases. Lie model, Lie algebra of derivations, Samelson product.

©2015 Korean Mathematical Society

1297
and $S$ is the $\mathbb{Q}$-graded vector spaces map of (degree 1) defined by
\[
\begin{align*}
S(v \otimes 1) &= 1 \otimes sv, \quad S(1 \otimes (Q \oplus sV)) = 0, \\
S(ax) &= (-1)^{|a|}aS(x), \quad \forall a \in T(V), x \in Q \oplus sV, \quad |x| > 0, \\
S(1 \otimes x) &= 0.
\end{align*}
\]

2. Preliminaries

Let $(L, d)$ be a differential graded Lie algebra. Given a map $\phi : (L, d_L) \to (K, d_K)$ of differential graded Lie algebras, define a $\phi$-derivation of degree $p$ to be a linear map $\theta : L \to K$ that increases degree by $p$ and satisfies $\theta([x, y]) = [\theta(x), \phi(y)] + (-1)^{|\phi||\theta|}[\phi(x), \theta(y)]$ for $x, y \in L$. Consider the vector space of $\phi$-derivations $\text{Der}_\phi(L, K; \phi)$ which are derivations that increase degree by some positive number $n$. When $n = 1$, we restrict to derivations $\theta$ such that $d_K \circ \theta + \theta \circ d_L = 0$. The differential $D$ is defined by $D(\theta) = d_K \circ \theta - (-1)^{|\phi|}\theta \circ d_L$. Following [5, p. 46], if $\phi : A \to B$ is a morphism of chain complexes, the mapping cone of $\phi$ is defined by $\text{Cone}(\phi) = sA \oplus B$ with the differential $d(sa, b) = (-sd_A(a), \phi(a) + d_B(b))$.

Let $(L, d)$ be a differential graded Lie algebra. Consider the adjoint mapping $\text{ad} : L \to \text{Der} L$, defined by $(\text{ad} x)(y) = [x, y]$. We have $\text{Cone}(\text{ad}) = sL \oplus \text{Der} L$ where $(sL \oplus \text{Der} L, D)$ is a differential graded Lie algebra with the bracket defined as follows [6]:
\[
\begin{align*}
[sx, sy] &= 0, \\
[sx, \theta] &= (-1)^{|x||\theta|}s\theta(x), \\
[sx_1 + \theta_1, sx_2 + \theta_2] &= (-1)^{|\theta_1|}s\theta_1(sx_2) - (-1)^{|\theta_2|}|x_1|s\theta_2(sx_1) + [\theta_1, \theta_2].
\end{align*}
\]
The differential is defined by $D(sx) = -s\delta x + ad x$. Note that $sx + \theta$ is a cycle if and only if $d(x) = 0$ and $[d, \theta] + ad x = 0$.

If $L = (L(V), d)$, there is a differential isomorphism of degree 1
\[
\Phi : \text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV, L)) \to sL \oplus \text{Der} L
\]
defined by $\Phi(f) = sf(1) + (-1)^{|f|}f \theta$ where $\theta(v) = f(sv)$ [1]. This gives a bracket (of degree 1) on $\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV, L(V)))$ defined on generators as follows: For $f, g \in \text{Hom}_{TV}(TV \otimes L(V), f(1), g(1)) \in L(V)$, and $f(sv) \in L(V) \subset T(V)$, then
\[
\begin{align*}
\{f, g\}(sv) &= -(-1)^{|f|}f(S(g(sv))) - (-1)^{|f|(|g|+1)}f(S(g(sv))))], \\
\{f, g\}(1) &= f(S(g(1))) - (-1)^{|f|(|g|+1)}S(f(1))).
\end{align*}
\]

3. Whitehead product in differential graded Lie algebras

Let $L$ be a differential graded Lie algebra and $sL$ the suspension on $L$. A bilinear pairing on $sL$ is defined by $[sx, sy] = (-1)^{|x||y|}s[x, y]$ where $x, y \in L$. A Whitehead product denoted by $[,]_W$ is a bilinear pairing satisfying the following identities:

(1) $|\alpha, \beta|_W = |\alpha| + |\beta| - 1,$
(2) $[\alpha, \beta]_W = (-1)^{|\alpha||\beta|}[\beta, \alpha]_W$ and
Lupton-Smith [4] defined a Whitehead product on its mapping cone (Cone(Φ), δq) which induces a Whitehead product [,]W on H∗(Cone(Φ), δq). The product is defined explicitly as follows:

Let a, b ∈ L; |a| = p − 1, |b| = q − 1,

\[[a, α] ∈ Cone_*(Φ), (b, β) ∈ Cone_*(Φ) \text{ where } α ∈ K_p, β ∈ K_q.\]

4. Samelson product on H∗(HomTV(TV ⊕ (Q ⊕ sV), L(W)))

Let f : X → Y be a map, Φ : L(V) → L(W) its Quillen model. Consider the map adΦ : L(W) → Der(L(V), L(W), Φ). The mapping cone of adΦ is (sL(W) ⊕ Der(L(V), L(W)), D) where D(θ) = δWθ − (−1)|θ|δV and D(sx) = −sδVx + adΦ(x). If (sa, θ), (sb, θ′) are cycles of degree p and q respectively, Lupton-Smith [4] defined a Whitehead product

\[H_p ⊗ H_q → H_{p+q−1}, \text{ where } H_p = H_p(sL(W) ⊕ Der(L(V), L(W); Φ))\]

...
where \( D(\theta) = D_\lambda(\theta) \) for \( \theta \in \text{Der}(L(V), L(V)(a, b)) \) and \( D(sx) = -s\delta x + \text{ad}_x x \) for \( x \in L(V)(a, b) \). We have \( D((-1)^{|x|}sx - S_x) = (-1)^{|x|}(-s\delta x + \text{ad}_x x) - (-1)^{|x|}\text{ad}_x x = 0 \) for \( x = a, b \). In the complex \((\text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(V)(a, b), D), \text{We consider } \phi_a \text{ and } \phi_b \text{ defined by}

\begin{align*}
\phi_a(1) &= -(-1)^{|a|}a; \\
\phi_a(sv) &= (-1)^{|a|}S_a(v)
\end{align*}

or in a condensed form \( \phi_a(1) = (-1)^{|x|}x; \phi_a(sv) = (-1)^{|x|}S_a(v), x = a, b. \)

**Proposition 1.** \( \phi_a \) and \( \phi_b \) are cycles of degrees \( p - 1 \) and \( q - 1 \) respectively.

**Proof.** It can be easily verified that \( (D\phi_a)(1) = 0 \) for \( x = a, b \). For \( (D\phi_a)(sv) \) we have

\[
(D\phi_a)(sv) = \delta\phi_a(sv) - (-1)^{|x|}\phi_a(d(sv))
\]

\[
= \delta((-1)^{|x|}S_x(v)) - (-1)^{|x|}\phi_a(v \otimes 1 - s(dv \otimes 1))
\]

\[
= (-1)^{|x|}((-1)^{|x|}[x, v] - (-1)^{|x|}S_x(dv)) - (-1)^{|x|}(-1)^{|v|}[v, \phi_x(1)]
\]

\[
+ (-1)^{|x|}\phi_a(s(dv \otimes 1))
\]

\[
= [x, v] - (-1)^{|x|}S_x(dv) + (-1)^{|x|}[v, x] + (-1)^{|x|}\phi_a(s(dv \otimes 1))
\]

\[
= 0.
\]

Define \( \theta_a, \theta_b \in \text{Der}(L(V)(a, b)) \) such that

\[
\begin{align*}
\theta_a(v) &= v^a; \quad \text{zero otherwise}, \\
\theta_b(v) &= v^b; \quad \text{zero otherwise}.
\end{align*}
\]

We have \( [\delta, \theta_x] = (-1)^{|x|}\text{ad } x \); \( x = a, b \).

Observe that \( S_a = \theta_a \) and \( S_b = \theta_b \) if restricted to \( L(V) \). We define a universal Samelson product \([\phi_a, \phi_b]\) in \((\text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(V)(a, b), D)\) as follows:

\[
[\phi_a, \phi_b](1) = (-1)^{|b|+1}[\phi_a(1), \phi_b(1)]
\]

\[
= (-1)^q[a, b],
\]

\[
[\phi_a, \phi_b](sv) = -(-1)^{|a|+1}[\theta_a, [\delta, \theta_b]](v)
\]

\[
= -(-1)^p[\theta_a, [\delta, \theta_b]](v).
\]

**Proposition 2.** \( \Upsilon = [\phi_a, \phi_b] \) is a cycle of degree \( p + q - 2 \).

**Proof.** There is a differential isomorphism of chain complexes

\[
F : \text{Hom}_{TV}(TV \otimes (Q \oplus sV), L(V)(a, b)) \longrightarrow sL(V)(a, b) \oplus \text{Der}(L(V), L(V)(a, b))
\]

defined by \( F(f) = sf(1) + (-1)^{|f|} \theta \) with \( \theta(v) = f(sv) \) see [1, Theorem 1]. This is an isomorphism of degree 1 \((DF = -FD)\). In particular

\[
F([\phi_a, \phi_b]) = (-1)^{p+q}s[\phi_a, \phi_b](1) - (-1)^{p+q}(-1)^p[\theta_a, [\delta, \theta_b]] \circ \lambda
\]

\[
= (-1)^p s[a, b] - (-1)^q[\theta_a, [\delta, \theta_b]] \circ \lambda.
\]
In order to check that $\mathcal{T}$ is a cycle, we need only to show that $(-1)^p s[a, b] - (-1)^q [\theta_a, [\delta, \theta_b]] \circ \lambda$ is a cycle in $sL(V)(a, b) \oplus \text{Der}(\mathbb{L}(V), \mathbb{L}(V)(a, b))$. Indeed
\[
D((-1)^p s[a, b] - (-1)^q [\theta_a, [\delta, \theta_b]] \circ \lambda) = -(-1)^p s[\delta, a, b] + (-1)^p \text{Ad}_\lambda[a, b] - (-1)^q [\delta, [\theta_a, [\delta, \theta_b]]] \circ \lambda
\]
\[
= (-1)^p \text{Ad}_\lambda[a, b] - (-1)^q [[\delta, \theta_a], [\delta, \theta_b]] \circ \lambda
\]
\[
= (-1)^p \text{Ad}_\lambda[a, b] - (-1)^q \text{Ad} a, \text{Ad} b \circ \lambda
\]
\[
= (-1)^p \text{Ad}_\lambda[a, b] - (-1)^q \text{Ad}_\lambda[a, b]
\]
\[
= 0.
\]

We are now ready to define Samelson’s products in $H_*(\text{Hom}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)))$. Recall that $f : X \rightarrow Y$ has a Quillen model $\Phi : \mathbb{L}(V) \rightarrow \mathbb{L}(W)$. Let $\alpha, \beta$ be cycles of respective degrees $p - 1, q - 1$ in $\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W))$. Define $\Psi_{\alpha, \beta} : \mathbb{L}(V)(a, b) \rightarrow \mathbb{L}(W)$ by:
\[
\Psi_{\alpha, \beta}(a) = \alpha(1), \quad \Psi_{\alpha, \beta}(b) = \beta(1),
\]
\[
\Psi_{\alpha, \beta}(v) = \Phi(v), \quad \Psi_{\alpha, \beta}(v^a) = \alpha(sv), \quad \Psi_{\alpha, \beta}(v^b) = \beta(sv).
\]

**Theorem 3.**
1. The composition

\[
TV \otimes (\mathbb{Q} \oplus sV) \xrightarrow{[\phi_a, \phi_b]} \mathbb{L}(V)(a, b) \xrightarrow{\Psi_{\alpha, \beta}} \mathbb{L}(W)
\]

defines a Samelson product on $H_*(\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)))$.
2. It is explicitly defined on the generators by

\[
[\alpha, \beta](1) = (-1)^{[\beta] + 1} [\alpha(1), \beta(1)],
\]
\[
[\alpha, \beta](sv) = (-1)^{[\alpha] + 1} [\alpha(1), \beta(sv)] + (-1)^{[\beta] + 1} [\beta(1), \alpha(sv)]
\]
\[
+ (-1)^{[\beta]} [\alpha, \beta](\theta_a \theta_a(dv)),
\]

where $\theta_a, \theta_b \in \text{Der}(\mathbb{L}(V)(a, b))$.

**Proof.** The first assertion is a direct consequence of the isomorphism

\[
\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W)) \cong sL(V) \oplus \text{Der}(\mathbb{L}(V), \mathbb{L}(W))
\]

and [4, Theorem 3.6.]. For the second assertion consider $\alpha, \beta$ cycles of respective degrees $p - 1$ and $q - 1$ in $\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(W))$ and $\phi_a, \phi_b$ corresponding cycles in $\text{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V)(a, b))$ (see Proposition 1), we have

\[
[\phi_a, \phi_b](sv) = -(-1)^p [\theta_a, [\delta, \theta_b]](v)
\]
\[
= -(-1)^p [\theta_a([\delta, \theta_b](v) - (-1)^p [\delta, [\theta_a, [\theta_a, [\delta, \theta_b]]]
\]
\[
= -(-1)^p [\theta_a(([-1]^{q-1}[b, v]) - (-1)^p (q-1)[\delta\theta_a\theta_a(v) - (-1)^q \theta_a\delta\theta_a(v)]
\]
\[
= -(-1)^p (q-1)(-1)^p [b, \theta_a](v)
\]
The homotopy Lie algebra of classifying spaces

J.-B. Gatsinzi

□ boundary.

J.-B. Gatsinzi and R. Kwashira

Rational homotopy groups of function spaces

G. Lupton and B. Smith

Rationalized evaluation subgroups of a map II: Quillen models and the adjoint maps

Proof.

Corollary 4. If $f : X \to Y$ is null homotopic, then

$[\alpha, \beta](sv) = (-1)^{q + 1} [1, \beta(1), \beta(sv)] + (-1)^p [\alpha(1), \beta(sv)]$

In particular $H_*(L(W))$ is a sub Lie algebra of $H_*(\text{Hom}(TV \otimes (Q \oplus sV), L(W)))$.

References


Jean-Baptiste Gatsinzi
University of Namibia
340 Mandume Ndemufayo Avenue
Private Bag 13301
Pionierspark, Windhoek, Namibia
E-mail address: jgatsinzi@unam.na

Rugare Kwashira
School of Mathematics
Faculty of Science
University of the Witwatersrand
Private Bag X3, Wits, 2050
South Africa
E-mail address: Rugare.Kwashira@wits.ac.za