THE LIMITING CASE OF SEMICONTINUITY OF AUTOMORPHISM GROUPS

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ABSTRACT. In this paper we study the semicontinuity of the automorphism groups of domains in multi-dimensional complex space. We give examples to show that known results are sharp (in terms of the required boundary smoothness).

1. Introduction

The paper [4] was the first work to study the semicontinuity of automorphism groups of domains in complex space. The main result there is as follows:

Theorem 1.1. Let \( \Omega^* \subseteq \mathbb{C}^n \) be a strongly pseudoconvex domain with smooth boundary. Then there is a neighborhood \( \mathcal{U} \) of \( \Omega^* \) in the \( C^\infty \) topology on domains (that is to say, \( \mathcal{U} \) is a collection of domains) so that, if \( \Omega \in \mathcal{U} \), then \( \text{Aut}(\Omega) \) is a subgroup of \( \text{Aut}(\Omega^*) \). Moreover, there is a \( C^\infty \) mapping \( \Psi \) from \( \Omega \) to \( \Omega^0 \) so that

\[
\text{Aut}(\Omega) \ni \varphi \mapsto \Psi \circ \varphi \circ \Psi^{-1}
\]

is an injective group homomorphism from \( \text{Aut}(\Omega) \) to \( \text{Aut}(\Omega^*) \).

Over the years, the hypothesis of smooth or \( C^\infty \) boundary in this theorem has been weakened. In the paper [5], the hypothesis was weakened (using an entirely different argument) to \( C^2 \) boundary smoothness. In the paper [3], yet another approach to the \( C^2 \) boundary smoothness situation was described. The paper [2] treats the case of \( C^1 \) boundary smoothness. Also the paper [1] treats other points of view, such as the dependence on the dimension of the automorphism group.

In the present paper we show that \( C^2 \) boundary smoothness is sharp for this type of result.
2. Notation, terminology, and enunciation of results

For us a domain in \( \mathbb{C}^n \) is a connected, open set. We generally denote a domain by \( \Omega \). We let the automorphism group of \( \Omega \), denoted by \( \text{Aut}(\Omega) \), be the collection of biholomorphic selfmaps of \( \Omega \). These form a group with the binary relation of composition. The topology on \( \text{Aut}(\Omega) \) is the compact-open topology (equivalently, the topology of uniform convergence on compact sets).

If \( \Omega \) is a domain with at least \( C^1 \) boundary, then we equip it with a defining function \( \rho \). This is a \( C^1 \) function defined on a neighborhood \( U \) of \( \partial \Omega \) so that \( \Omega \cap U = \{ z \in U : \rho(z) < 0 \} \).

We generally require that \( \nabla \rho \not= 0 \) on \( \partial \Omega \), so that the outward normal vector is well defined at each boundary point. We say that \( \Omega \) has \( C^k \) boundary if it has a defining function that is \( C^k \) (that is to say, \( k \)-times continuously differentiable).

Let \( \Omega \) be a domain with \( \partial \Omega \subset \mathbb{C}^2 \). We restrict attention to \( \mathbb{C}^2 \) just to simplify the notation a bit. The proof of an analogous result in \( n \) dimensions is quite similar.

Let \( \varphi \) be a \( C^\infty \) function supported in the Euclidean ball \( B(0, \delta) \) of center the origin and radius \( \delta > 0 \). We assume that \( \varphi \) is identically equal to 1 on \( B(0, \delta/2) \). Now, for \( K \geq 1 \) we specify the defining function

\[
\tilde{\rho}_K(w_1, w_2) = |w_2|^2 - \text{Im} w_1 - \sum_{k=0}^{K} \sum_{j=-\infty}^{\infty} 2^{2jk} \varphi \left( k + 2^{-2jk} w_1, 2^{-jk} w_2 \right).
\]
Let
\[ \tilde{\Omega}_K = \{(w_1, w_2) \in \mathbb{C}^2 : \tilde{\rho}_K(w_1, w_2) < 0\}. \]

This new domain should be compared to the defining function for the Siegel upper half space which is given by
\[ \tilde{\rho}_U = |w_2|^2 - \text{Im} w_1. \]

Of course it is well known that the Siegel upper half space \( U \) is biholomorphic to the unit ball \( B \). Indeed, the relevant mappings are
\[
\Phi : B \to U \\
(z_1, z_2) \mapsto \left( i \cdot \frac{1 - z_1}{1 + z_1}, \frac{z_2}{1 + z_1} \right)
\]
and
\[
\Phi^{-1} : U \to B \\
(w_1, w_2) \mapsto \left( \frac{i - w_1}{i + w_1}, \frac{2iw_2}{i + w_1 \cdot i + w_1} \right).
\]

So we think of \( \tilde{\rho}_K \) as defining a perturbation of the Siegel upper half space \( U \). Now we use \( \Phi^{-1} \) to pull this perturbed domain back to a perturbation of the unit ball \( B \). Now let
\[ \Omega_K = \Phi^{-1}(\tilde{\Omega}_K). \]

Of course, by inspection, \( \Omega_K \) has \( C^\infty \) smooth boundary except at the points \((\pm 1, 0)\) where the “bumps” coming from the translates of \( \varphi \) accumulate. We need to say something about the boundary smoothness at those two exceptional points, and we need to say something about the automorphism group of \( \Omega_K \) for each \( K \). Finally, we need to specify what the limit of the domains \( \Omega_K \) is as \( K \to +\infty \).

Examining our list of desiderata, we see that, with
\[ \tilde{\Omega}^* = \{(w_1, w_2) \in \mathbb{C}^2 : \tilde{\rho}^*(w_1, w_2) < 0\} \]
and
\[ \tilde{\rho}^* = |w_2|^2 - \text{Im} w_1 - \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} 2^{2jk} \varphi \left( k + 2^{-2jk} w_1, 2^{-jk} w_2 \right), \]
and
\[ \Omega^* = \Phi^{-1}(\tilde{\Omega}^*), \]
then \( \Omega_K \to \Omega^* \) in some sense as \( K \to +\infty \). In point of fact, the \( \Omega_K \) certainly converge to \( \Omega^* \) in the Hausdorff metric on sets. And it is also clear from inspection that the defining functions \( \rho_K \) for the \( \Omega_K \) (obtained by pulling back the defining functions for the \( \tilde{\Omega}_K \)) are bounded in the \( C^2 \) topology. In point of fact, a simple calculation with \( \Phi^{-1} \) shows that the \( j \)th bump in the \( k \)th group has height \( \approx 2^{-2|jk|} \) and diameter \( \approx 2^{-|jk|} \). The main point being that the decay is quadratic as the two extreme points are approached. That is why we get boundedness in the \( C^2 \) norm.
It follows then, by a version of the Landau inequalities (for which see [6]),
that $\Omega_K$ converges to $\Omega^*$ in the $C^{1,1}$ topology.

Now what about the automorphism group of $\Omega_K$? It is easiest to instead examine the automorphism group of $\tilde{\Omega}_K$ (which is of course the same group).

Thanks to work in [9], we know that any automorphism of $\tilde{\Omega}_K$ must be an automorphism of the Siegel upper half space $U$ which preserves the "bumps" that are created by the translates of $\varphi$. We conclude that the only possible automorphisms are dilations of the Siegel upper half space (see [8, Ch. 10]).

Examining the definition of $\tilde{\rho}_K$, we see that a dilation $\alpha_{\delta}(w_1, w_2)$ can leave this defining function invariant if and only if $\delta = 2^m$ and $m$ is divisible by $1, 2, \ldots, K$. In detail,

$$\tilde{\rho}_K(2^m w_1, 2^m w_2) = |2^m w_2|^2 - \text{Im} \ 2^m w_1$$

$$= \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} 2^{2j} \varphi \left( k + 2^{-2j} 2^m w_1, 2^{-j} 2^m w_2 \right)$$

$$= |2^m w_2|^2 - \text{Im} \ 2^m w_1$$

$$- \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} 2^{2j} \varphi \left( k + 2^{m-2j} w_1, 2^{m-j} w_2 \right).$$

Multiplying by $2^{-2m}$, we see that we must examine

$$|w_2|^2 - \text{Im} w_1 - \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} 2^{2j} 2^{-2m} \varphi \left( k + 2^{-2j} 2^m w_1, 2^{-j} 2^m w_2 \right).$$

We want to shift the index of summation by replacing $j$ with $j + m/k$, but we can only do so if $m$ is divisible by $k$ for every $k = 1, 2, \ldots, K$. The result of this shift is

$$|w_2|^2 - \text{Im} w_1 - \sum_{k=0}^{K} \sum_{j=-\infty}^{\infty} 2^{2j} \varphi \left( k + 2^{-2j} w_1, 2^{-j} w_2 \right).$$

So we see that the defining function has been preserved under the dilation. Hence $w \mapsto \alpha_{2^m}(w)$ is an automorphism of $\tilde{\Omega}_K$ provided that $k \mid m$ for $k = 1, 2, \ldots, K$. In particular, we must demand that $m \geq K$. Since iterates of this dilation are also automorphisms, we conclude that the automorphism group of $\tilde{\Omega}_K$ contains a copy of $\mathbb{Z}$.

The analysis in the last paragraph also shows that the automorphism group of $\tilde{\Omega}^*$ does not contain any nontrivial dilations. For, if it did, then it would have to be a dilation of magnitude $2^m$ with $m \geq K$ for every positive $K$. And that is impossible. So the automorphism group of $\tilde{\Omega}^*$ is trivial—it contains only the identity map.
In conclusion, we have verified all the required properties of the $\tilde{\Omega}_K$ (and hence also of the $\Omega_K$) and of $\Omega^*$. Thus the theorem is proved.

References


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