

## NON-DEVELOPABLE RULED SURFACES WITH TIMELIKE RULING IN MINKOWSKI 3-SPACE

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**ABSTRACT.** In this paper, using pseudo-spherical Frenet frame of pseudo-spherical curves in hyperbolic space, we define the notion of the structure functions on the non-developable ruled surfaces with timelike ruling. Then we obtain the properties of the structure functions and a complete classification of the non-developable ruled surfaces with timelike ruling in Minkowski 3-space by the theories of the structure functions.

### 1. Introduction

H. R. Müller introduced the concepts of the pitch and the angle of the pitch on a closed ruled surface in Euclidean 3-space [4]. The authors of [2] defined the structure functions of the non-developable ruled surfaces in Euclidean 3-space, and they also gave the deep relationship between the structure functions and the pitch, angle functions. Especially, non-developable ruled surfaces have been characterized by the structure functions [5].

In this paper we consider non-developable ruled surfaces with timelike ruling in Minkowski 3-space and the angle of pitch of the ruled surfaces in Minkowski 3-space. Firstly, we define the structure functions of non-developable ruled surfaces with timelike ruling, and then we generalize the notion of the angle of pitch of the closed ruled surfaces to any non-developable ruled surfaces with timelike ruling, called angle (density) function of pitch, or according to its kinematics meaning, self spinning function of non-developable ruled surfaces with timelike ruling in Minkowski 3-space. Exactly, we will show the kinematics meaning of the structure functions of the ruled surfaces with timelike ruling. Then we obtain some properties of the structure functions, pitch function and self spinning function. Finally, we give a kind of classification of non-developable

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Received September 12, 2014; Revised December 16, 2014.

2010 *Mathematics Subject Classification.* 53A35, 53A55, 53B30.

*Key words and phrases.* pseudo-spherical curve, Frenet formula, ruled surface, structure function, pitch function.

This work was supported by NSFC (Nos. 11201056 and 11371080) and the Fundamental Research Funds for Ministry of Education of China (No. N130405006). The second author was financial supported by the China Scholarship Council.

ruled surfaces with timelike ruling in Minkowski 3-space according to the theories of the structure functions, pitch function and self spinning function of the non-developable ruled surfaces with timelike ruling in Minkowski 3-space.

## 2. Pseudo-spherical curves in $\mathbb{H}^2$

Let  $\mathbb{R}^3$  be a 3-dimensional real vector space with its usual vector structure. Denote by  $B = \{e_1, e_2, e_3\}$  the usual base of  $\mathbb{R}^3$ , that is,

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$

The 3-dimensional Minkowski space is the metric space  $\mathbb{E}_1^3 = (\mathbb{R}^3, \langle, \rangle)$  with the metric  $\langle, \rangle$  given by

$$\langle x, y \rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

where  $x = (x_1, x_2, x_3)$ ,  $y = (y_1, y_2, y_3)$ . Here the metric  $\langle, \rangle$  is called the *Lorentzian metric*, which can also be written as:

$$\langle x, y \rangle = x \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} y^{\text{Tran}}.$$

The *pseudo-norm* of the vector  $v \in \mathbb{E}_1^3$  is defined by  $\|v\| = \sqrt{|\langle v, v \rangle|} = \sqrt{|v^2|}$ .

If  $u = (u_1, u_2, u_3)$ ,  $v = (v_1, v_2, v_3) \in \mathbb{E}_1^3$ , the *Lorentzian vector product* of  $u$  and  $v$  is defined as the unique vector denoted by  $u \times v$  that satisfies

$$(2.1) \quad u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_2v_1 - u_1v_2).$$

By simple verification, it follows that

$$(2.2) \quad u \times v \times w = \langle v, w \rangle u - \langle u, w \rangle v, \quad \forall u, v, w \in \mathbb{E}_1^3.$$

Naturally an arbitrary vector  $v \in \mathbb{E}_1^3$  has one of the following three Lorentz causal characters:

- spacelike if  $\langle v, v \rangle > 0$  or  $v = 0$ ;
- timelike if  $\langle v, v \rangle < 0$ ;
- lightlike (null) if  $\langle v, v \rangle = 0$  and  $v \neq 0$ .

Similarly, an arbitrary curve  $r = x(s) \in \mathbb{E}_1^3$ ,  $s \in \mathbf{I} \subset \mathbb{R}$ , is called spacelike (resp. timelike, null), if its velocity vectors  $x'(s)$  is spacelike (resp. timelike, null). Without special declaration, in this paper, we always assume the curve  $r$  is not a null curve. For the (space) curve  $r = x(s) : \mathbf{I} \rightarrow \mathbb{E}_1^3$  parameterized by its pseudo-arc length  $s$ , especially in the following, we will use the dot notation “.” to denote the derivative with respect to the arc length  $s$  of a curve, while “ ’ ” represents the differentiation with respect to the parameter  $t$ .

Let  $\{\alpha(s), \beta(s), \gamma(s)\}$  be the Frenet frame field along curve  $x(s)$ , where  $\alpha(s)$  is the tangent vector field,  $\beta(s)$  is the normal vector field and  $\gamma(s)$  is the binormal vector field, respectively. Certainly, it is easy to verify that the Frenet

frame  $\{\alpha(s), \beta(s), \gamma(s)\}$  must contain two spacelike vectors and one timelike vector. Therefore, we can assume

$$\langle \alpha, \alpha \rangle = \varepsilon_1, \langle \beta, \beta \rangle = \varepsilon_2, \langle \gamma, \gamma \rangle = \varepsilon_3,$$

where  $\varepsilon_1\varepsilon_2\varepsilon_3 = -1$ ,  $\varepsilon_i = \pm 1$ ,  $i = 1, 2, 3$ . Accordingly, the Frenet formulas of  $x(s)$  could be given by

$$(2.3) \quad \begin{cases} \dot{x}(s) = \alpha(s), \\ \dot{\alpha}(s) = \kappa(s)\beta(s), \\ \dot{\beta}(s) = -\varepsilon_1\varepsilon_2\kappa(s)\alpha(s) + \tau(s)\gamma(s), \\ \dot{\gamma}(s) = -\varepsilon_2\varepsilon_3\tau(s)\beta(s), \end{cases}$$

where the functions  $\kappa(s)$  and  $\tau(s)$  are the (Frenet) curvature function and torsion function of  $x(s)$  respectively.

Now on the base of the different values of  $\varepsilon_1, \varepsilon_2, \varepsilon_3$ , we provide three detailed classifications for a curve  $x(s)$ :

- Remark 2.1.* (1)  $x(s)$  is timelike curve if  $-\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ ;  
 (2)  $x(s)$  is called *the first kind of spacelike curve* if  $\varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = 1$ ;  
 (3)  $x(s)$  is called *the second kind of spacelike curve* if  $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$ .

Furthermore, if  $x(s)$  is a pseudo-spherical curve in  $\mathbb{H}^2$ , by a translation in  $\mathbb{E}_1^3$  if necessary, we may assume that

$$\langle x(s), x(s) \rangle = \langle x, x \rangle = -a^2,$$

where  $a$  is a constant. Without loss of generality we may assume that  $a^2 = 1$ .

Then, we define a vector field

$$(2.4) \quad y(s) := \alpha(s) \times x(s).$$

It is easy to see that  $\alpha(s), x(s)$  and  $y(s)$  form an orthonormal basis along the curve  $x(s)$  in  $\mathbb{E}_1^3$ , where  $\times$  denotes the vector product of two vectors in  $\mathbb{E}_1^3$  defined by Eq. (2.1). Here, it is convenient to assume  $\langle \alpha, \alpha \rangle = 1, \langle y, y \rangle = 1$ . Hence, Eq. (2.4) yields

$$(2.5) \quad \alpha = x \times y, \quad x = -y \times \alpha.$$

In the following a similar classification as Remark 2.1 could be given:

*Remark 2.2.* The pseudo-spherical curve  $x(s)$  in  $\mathbb{H}^2$  is the second spacelike pseudo-spherical curve.

Since  $\alpha(s), x(s)$  and  $y(s)$  form an orthonormal basis along the curve  $x(s)$ , we call  $\{\alpha(s), x(s), y(s)\}$  the *pseudo-spherical Frenet frame* of the pseudo-spherical curve  $x(s)$  in  $\mathbb{H}^2$ . By a direct computation, we conclude that there exists a function  $\kappa_g(s)$  satisfying that

$$(2.6) \quad \begin{cases} \dot{\alpha}(s) = x(s) + \kappa_g(s)y(s), \\ \dot{x}(s) = \alpha(s), \\ \dot{y}(s) = -\kappa_g(s)\alpha(s). \end{cases}$$

Moreover, we call  $\kappa_g(s)$  the *pseudo-spherical curvature function* of the pseudo-spherical curve  $x(s)$  in  $\mathbb{H}^2$ .

*Remark 2.3.* If  $\langle x(s), x(s) \rangle = \langle x, x \rangle = a \neq -1$ , the pseudo-spherical Frenet frame of the non unit pseudo-spherical curve can be written as  $\{\alpha(s), \frac{1}{\sqrt{|a|}}x(s), \frac{1}{\sqrt{|a|}}y(s)\}$ . Therefore, in this following, the pseudo-spherical curve always means the unit pseudo-spherical curve in  $\mathbb{H}^2$ .

In fact, it is important to consider the relations between the pseudo-spherical curvature  $\kappa_g(s)$  and the curvature  $\kappa(s)$ , the torsion  $\tau(s)$  of a pseudo-spherical curve  $x(s)$  in  $\mathbb{H}^2$ . If  $x(s)$  is a pseudo-spherical curve, from (2.3) and (2.6) we have

$$(2.7) \quad \kappa(s)\beta(s) = \dot{\alpha}(s) = x(s) + \kappa_g(s)y(s).$$

By computing the inner products with both sides of the above equation respectively, we obtain

$$(2.8) \quad \kappa^2(s) = \kappa_g^2(s) - 1.$$

Differentiating both sides of Eq. (2.8), it is direct to generate

$$(2.9) \quad \kappa\dot{\kappa} = \kappa_g\dot{\kappa}_g.$$

Differentiating both sides of Eq. (2.7) and using formulas (2.3)-(2.6), it follows that

$$(2.10) \quad \dot{\kappa}(s)\beta(s) - \varepsilon_1\varepsilon_2\kappa^2(s)\alpha(s) + \kappa(s)\tau(s)\gamma(s) = \alpha(s) + \dot{\kappa}_g(s)y(s) - \kappa_g^2(s)\alpha(s).$$

Since  $\langle \alpha, \beta \rangle = 0$ ,  $\langle \alpha, \gamma \rangle = 0$ , the above equation gives

$$(2.11) \quad \dot{\kappa}(s)\beta(s) + \kappa(s)\tau(s)\gamma(s) = \dot{\kappa}_g(s)y(s).$$

Similarly, by computing the inner products with both sides of Eq. (2.11), we get

$$(2.12) \quad \dot{\kappa}^2(s) - \kappa^2(s)\tau^2(s) = \dot{\kappa}_g^2(s).$$

Then Eqs. (2.8), (2.9) and (2.12) lead to

$$\begin{aligned} \tau^2 &= \frac{\dot{\kappa}^2 - \dot{\kappa}_g^2}{\kappa^2}, \\ &= \frac{\frac{\kappa_g^2 \dot{\kappa}_g^2}{\kappa^2} - \dot{\kappa}_g^2}{\kappa^2}, \\ &= \frac{\dot{\kappa}_g^2}{\kappa^4}, \end{aligned}$$

and

$$(2.13) \quad \tau(s) = \frac{\pm \dot{\kappa}_g(s)}{\kappa_g^2(s) - 1}.$$

Therefore, combining of Eqs. (2.8) and (2.13) generates the following proposition.

**Proposition 2.1.** *For a pseudo-spherical curve  $x(s)$  in  $\mathbb{H}^2$ , its pseudo-spherical curvature function  $\kappa_g(s)$ , the curvature function  $\kappa(s)$  and the torsion function  $\tau(s)$  satisfy that*

$$(2.14) \quad \begin{cases} \kappa(s) = \sqrt{|\kappa_g^2(s) - 1|}, \\ \tau(s) = \frac{\pm \dot{\kappa}_g(s)}{\kappa_g^2(s) - 1}. \end{cases}$$

Furthermore, we get:

**Corollary 2.1.** *If  $\kappa_g(s)$  is a constant, then the pseudo-spherical curve  $x(s)$  is a circle or an ellipse.*

*Proof.* From (2.14) we can figure out that  $\kappa_g = \text{constant}$  yields that  $\tau(s) \equiv 0$  and  $\kappa(s) = \text{constant}$  directly, which implies  $x(s)$  is a circle, a hyperbola or a parabola. □

On the other hand, if  $\kappa_g$  is a constant, from (2.6) we have

$$(2.15) \quad \dot{x}(s) = \alpha(s),$$

$$(2.16) \quad \ddot{x}(s) = \dot{\alpha}(s) = x(s) + \kappa_g y(s),$$

$$(2.17) \quad \ddot{x}(s) = \dot{x}(s) + \kappa_g \dot{y}(s) = (1 - \kappa_g^2) \dot{x}(s).$$

Obviously Eq. (2.17) also describes in more detail the curve  $x(s)$  is a circle (when  $|\kappa_g| > 1$ ) or an ellipse (when  $|\kappa_g| < 1$ ). At the end of this section, we will prove the following proposition.

**Proposition 2.2.** *Let a curve  $a(s) = \int_{s_0}^s y(s)ds$ . Then  $a(s)$  is a curve with constant torsion  $\tilde{\tau}(s) = -1$ , curvature  $\tilde{\kappa}(s) = |\kappa_g|$ , and the Frenet frame  $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}$  of  $a(s)$  can be written as  $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\} = \{y, \varepsilon\alpha, -\varepsilon x\}$ , where  $\varepsilon = -\text{sign}(\kappa_g)$ .*

*Proof.* By a direct derivation with Eqs. (2.5) and (2.6), we obtain

$$\tilde{\alpha} \frac{d\tilde{s}}{ds} = a' = y.$$

Here we can assume  $\frac{d\tilde{s}}{ds} = 1$ , and it follows that

$$\tilde{\beta} = \frac{y'}{|y'|} = \varepsilon\alpha, \quad \tilde{\gamma} = \tilde{\alpha} \times \tilde{\beta} = -\varepsilon x,$$

where  $\varepsilon = -\text{sign}(\kappa_g)$ .

Then it is easy to obtain the curvature  $\tilde{\kappa}$  and torsion  $\tilde{\tau}$  of  $a(s)$ , that is,

$$\begin{cases} \tilde{\kappa}(s) = |y'| = |\kappa_g|, \\ \tilde{\tau}(s) = \frac{\langle a'(s), a''(s), a'''(s) \rangle}{(a'(s) \times a''(s))^2} = \frac{\kappa_g^2}{-\kappa_g^2} = -1. \end{cases}$$

Therefore, there is no doubt that  $\tilde{\tau}$  is a constant. □

**3. Structure functions for ruled surfaces with time-like ruling**

Let  $X(u, v) = a(u) + vb(u)$  be a non-developable ruled surface in Minkowski 3-space  $\mathbb{E}_1^3$  with timelike ruling  $b^2(u) = -1$ , where  $u$  is the pseudo-arc length parameter of the curve  $b(u)$ . For convenience, we further assume that the base curve  $a(u)$  is the striction line of the surface  $X(u, v)$ , that is,  $\langle a'(u), b'(u) \rangle = 0$ , where  $a' = \frac{da}{du}$  according to previous notations. Generally, we call such an expression the *standard equation* of the non-developable ruled surfaces with timelike ruling in Minkowski 3-space  $\mathbb{E}_1^3$ .

Here we take the notations  $x(u) = b(u)$ ,  $\alpha(u) = x'(u)$ ,  $y(u) = \alpha(u) \times x(u)$  and  $\Omega(u) = \kappa_g(u)x(u) + y(u)$ . Since  $\langle a'(u), b'(u) \rangle = 0$ , it is reasonable to assume

$$(3.1) \quad a'(u) = \lambda(u)x(u) + \mu(u)y(u)$$

for two functions  $\lambda(u)$  and  $\mu(u)$ .

*Remark 3.1.* For a non-developable ruled surface  $X(u, v) = a(u) + vb(u)$  with  $b^2(u) = -1$  and  $\langle a'(u), b'(u) \rangle = 0$ , where the parameter  $u$  is the pseudo-arc length parameter of  $b(u)$ , the functions  $\lambda(u)$  and  $\mu(u)$  are uniquely determined by pseudo-spherical Frenet frame  $\{\alpha(u), x(u), y(u)\}$ .

- Remark 3.2.*
- (1)  $a(u)$  is a timelike curve if  $|\mu(u)| < |\lambda(u)|$ ;
  - (2)  $a(u)$  is a space-like curve if  $|\mu(u)| > |\lambda(u)|$ ;
  - (3)  $a(u)$  is a null curve if  $|\mu(u)| = |\lambda(u)|$ .

From Eqs. (2.6), (3.1) and existence theorems for system of differential equations, by a trivial computation, we can obtain:

**Proposition 3.1.** *The ruled surface  $X(u, v) = a(u) + vb(u)$  with timelike ruling is determined by  $\{\kappa_g(u), \lambda(u), \mu(u)\}$  up to a transformation in Minkowski 3-space  $\mathbb{E}_1^3$ .*

Then we give the definition of the structure functions.

**Definition 3.1.** The functions  $\kappa_g(u)$ ,  $\lambda(u)$  and  $\mu(u)$  in Eqs. (2.6)-(3.1) are called structure functions of the non-developable ruled surface  $X(u, v)$  with timelike ruling in Minkowski 3-space  $\mathbb{E}_1^3$ .

The notions of structure functions for the non-developable ruled surfaces with timelike ruling in Minkowski 3-space are as follows.

**Definition 3.2.** Let  $X(u, v) = a(u) + vb(u)$  be a non-developable ruled surface with timelike ruling in  $\mathbb{E}_1^3$ . We denote the translation of the points on the ruling from  $X(u_0, v)$  to  $X(u_0 + \Delta u, v)$  by  $\text{trans}(u_0, u_0 + \Delta u)$ . Then

$$(3.2) \quad \text{trans}_X(u_0) = \lim_{\Delta u \rightarrow 0} \frac{\text{trans}(u_0, u_0 + \Delta u)}{\Delta u} = \pm \lim_{\Delta u \rightarrow 0} \frac{\langle a(u_0 + \Delta u) - a(u_0), b(u_0) \rangle}{\Delta u}$$

is called translation density at  $u_0$ . The function  $\text{trans}_X(u)$  is called translation density function of  $X(u, v)$ .

Through a simple calculation as [5], we can get:

**Theorem 3.1.** *The function  $\lambda(u)$  of the structure functions of the non-developable ruled surface  $X(u, v)$  with timelike ruling is the density function of the signed translation of the rulings of  $X(u, v)$ . That is, the translation distance of the points on the ruling from  $X(u_1, v)$  to  $X(u_2, v)$  is given by*

$$(3.3) \quad \text{trans}(u_1, u_2) = \pm \int_{u_1}^{u_2} \lambda(u) du.$$

We call  $\lambda(u)$  the pitch function (or pitch density function) of the (non-developable) ruled surface  $X(u, v)$ .

Similarly, we have:

**Definition 3.3.** Let  $X(u, v)$  be a non-developable ruled surface with timelike ruling in  $\mathbb{E}_1^3$ . We denote the distance of two rulings  $X(u_0 + \Delta u, v)$  and  $X(u_0, v)$  by  $\text{dist}(u_0, u_0 + \Delta u)$ . Then

$$(3.4) \quad \text{dist}_X(u_0) = \lim_{\Delta u \rightarrow 0} \frac{\text{dist}(u_0, u_0 + \Delta u)}{\Delta u}$$

is called distance density at  $u_0$ . The function  $\text{dist}_X(u)$  is called distance density function of  $X(u, v)$ .

The following theorem shows that Definition 3.3 has the parallel meaning with [5].

**Theorem 3.2.** *The function  $\mu(u)$  of the structure functions of the non-developable ruled surface  $X(u, v)$  with timelike ruling is the density function of the signed distance of the ruling of  $X(u, v)$ . That is, the distance of two rulings  $X(u_1, v)$  and  $X(u_2, v)$  is given by*

$$(3.5) \quad \text{dist}(u_1, u_2) = \pm \int_{u_1}^{u_2} \mu(u) du.$$

Next, we have:

**Definition 3.4.** Let  $X(u, v) = a(u) + vb(u)$  be a non-developable ruled surface with timelike ruling in  $\mathbb{E}_1^3$ . For two rulings at  $X(u_0, v)$  and  $X(u_0 + \Delta u, v)$  we define

$$(3.6) \quad \text{normal}_X(u_0) = \lim_{\Delta u \rightarrow 0} \frac{b(u_0) \times b(u_0 + \Delta u)}{|b(u_0) \times b(u_0 + \Delta u)|} = \frac{b(u_0) \times b'(u_0)}{|b(u_0) \times b'(u_0)|}.$$

The spacelike vector  $\text{normal}_X(u_0)$  is called unit common perpendicular vector of the ruling at  $u_0$ . The vector field  $\text{normal}_X(u)$  is called unit common perpendicular vector field of the ruling of  $X(u, v)$ .

Comparing the definition of  $\text{normal}_X(u)$  with the definition of  $\kappa_g$  in (2.6), we see:

**Theorem 3.3.** *The function  $\kappa_g(u)$  of the structure functions of the non-developable ruled surface  $X(u, v)$  with timelike ruling is signed spinning speed function of the unit common perpendicular vector field of the ruling of  $X(u, v)$ .*

Therefore, we could get some relations between the Gaussian curvature, the mean curvature and the structure functions.

**Proposition 3.2.**

$$(3.7) \quad K(u, v) = \frac{\mu^2(u)}{(\mu^2(u) + v^2)^2},$$

$$(3.8) \quad H(u, v) = \frac{(\mu^2(u) + v^2)\kappa_g(u) + \mu'(u)v - \lambda(u)\mu(u)}{2\sqrt{(\mu^2(u) + v^2)^3}},$$

where  $K(u, v)$ ,  $H(u, v)$  are the Gaussian and the mean curvatures of the ruled surface  $X(u, v)$  with timelike ruling.

*Proof.* From  $X(u, v) = a(u) + vx(u)$ , Eqs. (3.1) and (2.6), it follows that

$$\begin{aligned} X_u &= \lambda x + \mu y + v\alpha, \\ X_v &= x, \\ X_{uu} &= (\lambda' + v)x + (\lambda - \mu\kappa_g)\alpha + (\mu' + v\kappa_g)y, \\ X_{uv} &= \alpha, \\ X_{vv} &= 0, \\ n &= \frac{-\mu\alpha + vy}{\sqrt{\mu^2 + v^2}}, \end{aligned}$$

where  $n$  is the unit normal vector of the ruled surface.

Then we can get

$$(3.9) \quad E = \langle X_u, X_u \rangle = -\lambda^2(u) + \mu^2(u) + v^2,$$

$$(3.10) \quad F = \langle X_u, X_v \rangle = -\lambda(u),$$

$$(3.11) \quad G = \langle X_v, X_v \rangle = -1,$$

$$(3.12) \quad L = \langle X_{uu}, n \rangle = \frac{-\lambda\mu + \kappa_g(u^2 + v^2) + v\mu'}{\sqrt{\mu^2 + v^2}},$$

$$(3.13) \quad M = \langle X_{uv}, n \rangle = \frac{-\mu}{\sqrt{\mu^2 + v^2}},$$

$$(3.14) \quad N = \langle X_{vv}, n \rangle = 0.$$

Therefore,

$$(3.15) \quad \begin{aligned} K &= \frac{LN - M^2}{EG - F^2} = \frac{\mu^2(u)}{(\mu^2(u) + v^2)^2}, \\ H &= \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{(\mu^2(u) + v^2)\kappa_g(u) + \mu'(u)v - \lambda(u)\mu(u)}{2\sqrt{(\mu^2(u) + v^2)^3}}. \end{aligned}$$

Thus we complete the proof of Proposition 3.2.  $\square$

*Remark 3.3.* From (3.7) we know that the structure function  $\mu(u)$  is nonzero for the non-developable ruled surfaces with timelike ruling in  $\mathbb{E}_1^3$ .

**4. Properties and classification of non-developable ruled surface with timelike ruling**

In this section we consider the properties and classification of non-developable ruled surfaces with timelike ruling.

**Definition 4.1.** Let  $X(u, v) = a(u) + vb(u)$  be any non-developable ruled surface with timelike ruling in  $\mathbb{E}_1^3$ .  $X(u, v)$  is called pitched surface if its pitch function  $\lambda \neq 0$ . Otherwise,  $X(u, v)$  is called non-pitched ruled surface.

From Eq. (3.1), we have

$$a(u) = \int_{u_0}^u [\lambda(t)x(t) + \mu(t)y(t)]dt.$$

The non-developable ruled surface  $X(u, v)$  with timelike ruling can be rewritten as

$$X(u, v) = \int_{u_0}^u [\lambda(t)b(t) + \mu(t)y(t)]dt + vb(u).$$

Especially, if it is non-pitched,

$$X(u, v) = \int_{u_0}^u \mu(t)y(t)dt + vb(u),$$

which is the binormal surface of its striction line. Moreover, if the structure function  $\mu(u)$  is a constant, from Proposition 2.2, this surface is a binormal surface of the constant torsion curve  $\int_{u_0}^u [\mu y(t)]dt$ .

The following conclusion shows a characterization of the pitch function of the ruled surface with timelike ruling in  $\mathbb{E}_1^3$ .

**Theorem 4.1.** *The pitch function  $\lambda(u)$  of a non-developable ruled surface  $X(u, v) = a(u) + vb(u)$  with  $b^2(u) = -1$  vanishes identically if and only if the surface  $X(u, v)$  is the binormal surface of its striction line.*

*Proof.* If  $\lambda = 0$ , let  $\kappa_1$  and  $\tau_1$  be the curvature function and torsion function of the curve

$$(4.1) \quad a(u) = \int_{u_0}^u \mu(t)y(t)dt$$

with the pseudo-arc length parameter  $s_1$  and the Frenet frame  $\{\alpha_1, \beta_1, \gamma_1\}$ . From Remark 3.2,  $a(u)$  is a spacelike curve. Differentiating both sides of the above equation with respect to  $u$  gives us that

$$(4.2) \quad \alpha_1 \frac{ds_1}{du} = \mu(u)y(u).$$

Without loss of generality we could assume that

$$(4.3) \quad \frac{ds_1}{du} = \mu(u).$$

Thus it is easy to see

$$(4.4) \quad \alpha_1 = y.$$

Again differentiating both sides of the above equation with respect to  $u$ , from Eqs. (2.6) and (4.3) we have

$$(4.5) \quad \kappa_1 \beta_1 \frac{ds_1}{du} = \kappa_1 \beta_1 \mu = -\kappa_g \alpha.$$

Therefore,

$$(4.6) \quad \kappa_1^2 \mu^2 \varepsilon_2 = \kappa_g^2.$$

Eq. (4.6) implies  $\varepsilon_2 = 1$ , that is,  $a(u)$  is a second space-like curve. Then

$$(4.7) \quad \beta_1(s) = \alpha(u).$$

Substituting this in Eq. (4.4), we can establish

$$(4.8) \quad \gamma_1(s) = b(u) = x(u).$$

Conversely, if the surface  $X(u, v)$  is the binormal surface of its striction line, by differentiating both sides of (4.8), we know

$$(4.9) \quad \tau_1(s) \beta_1(s) \frac{ds}{du} = \alpha(u).$$

Thus Eqs. (2.5), (4.8), (4.9) take

$$\tau_1(s) \frac{ds}{du} \alpha_1(s) = y(u),$$

namely,  $a'(u) = \alpha_1(s) \frac{ds}{du} = \frac{1}{\tau_1(s)} y(u)$ . Comparing with Eq. (3.1), we can conclude  $\lambda = 0$ .  $\square$

In this case, the parameter  $u$  is the pseudo-arc length parameter of the curve  $b(u)$ , and not the pseudo-arc length parameter of the curve  $a(u)$ . If the parameter  $u$  is the pseudo-arc length parameter of the curve  $b(u)$  and the curve  $a(u)$ , we may now state:

**Proposition 4.1.** *Let  $X(u, v) = a(u) + vb(u)$  be a non-developable ruled surface in  $\mathbb{E}_1^3$  with  $b^2(u) = -1$  and  $a(u)$  the striction line of  $X(u, v)$ . If the parameter  $u$  is the pseudo-arc length parameter of both  $a(u)$  and  $b(u)$ , the structure functions  $\lambda(u)$  and  $\mu(u)$  of  $X(u, v)$  satisfy that  $-\lambda^2(u) + \mu^2(u) = \pm 1$ .*

*Proof.* Since  $u$  is the pseudo-arc length parameter of the curve  $a(u)$ , we have  $\langle a'(u), a'(u) \rangle = \pm 1$ . From (3.1), we can get

$$\langle a'(u), a'(u) \rangle = \langle \lambda(u)x + \mu(u)y, \lambda(u)x + \mu(u)y \rangle = -\lambda^2(u) + \mu^2(u) = \pm 1.$$

This completes the proof of Proposition 4.1.  $\square$

Immediately, we can deduce the following conclusions with the structure functions and pitch function of the ruled surfaces  $X(u, v)$ .

**Theorem 4.2.** *If a ruled surface  $X(u, v)$  is non-pitched, it can be represented as the binormal surface of its striction line. Moreover, if this surface is non-pitched with constant structure function, it can be represented as a binormal surface of a constant torsion curve.*

Additionally, we obtain the following classification.

**Theorem 4.3.** *For a non-pitched ruled surface  $X(u, v)$ , if its structure function*  
 (1)

$$(4.10) \quad \mu(u) = \kappa_g(u)(c_1 \exp(u) + c_2 \exp(-u)),$$

*it is a binormal surface of a pseudo-spherical curve;*  
 (2)

$$(4.11) \quad \kappa_g(u) = \frac{1}{c_3 u + c_4},$$

*it is a binormal surface of a rectifying curve;*  
 (3)

$$(4.12) \quad \kappa_g(u) = c_5,$$

*it is a binormal surface of a helix;*  
 (4)

$$(4.13) \quad \mu(u) = c_6 \kappa_g(u) + c_7,$$

*it is a binormal surface of a Bertrand curve, where  $c_1, \dots, c_7$  are constants,  $c_1^2 + c_2^2 \neq 0$  and  $c_3 c_4 c_5 c_6 c_7 \neq 0$ .*

*Proof.* (4.6) to (4.7) imply  $\kappa_1 = -\frac{\kappa_g}{\mu}$  and

$$(4.14) \quad (-\kappa_1 \alpha_1 + \tau_1 \gamma_1) \frac{ds_1}{du} = (-\kappa_1 \alpha_1 + \tau_1 \gamma_1) \mu = x + \kappa_g y.$$

Finally, Eq. (4.9) generates  $\tau_1 = \frac{1}{\mu}$ . To sum up, the curvature function  $\kappa_1(u)$  and the torsion function  $\tau_1(u)$  of the curve (4.1) are

$$(4.15) \quad \begin{cases} \kappa_1(u) = -\frac{\kappa_g(u)}{\mu(u)}, \\ \tau_1(u) = \frac{1}{\mu(u)}. \end{cases}$$

Inserting one of Eqs. (4.10)-(4.13) into Eq. (4.15) in proper sequence, according to the properties of curves with respect to curvature  $\kappa_1$  and torsion  $\tau_1$ , from Theorem 4.2 we can obtain the conclusion directly.  $\square$

By a trivial calculation, we carry out:

**Theorem 4.4.** *Let  $a(s)$  be a spacelike curve with the timelike principal normal vector, where  $s$  is its pseudo-arc length parameter. Here  $\kappa_1(s)$  and  $\tau_1(s)$  are its curvature and torsion respectively, and  $X(s, v) = a(s) + v\beta_1(s)$  is its normal surface. Then the striction line of  $X(s, v)$  is*

$$(4.16) \quad A(u) = a(u) - \frac{\kappa_1(u)}{\kappa_1^2(u) + \tau_1^2(u)}\beta_1(u).$$

At the same time, the structure functions can be given by

$$(4.17) \quad \begin{cases} \lambda(u) = -\left(\frac{\kappa_1}{\kappa_1^2 + \tau_1^2}\right)', \\ \mu(u) = -\left(\frac{\tau_1}{\kappa_1^2 + \tau_1^2}\right), \\ \kappa_g(u) = \varepsilon \frac{\kappa_1(u)\tau_1'(u) - \tau_1(u)\kappa_1'(u)}{\sqrt{(\kappa_1^2(u) + \tau_1^2(u))^3}}, \end{cases}$$

where

$$(4.18) \quad u = \int_{s_0}^s [\kappa_1(s)^2 + \tau_1(s)^2] ds$$

is the pseudo-arc length parameter of  $\beta_1$ .

*Proof.* The striction line of  $X(s, v)$  is

$$A(u) = a(u) - \frac{\langle a'(u), \dot{\beta}_1(u) \rangle}{\langle \dot{\beta}_1(u), \dot{\beta}_1(u) \rangle} \beta_1(u).$$

Differentiating both sides of

$$(4.19) \quad \beta_1(u) = x(u),$$

we see

$$(4.20) \quad (\kappa_1(s)\alpha_1(s) + \tau_1(s)\gamma_1(s)) \frac{ds}{du} = \alpha(u).$$

From Eqs. (4.19) and (4.20), we determine

$$(4.21) \quad \kappa_1^2(s) + \tau_1^2(s) = \left(\frac{du}{ds}\right)^2,$$

and

$$(4.22) \quad (-\kappa_1(s)\gamma_1(s) + \tau_1(s)\alpha_1(s)) \frac{ds}{du} = -y(u).$$

Eqs. (4.20), (4.21) and (4.22) give

$$(4.23) \quad \alpha_1(s) = \frac{1}{\sqrt{\kappa_1^2 + \tau_1^2}} (\kappa_1\alpha - \tau_1y).$$

Then

$$\langle \alpha_1'(u), \dot{\beta}_1(u) \rangle = \frac{\kappa_1}{\kappa_1^2 + \tau_1^2},$$

and  $\langle \dot{\beta}_1(u), \dot{\beta}_1(u) \rangle = \langle \alpha(u), \alpha(u) \rangle = 1$ . By differentiating Eq. (4.16), Theorem 4.4 can be proved completely.  $\square$

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