SOME RESULTS OF $p$-BIHARMONIC MAPS INTO A NON-POSITIVELY CURVED MANIFOLD

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Abstract. In this paper, we investigate $p$-biharmonic maps $u : (M, g) \rightarrow (N, h)$ from a Riemannian manifold into a Riemannian manifold with non-positive sectional curvature. We obtain that if $\int_M |\tau(u)|^{a+p}dv_g < \infty$ and $\int_M |d(u)|^2dv_g < \infty$, then $u$ is harmonic, where $a \geq 0$ is a nonnegative constant and $p \geq 2$. We also obtain that any weakly convex $p$-biharmonic hypersurfaces in space form $N(c)$ with $c \leq 0$ is minimal. These results give affirmative partial answer to Conjecture 2 (generalized Chen’s conjecture for $p$-biharmonic submanifolds).

1. Introduction

Harmonic maps play a central role in geometry. They are critical points of the energy $E(u) = \int_M \frac{|du|^2}{2}dv_g$ for smooth maps between manifolds $u : (M, g) \rightarrow (N, h)$ and the Euler-Lagrange equation is that tension field $\tau(u)$ vanishes. Extensions to the notions of $p$-harmonic maps, $F$-harmonic maps and $f$-harmonic maps were introduced and many results have been carried out (for instance, see [1, 2, 3, 8, 23]). In 1983, J. Eells and L. Lemaire [10] proposed the problem to consider the biharmonic maps: they are critical maps of the functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2}dv_g.$$

We see that harmonic maps are biharmonic maps and even more, minimizers of the bienergy functional. After G. Y. Jiang [15] studied the first and second variation formulas of the bienergy $E_2$, there have been extensive studies on biharmonic maps (for instance, see [9, 15, 16, 17, 21, 22, 24, 25]). Recently the first author and S. X. Feng in [13] introduced the following functional

$$E_{F,2}(u) = \int_M F\left(\frac{|\tau(u)|^2}{2}\right)dv_g,$$

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where \( \tau(u) = -\delta du = \text{trace} \bar{\nabla}(du) \). The map \( u \) is called an \( F \)-biharmonic map if it is a critical point of that \( F \)-bienergy \( E_{F,2}(u) \), which is a generalization of biharmonic maps, \( p \)-biharmonic maps [14] or exponentially biharmonic maps. Notice that harmonic maps are always \( F \)-biharmonic by definition. When \( F(t) = (2t)^2 \), we have a \( p \)-bienergy functional
\[
E_{p,2}(u) = \int_M |\tau(u)|^p dv_g,
\]
where \( p \geq 2 \). The Euler-Lagrange equation of \( E_{p,2} \) is \( \tau_{p,2}(u) = 0 \), where \( \tau_{p,2}(u) \) is given by (13). A map \( u : (M, g) \to (N, h) \) is called a \( p \)-biharmonic map if \( \tau_{p,2}(u) = 0 \). When \( u : (M, g) \to (N, h) \) is a \( p \)-biharmonic isometric immersion, then \( M \) is called a \( p \)-biharmonic submanifold in \( N \).

Recently, N. Nakaiuchi, H. Urakawa and S. Gudmundsson [21] proved that biharmonic maps from a complete Riemannian manifold into a non-positive curved manifold with finite bienergy and energy are harmonic. S. Maeta [20] proved that biharmonic maps from a complete Riemannian manifold into a non-positive curved manifold with finite \((a+2)\)-bienergy \( \int_M |\tau(u)|^{a+2} dv_g < \infty \) \( (a \geq 0) \) and energy are harmonic. In this paper, we first obtain the following result:

**Theorem 1.1** (cf. Theorem 3.1). Let \( u : (M^m, g) \to (N^n, h) \) be a \( p \)-biharmonic map from a Riemannian manifold \( (M, g) \) into a Riemannian manifold \( (N, h) \) with non-positive sectional curvature and let \( a \geq 0 \) be a non-negative real constant.

(i) If
\[
\int_M |\tau(u)|^{a+p} dv_g < \infty,
\]
and the energy is finite, that is,
\[
\int_M |du|^2 dv_g < \infty,
\]
then \( u \) is harmonic.

(ii) If \( \text{Vol}(M, g) = \infty \), and
\[
\int_M |\tau(u)|^{a+p} dv_g < \infty,
\]
then \( u \) is harmonic, where \( p \geq 2 \).

One of the most interesting problems in the biharmonic theory is Chen’s conjecture. In 1988, Chen raised the following problem:

**Conjecture 1** ([7]). Any biharmonic submanifold in \( E^n \) is minimal.

There are many affirmative partial answers to Chen’s conjecture. On the other hand, Chen’s conjecture was generalized as follows (cf. [6]): “Any biharmonic submanifolds in a Riemannian manifold with non-positive sectional curvature is minimal”. There are also many affirmative partial answers to this conjecture.
(a) Any biharmonic submanifold in $H^3(-1)$ is minimal (cf. [5]).
(b) Any biharmonic hypersurfaces in $H^4(-1)$ is minimal (cf. [4]).
(c) Any weakly biharmonic hypersurfaces in space form $N^{m+1}(c)$ with $c \leq 0$ is minimal (cf. [18]).
(d) Any compact biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [15]).
(e) Any compact $F$-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal (cf. [13]).

For $p$-biharmonic submanifolds, it is natural to consider the following problem.

**Conjecture 2.** Any $p$-biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal.

For $p$-biharmonic submanifolds, we obtain the following result:

**Theorem 1.2** (cf. Theorem 4.1). Let $u: (M^m, g) \to (N^{m+1}, \langle , \rangle)$ be a weakly convex $p$-biharmonic hypersurface in a space form $N^{m+1}(c)$ with $c \leq 0$. Then $u$ is minimal, where $p \geq 2$.

These results give affirmative partial answers to the generalized Chen’s conjecture for $p$-biharmonic submanifold.

**2. Preliminaries**

In this section we give more details for the definitions of harmonic maps, biharmonic maps, $p$-biharmonic maps and $p$-biharmonic submanifolds.

Let $u: (M^m, g) \to (N^n, h)$ be a map from an $m$-dimensional Riemannian manifold $(M, g)$ to an $n$-dimensional Riemannian manifold $(N, h)$. The energy of $u$ is defined by

$$E(u) = \int_M \frac{|du|^2}{2} dv_g.$$ 

The Euler-Lagrange equation of $E$ is

$$\tau(u) = \sum_{i=1}^m \{\tilde{\nabla}_e du(e_i) - du(\nabla_e e_i)\} = 0,$$

where we denote by $\nabla$ the Levi-Civita connection on $(M, g)$ and $\tilde{\nabla}$ the induced Levi-civita connection on $u^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame field on $(M, g)$. $\tau(u)$ is called the tension field of $u$. A map $u: (M, g) \to (N, h)$ is called a harmonic map if $\tau(u) = 0$.

To generalize the notion of harmonic maps, in 1983 J. Eells and L. Lemaire [10] proposed considering the bienergy functional

$$E_2(u) = \int_M \frac{|\tau(u)|^2}{2} dv_g.$$
In 1986, G. Y. Jiang [15] studied the first and second variation formulas of the bienergy $E_2$. The Euler-Lagrange equation of $E_2$ is

$$
\tau_2(u) = -\tilde{\Delta}(\tau(u)) - \sum_i R^N(\tau(u), du(e_i))du(e_i) = 0,
$$

where $\tilde{\Delta} = \sum_i (\bigtriangledown_{e_i} \bigtriangledown_{e_i} - \bigtriangledown_{\bigtriangledown_{e_i}} e_i)$ is the rough Laplacian on the section of $u^{-1}TN$ and $R^N(X, Y) = [N\nabla_X, N\nabla_Y] - N\nabla_{[X,Y]}$ is the curvature operator on $N$. A map $u : (M, g) \to (N, h)$ is called a biharmonic map if $\tau_2(u) = 0$.

To generalize the notion of biharmonic maps, the first author and S. X. Feng [13] introduced the $F$-bienergy functional

$$
E_{F, 2}(u) = \int_M F\frac{|\tau(u)|^2}{2}dv_g,
$$

where $F : [0, \infty) \to [0, \infty)$ is a $C^3$ function such that $F'' > 0$ on $(0, \infty)$. The Euler-Lagrange equation of $E_{F, 2}$ is

$$
\tau_{F, 2}(u) = -\tilde{\Delta}(F'\frac{|\tau(u)|^2}{2})\tau(u) - \sum_i R^N(F'\frac{|\tau(u)|^2}{2})\tau(u), du(e_i))du(e_i) = 0.
$$

A map $u : (M, g) \to (N, h)$ is called a $F$-biharmonic map if $\tau_{F, 2}(u) = 0$.

When $F(t) = (2t)^{\tilde{\tau}}$, we have a $p$-bienergy functional

$$
E_{p, 2}(u) = \int_M |\tau(u)|^p dv_g,
$$

where $p \geq 2$. The Euler-Lagrange equation of $E_{p, 2}$ is

$$
\tau_{p, 2}(u) = -\tilde{\Delta}(p|\tau(u)|^{p-2}\tau(u)) - \sum_i R^N(p|\tau(u)|^{p-2}\tau(u), du(e_i))du(e_i) = 0.
$$

A map $u : (M, g) \to (N, h)$ is called a $p$-biharmonic map if $\tau_{p, 2}(u) = 0$.

Now we recall the definition of $p$-biharmonic submanifolds (cf. [12]).

Let $u : (M, g) \to (N, h = \langle \cdot, \cdot \rangle)$ be an isometric immersion from an $m$-dimensional Riemannian manifold into an $m + \ell$-dimensional Riemannian manifold. We identify $du(X)$ with $X \in \Gamma(TM)$ for each $x \in M$. We also denote by $\langle \cdot, \cdot \rangle$ the induced metric $u^{-1}h$. The Gauss formula is given by

$$
N\nabla_X Y = \nabla_X Y + B(X, Y), \quad X, Y \in \Gamma(TM),
$$

where $B$ is the second fundamental form of $M$ in $N$. The Weingarten formula is given by

$$
N\nabla_X \xi = -A_\xi X + \nabla_X^\perp \xi, \quad X \in \Gamma(TM), \ \xi \in \Gamma(T^\perp M),
$$

where $A_\xi$ is the shape operator for a unit normal vector field $\xi$ on $M$, and $\nabla^\perp$ denotes the normal connection on the normal bundle of $M$ in $N$. For any $x \in M$, the mean curvature vector field $H$ of $M$ at $x$ is given by

$$
H = \frac{1}{m} \sum_{i=1}^m B(e_i, e_i).$$
If an isometric \(u: (M, g) \to (N, h)\) is \(p\)-biharmonic, then \(M\) is called a \(p\)-biharmonic submanifold in \(N\). In this case, we remark that the tension field \(\tau(u)\) of \(u\) is written \(\tau(u) = mH\), where \(H\) is the mean curvature vector field of \(M\). The necessary and sufficient condition for \(M\) in \(N\) to be \(p\)-biharmonic is the following:

\[
-\tilde{\Delta}(|H|^{p-2}H) - \sum_{i} R^N(|H|^{p-2}H, e_i)e_i = 0.
\]

From (1), we obtain the necessary and sufficient condition for \(M\) in \(N\) to be \(p\)-biharmonic as follows:

\[
\Delta^\perp(|H|^{p-2}H) - \sum_{i=1}^m B(e_i, A|H|^{p-2}H(e_i)) + \sum_{i=1}^m R^N(|H|^{p-2}H, e_i)e_i = 0,
\]

\[
Tr_g(\nabla A|H|^{p-2}H) + Tr_g[A\nabla^\perp(|H|^{p-2}H)] - \sum_{i=1}^m R^N(|H|^{p-2}H, e_i)e_i) = 0,
\]

where \(\Delta^\perp = \sum_{i=1}^m \nabla^\perp_{e_i} \nabla^\perp_{e_i} - \nabla^\perp_{\nabla^\perp_{e_i} e_i}\) is the Laplace operator associated with the normal connection \(\nabla^\perp\).

We also need the following lemma.

**Lemma 2.1** (Gaffney, [11]). Let \((M, g)\) be a complete Riemannian manifold. If a \(C^1\) -form \(\alpha\) satisfies that \(\int_M |\alpha| dv_g < \infty\) and \(\int_M (\delta \alpha) dv_g < \infty\), or equivalently, a \(C^1\) vector \(X\) defined by \(\alpha(Y) = \langle X, Y \rangle \) (\(\forall Y \in \Gamma(TM)\)) satisfies that \(\int_M |X| dv_g < \infty\) and \(\int_M \text{div}(X) dv_g < \infty\), then

\[
\int_M (-\delta \alpha) dv_g = \int_M \text{div}(X) dv_g = 0.
\]

### 3. Main results of \(p\)-biharmonic maps

In this section, we obtain the following result.

**Theorem 3.1.** Let \(u: (M^m, g) \to (N^n, h)\) be a \(p\)-biharmonic map from a Riemannian manifold \((M, g)\) into a Riemannian manifold \((N, h)\) with non-positive sectional curvature and let \(a \geq 0\) be a non-negative real constant.

(i) If

\[
\int_M |\tau(u)|^{a+p} dv_g < \infty,
\]

and the energy is finite, that is,

\[
\int_M |du|^2 dv_g < \infty,
\]

then \(u\) is harmonic.

(ii) If \(\text{Vol}(M, g) = \infty\), and

\[
\int_M |\tau(u)|^{a+p} dv_g < \infty,
\]

then \(u\) is harmonic.
then \( u \) is harmonic, where \( p \geq 2 \).

**Proof.** Take a fixed point \( x_0 \in M \) and for every \( r > 0 \), let us consider the following cut off function \( \lambda(x) \) on \( M \):

\[
\begin{cases}
0 \leq \lambda(x) \leq 1, & x \in M, \\
\lambda(x) = 1, & x \in B_r(x_0), \\
\lambda(x) = 0, & x \in M - B_{2r}(x_0), \\
|\nabla \lambda| \leq \frac{C}{r}, & x \in M,
\end{cases}
\]

where \( B_r(x_0) = \{ x \in M : d(x, x_0) < r \} \), \( C \) is a positive constant and \( d \) is the distance of \( (M, g) \). From (13), we have

\[
\int_M \langle -\tilde{\Delta}(|\tau(u)|^{p-2}\tau(u)), \lambda^2|\tau(u)|^a\tau(u) \rangle dv_g
\]

(6) \[
= \int_M \lambda^2|\tau(u)|^{a+p-2} \sum_{i=1}^{m} \langle R^N(\tau(u), du(e_i))du(e_i), \tau(u) \rangle dv_g \leq 0,
\]

since the sectional curvature of \( (N, h) \) is non-positive. From (6), we have

\[
0 \geq \int_M \langle -\tilde{\Delta}(|\tau(u)|^{p-2}\tau(u)), \lambda^2|\tau(u)|^a\tau(u) \rangle dv_g
\]

\[
= \int_M \langle \tilde{\nabla}(|\tau(u)|^{p-2}\tau(u)), \tilde{\nabla}(\lambda^2|\tau(u)|^a\tau(u)) \rangle dv_g
\]

\[
= \int_M \sum_{i=1}^{m} \langle \tilde{\nabla}_{e_i}(|\tau(u)|^{p-2}\tau(u)), \tilde{\nabla}_{e_i}(\lambda^2|\tau(u)|^a\tau(u)) \rangle dv_g
\]

\[
= \int_M \sum_{i=1}^{m} \left[ \langle |\tau(u)|^{p-2}\tilde{\nabla}_{e_i}\tau(u) + (p-2)|\tau(u)|^{p-4}\langle \tilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle \tau(u) \rangle, \lambda^2|\tau(u)|^a\tau(u) \rangle dv_g
\]

\[
= \int_M \sum_{i=1}^{m} \left[ \langle |\tau(u)|^{p-2}\tilde{\nabla}_{e_i}\tau(u) + (p-2)|\tau(u)|^{p-4}\langle \tilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle \tau(u) \rangle, \lambda^2|\tau(u)|^a\tau(u) \rangle dv_g
\]

\[
+ 2\lambda e_i(\lambda)|\tau(u)|^a\tau(u) + a\lambda^2|\tau(u)|^{a+2}\langle \tilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle \tau(u) \rangle dv_g
\]

\[
+ \lambda^2|\tau(u)|^a\tilde{\nabla}_{e_i}\tau(u) \rangle dv_g
\]

\[
= \int_M \sum_{i=1}^{m} 2(p-1)\lambda e_i(\lambda)|\tau(u)|^{a+p-2}\langle \tilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle dv_g
\]

\[
+ \int_M \sum_{i=1}^{m} [a(p-1) + (p-2)]\lambda^2|\tau(u)|^{a+p-4}\langle \tilde{\nabla}_{e_i}\tau(u), \tau(u) \rangle dv_g
\]

\[
+ \int_M \sum_{i=1}^{m} \lambda^2|\tau(u)|^{a+p-2}\langle \tilde{\nabla}_{e_i}\tau(u), \tilde{\nabla}_{e_i}\tau(u) \rangle dv_g
\]
By assumption
\begin{equation}
\int_M \sum_{i=1}^m 2(p-1)\lambda e_i(\lambda)|\tau(u)|^{a+p-2}\langle \tilde{\nabla}_{e_i} \tau(u), \tau(u) \rangle dv_g \\
(7) + \int_M \sum_{i=1}^m \lambda^2 |\tau(u)|^{a+p-2}\langle \tilde{\nabla}_{e_i} \tau(u), \tilde{\nabla}_{e_i} \tau(u) \rangle dv_g,
\end{equation}
where the inequality follows from \(|a(p-1) + (p-2)|\lambda^2|\tau(u)|^{a+p-4}\langle \tilde{\nabla}_{e_i} \tau(u), \tau(u) \rangle^2 \geq 0.
\end{equation}
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From (7), we have
\begin{equation}
\int_M \sum_{i=1}^m \lambda^2 |\tau(u)|^{a+p-2}\langle \tilde{\nabla}_{e_i} \tau(u), \tilde{\nabla}_{e_i} \tau(u) \rangle dv_g
\end{equation}
\begin{equation}
\leq -2(p-1) \int_M \sum_{i=1}^m \langle \tilde{\nabla}_{e_i} \tau(u), \lambda e_i(\lambda) |\tau(u)|^{a+p-2} \tau(u) \rangle dv_g.
\end{equation}
By using Young's inequality, we have
\begin{equation}
-2(p-1) \int_M \sum_{i=1}^m \langle \tilde{\nabla}_{e_i} \tau(u), \lambda e_i(\lambda) |\tau(u)|^{a+p-2} \tau(u) \rangle dv_g \\
\leq \frac{1}{2} \int_M \sum_{i=1}^m \lambda^2 |\tau(u)|^{a+p-2} \langle \tilde{\nabla}_{e_i} \tau(u), \tilde{\nabla}_{e_i} \tau(u) \rangle^2 dv_g + 2(p-1)^2 \int_M |\nabla \lambda|^2 |\tau(u)|^{a+p} dv_g.
\end{equation}
From (8) and (9), we have
\begin{equation}
\int_M \sum_{i=1}^m \lambda^2 |\tau(u)|^{a+p-2}\langle \tilde{\nabla}_{e_i} \tau(u), \tilde{\nabla}_{e_i} \tau(u) \rangle dv_g \\
\leq 4(p-1)^2 \int_M |\nabla \lambda|^2 |\tau(u)|^{a+p} dv_g \\
\leq \frac{4(p-1)^2 C^2}{r^2} \int_M |\tau(u)|^{a+p} dv_g \\
(10) \leq \frac{4(p-1)^2 C^2}{r^2} \int_M |\tau(u)|^{a+p} dv_g.
\end{equation}
By assumption \int_M |\tau(u)|^{a+p} dv_g < \infty, letting \(r \to \infty\) in (10), we have
\begin{equation}
\int_M \sum_{i=1}^m |\tau(u)|^{a+p-2}\langle \tilde{\nabla}_{e_i} \tau(u), \tilde{\nabla}_{e_i} \tau(u) \rangle dv_g = 0.
\end{equation}
Therefore, we obtain that \(|\tau(u)|\) is constant and \(\nabla_X \tau(u) = 0\) for any vector field \(X\) on \(M\).
Therefore, if \(Vol(M) = \infty\) and \(|\tau(u)| \neq 0\), then
\begin{equation}
\int_M |\tau(u)|^{a+p} dv_g = |\tau(u)|^{a+p} Vol(M) = \infty,
\end{equation}
which yields a contradiction. Thus, we have \(|\tau(u)| = 0\), i.e., \(u\) is harmonic. We have (ii).
For (i), assume both $\int_M |\tau(u)|^{a+p}dv_g < \infty$ and $\int_M |du|^2dv_g < \infty$. Define a 1-from $\alpha$ on $M$ defined by
\begin{equation}
\alpha(X) = |\tau(u)|^{\frac{a+p}{2p}}(du(X), \tau(u))
\end{equation}
for any vector $X \in \Gamma(TM)$.

Note here that
\[\int_M |\alpha|^2dv_g = \int_M \left[ \sum_{i=1}^m |\alpha(e_i)|^2 \right]^{\frac{2}{3}}dv_g = \int_M \left[ \sum_{i=1}^m |\tau(u)|^{\frac{a+p}{2p}}(du(e_i), \tau(u)) \right]^{\frac{2}{3}}dv_g \leq \int_M |\tau(u)|^{\frac{a+p}{p}}|du|dv_g \leq \int_M |\tau(u)|^{a+p}dv_g \leq \int_M |du|^2dv_g^{\frac{2}{p}} < \infty.
\]

Now we compute
\[-\delta \alpha = \sum_{i=1}^m (\nabla_{e_i} \alpha)(e_i) = \sum_{i=1}^m (\nabla_{e_i} \alpha(e_i) - \alpha(\nabla_{e_i} e_i)) = \sum_{i=1}^m \nabla_{e_i} [|\tau(u)|^{\frac{a+p}{2p}}(du(e_i), \tau(u))] - \sum_{i=1}^m |\tau(u)|^{\frac{a+p}{2p}}(du(\nabla_{e_i} e_i), \tau(u)) = \sum_{i=1}^m |\tau(u)|^{\frac{a+p}{2p}}(\tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i), \tau(u)) = |\tau(u)|^{\frac{a+p}{2p}+1},\]

where the fourth equality follows from that $|\tau(u)|$ is constant and $\tilde{\nabla}_X \tau(u) = 0$ for $X \in \Gamma(TM)$. Since $\int_M |\tau(u)|^{a+p}dv_g < \infty$ and $|\tau(u)|$ is constant, the function $-\delta \alpha$ is also integrable over $M$. From this and (12), we can apply Lemma 2.1 for the 1-from $\alpha$. Therefore we have
\[0 = \int_M (-\delta \alpha)dv_g = \int_M |\tau(u)|^{\frac{a+p}{2p}+1}dv_g,\]
so we have $\tau(u) = 0$, that is, $u$ is harmonic. \hfill \Box

4. Main results of $p$-biharmonic hypersurfaces

In this section, we obtain the following result.
Theorem 4.1. Let \( u : (M^n, g) \to (N^{m+1}, (,)) \) be a weakly convex \( p \)-biharmonic hypersurface in a space form \( N^{m+1}(c) \) with \( c \leq 0 \). Then \( u \) is minimal, where \( p \geq 2 \).

Proof. Assume that \( H = h\nu \), where \( \nu \) is the unit normal vector field on \( M \). Since \( M \) is weakly convex, we have \( h \geq 0 \). Set \( C = \{ q \in M : h(q) > 0 \} \). We will prove that \( A \) is an empty set.

If \( C \) is not empty, we see that \( C \) is an open subset of \( M \). We assume that \( C_1 \) is a nonempty connect component of \( C \). We will prove that \( h \equiv 0 \) in \( C_1 \), thus a contradiction.

Firstly, we prove that \( h \) is a constant in \( C_1 \).

Let \( q \in C_1 \) be a point. Choose a local orthonormal frame \( \{ e_i, i = 1, \ldots, m \} \) around \( q \) such that \( (B, \nu) \) is a diagonal matrix and \( \nabla e_i, e_j|_q = 0 \).

From equation (3), we have at \( q \)

\[
0 = \sum_{i=1}^{m} (\nabla e_i, A_{(h^{p-2}H)}(e_i), e_k) + \sum_{i=1}^{m} A_{\nabla e_i} \perp (h^{p-2}H)(e_i), e_k)
\]

\[
= \sum_{i=1}^{m} e_i (A_{(h^{p-2}H)}(e_i), e_k) + \sum_{i=1}^{m} (B(e_i, e_k), \nabla e_i \perp (h^{p-2}H))
\]

\[
= \sum_{i=1}^{m} e_i (h^{p-2}H, B(e_i, e_k)) + \sum_{i=1}^{m} (B(e_i, e_k), \nabla e_i \perp (h^{p-2}H))
\]

\[
= \sum_{i=1}^{m} (h^{p-2}H, \nabla e_i \perp B(e_i, e_i)) + 2 \sum_{i=1}^{m} (B(e_i, e_k), \nabla e_i \perp (h^{p-2}H))
\]

\[
= \sum_{i=1}^{m} \lambda_k e_k(h) + 2(p-1)h^{p-2}\lambda_0 e_k(h)
\]

where \( \lambda_k \) is the \( k \)th principle curvature of \( M \) at \( q \), which is nonnegative by the assumption that \( M \) is weakly convex. Since \( (mh + 2(p-1)\lambda_k)h^{p-2} > 0 \) at \( q \), we have \( e_k(h) = 0 \) at \( q \), for \( k = 1, \ldots, m \), which implies that \( \nabla h = 0 \) at \( q \). Because \( q \) is an arbitrary point in \( C_1 \), we have \( \nabla h = 0 \) in \( C_1 \). Therefore we obtain that \( h \) is constant in \( C_1 \).

Secondly, we prove that \( h \) is zero in \( C_1 \).

From the equation (2), we have

\[
\Delta h^{2p-2} = \Delta (h^{p-2}H, h^{p-2}H)
\]

\[
= 2(\Delta^\perp (h^{p-2}H), h^{p-2}H) + 2||\nabla^\perp (h^{p-2}H)||^2
\]
\[= 2|\nabla^\perp (h^{p-2}H)|^2 + 2 \sum_{i=1}^{m} \langle B(A_{h^{p-2}H} e_i, e_i), h^{p-2}H \rangle - \sum_{i=1}^{m} \langle R^N(h^{p-2}H, e_i)e_i, h^{p-2}H \rangle \geq |\nabla^\perp (h^{p-2}H)|^2 + 2 \sum_{i=1}^{m} \langle B(A_{h^{p-2}H} e_i, e_i), h^{p-2}H \rangle,\]

where the inequality follows from the sectional curvature of \((N, h)\) is non-positive. Now we state two inequalities:

\[(14) \quad |\nabla^\perp (h^{p-2}H)|^2 \geq h^{2p-4}|\nabla^\perp H|^2 \]

and

\[(15) \quad \sum_{i=1}^{m} \langle B(A_{h^{p-2}H} e_i, e_i), h^{p-2}H \rangle \geq mh^{2p}.\]

In fact,

\[
|\nabla^\perp (h^{p-2}H)|^2 = |(p-2)h^{p-4}\langle \nabla^\perp H, H \rangle \bar{H} + h^{p-2}\nabla^\perp H|^2 \\
= (p-2)^2h^{2p-6}\langle \nabla^\perp H, H \rangle^2 + h^{2p-4}|\nabla^\perp H|^2 \\
+ 2(p-2)h^{3p-6}\langle \nabla^\perp H, H \rangle^2 \\
\geq h^{2p-4}|\nabla^\perp H|^2,
\]

and

\[
\sum_{i=1}^{m} \langle B(A_{h^{p-2}H} e_i, e_i), h^{p-2}H \rangle \\
= \sum_{i=1}^{m} h^{2p-2}\langle B(A_{\nu} e_i, e_i), \nu \rangle \\
= \sum_{i=1}^{m} h^{2p-2}\langle A_{\nu} e_i, A_{\nu} e_i \rangle \\
= \sum_{i,j=1}^{m} h^{2p-2}|\langle B(e_i, e_j), \nu \rangle|^2 \\
\geq mh^{2p}.
\]

From (13), (14) and (15), we have

\[\Delta h^{2p-2} \geq 2h^{2p-4}|\nabla^\perp H|^2 + 2mh^{2p}.\]

So we have

\[(16) \quad \Delta h^{2p-2} \geq 2mh^{2p}.\]
From equation (16), we have in $C_1$

$$0 = \triangle h^{2p-2} \geq 2m h^{2p}.$$

We know that $h \equiv 0$ in $C_1$. This is a contradiction. □

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