CIRCLE-FOLIATED MINIMAL SURFACES IN 4-DIMENSIONAL SPACE FORMS

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Abstract. Catenoid and Riemann’s minimal surface are foliated by circles, that is, they are union of circles. In \( \mathbb{R}^3 \), there is no other non-planar example of circle-foliated minimal surfaces. In \( \mathbb{R}^4 \), the graph \( G_c \) of \( w = c/z \) for real constant \( c \) and \( \zeta \in \mathbb{C} \setminus \{0\} \) is also foliated by circles. In this paper, we show that every circle-foliated minimal surface in \( \mathbb{R}^n \) is either a catenoid or Riemann’s minimal surface in some 3-dimensional Affine subspace or a graph surface \( G_c \) in some 4-dimensional Affine subspace. We use the property that \( G_c \) is circle-foliated to construct circle-foliated minimal surfaces in \( S^4 \) and \( H^4 \).

1. Introduction

A surface \( M \in \mathbb{R}^n \) is said to be circle-foliated if there is a one-parameter family of planes whose intersection with \( M \) are circles. The catenoid and Riemann’s minimal surface are examples of circle-foliated minimal surfaces in \( \mathbb{R}^3 \). Enneper proved that the planes containing the circles of a circle-foliated minimal surface in \( \mathbb{R}^3 \) should be parallel [2] and [7]. Then it is easy to see that the plane, catenoid and Riemann’s minimal surface are the only circle-foliated minimal surfaces in \( \mathbb{R}^3 \) [7].

One may consider \( \mathbb{R}^4 \) as \( \mathbb{C}^2 \) with complex coordinates \( (z, w) \). For a real constant \( c \neq 0 \), the graph \( G_c = \{(w, z) \in \mathbb{C}^2 | wz = c\} \) is circle-foliated. In fact, the image \( g_r \) of the circle \( \{|z| = r\} \) on the \( z \)-plane is \( \{(z, c/z) | |z| = r\} \). Considering \( \mathbb{C}^2 \) as \( \mathbb{R}^4 \), \( g_r \) lies on the plane through \( (0, 0, 0, 0) \), \( (1, 0, -r^2, 0) \) and \( (0, 1, 0, r^2) \) (cf. Remark 2). Since \( |(z, c/z)|^2 = r^2 + c^2/r^2 \), \( g_r \) is a circle. Therefore \( G_c \) is circle-foliated. Since every complex submanifold of a Kaehler manifold is minimal [6], \( G_c \) is minimal. Moreover, \( G_c \) is complete, doubly-connected and has finite total curvature \(-4\pi\) with two planar ends, which are asymptotic to the planes \( \{z = 0\} \) and \( \{w = 0\} \) (cf. Remark 2). Hoffman and Osserman classified complete simply-connected and doubly-connected minimal surfaces in \( \mathbb{R}^n \) with total curvature \(-4\pi\) including \( G_c \) [4]. They showed that

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such minimal surfaces are foliated by ellipses or circles, and called them as the\textit{ skew catenoids}.

In this paper, we show that every circle-foliated minimal surface in $\mathbb{R}^n$ is either a catenoid or a Riemann's minimal surface in a 3-dimensional Affine subspace or the graph surface $G_c$ in a 4-dimensional Affine subspace. Therefore there is no counterpart of the Riemann's minimal surface in $\mathbb{R}^n$, for $n \geq 4$. We then use this property of $G_c$ to construct circle-foliated minimal surfaces in $S^4$ and $\mathbb{H}^4$.

2. Circle-foliated minimal surfaces in $\mathbb{R}^n$

Let $\Sigma$ be a circle-foliated surface in $\mathbb{R}^n$. Let $\{P_t\}$ be the one-parameter family of planes on which the circles of the foliation is on. Let $\tilde{P}_t$ be the plane parallel to $P_t$ and passes through the origin of $\mathbb{R}^n$. There is a one-parameter family of orthonormal basis of $\mathbb{R}^n$ satisfying Frenet type equations \cite{3}.

\textbf{Theorem A.} Let $\{\tilde{P}_t\}$ be a smooth one-parameter family of planes in $\mathbb{R}^n$. There is a one-parameter family of orthonormal basis $e_1(t), e_2(t), \ldots, e_n(t)$ of $\mathbb{R}^n$ such that $e_1(t)$ and $e_2(t)$ span $\tilde{P}_t$, and the following equations hold

\begin{align}
&\begin{align*}
&\dot{e}'_1 = \alpha'_1 e_j + \kappa'_1 e_{2+i} & (j, l = 1, 2) \\
&\dot{e}'_{2+i} = -\kappa'_1 e_i + \tau'_1 e_{2+i} + \gamma'_1 e_{4+\lambda} & (\lambda = 1, \ldots, n-4) \\
&\dot{e}'_{4+\xi} = -\gamma'_1 e_{2+i} + \beta'_1 e_{4+\lambda} & (\alpha'_1 = -\alpha'_j, \tau'_1 = -\tau'_i, \beta'_1 = -\beta'_\lambda),
\end{align*}
\end{align}

where $(\kappa^1)^2 \geq (\kappa^2)^2$, and \( t = \frac{4}{\kappa^1} \).

Using the above orthonormal basis of $\mathbb{R}^n$, we can parameterize a circle-foliated surface by

\begin{align}
X(t, \theta) = c(t) + r(t)(\cos \theta e_1 + \sin \theta e_2),
\end{align}

where $c(t)$ and $r(t)$ are the center and the radius of the circle on $P_t$.

\textbf{Theorem 1.} Circle-foliated minimal surface in $\mathbb{R}^n$ is either i) a catenoid or a Riemann's minimal surface in 3-dimensional Affine subspace or ii) a graph surface $G_c$ (defined in §1) in some 4-dimensional Affine subspace.

To prove the above theorem, we have to show that every circle-foliated minimal surface in $\mathbb{R}^n$, $n \geq 5$, actually lies in a (at most) 4-dimensional Affine subspace. First, let us assume that a circle-foliated surface $X$ lies in $\mathbb{R}^5$. (The case of $n \geq 6$ is analogous to the case of $\mathbb{R}^5$.) For the simplicity of notations, we write (1) as

\begin{align}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5
\end{pmatrix}' = \begin{pmatrix}0 & -\beta & -\kappa & 0 & 0 \\
\beta & 0 & 0 & \tau & 0 \\
\kappa & 0 & 0 & -\eta & -\nu \\
0 & -\tau & \eta & 0 & -\mu \\
0 & 0 & \nu & \mu & 0
\end{pmatrix} \begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5
\end{pmatrix}.
\end{align}
Let \( c'(t) = \sum_{i=1}^{5} \alpha_i e_i \), where \( \alpha_i \)'s are smooth functions. Then we have
\[
X_t = (\alpha_1 + r \cos \theta + r \beta \sin \theta)e_1 + (\alpha_2 + r \sin \theta - r \beta \cos \theta)e_2 \\
+ (\alpha_3 - r \kappa \cos \theta)e_3 + (\alpha_4 + r \tau \sin \theta)e_4 + \alpha_5 e_5,
\]
\[
X_\theta = -r \sin \theta e_1 + r \cos \theta e_2.
\]
Let \( N \) be a normal vector of \( X \) given by
\[
N = \cos \theta e_1 + \sin \theta e_2 + \gamma e_3 + \delta e_4 + \rho e_5.
\]
Then \( \gamma, \delta \) and \( \rho \) satisfy
\[
X_t \cdot N = \alpha_1 \cos \theta + \alpha_2 \sin \theta + r' + \gamma (\alpha_3 - r \kappa \cos \theta) \\
+ \delta(\alpha_4 + r \tau \sin \theta) + \rho \alpha_5 = 0.
\]
Let \( E, F, G \) be the coefficients of the first fundamental form of \( X \). Then
\[
E = |X_t|^2 = \sum_{i=1}^{5} \alpha_i^2 + r'^2 + r^2 \beta^2 + 2r' \alpha_1 \cos \theta + 2r' \alpha_2 \sin \theta \\
+ 2r \alpha_1 \beta \sin \theta - 2r \alpha_2 \beta \cos \theta - 2r \alpha_3 \kappa \cos \theta \\
+ r^2(\kappa^2 - \tau^2)(\cos \theta)^2 + 2r \alpha_4 \tau \sin \theta + r^2 \tau^2,
\]
\[
F = X_t \cdot X_\theta = -r \alpha_1 \sin \theta + r \alpha_2 \cos \theta - r^2 \beta,
\]
\[
G = |X_\theta|^2 = r^2.
\]

**Lemma 1.** The surface \( X(t, \theta) \) defined by (2) with \( \tau \neq 0 \) is minimal only if

i) \( \alpha_i = 0 \) for all \( i = 1, \ldots, 5 \), ii) \( \mu = \nu = 0 \) and iii) \( \kappa^2 = \tau^2 \), \( \beta \kappa = \tau \eta \) and \( \beta \tau = \kappa \tau \). Hence \( X(t, \theta) \) lies in a 4-dimensional Affine subspace.

**Proof.** Let \( l = X_t \cdot N, m = X_\theta \cdot N \) and \( n = X_{\theta \theta} \cdot N \), where \( N \) is given by (4).
Since \( X(t, \theta) \) is minimal, we must have
\[
\mathcal{H} := lG + nE - 2mF = 0.
\]
Direct computation shows that
\[
\mathcal{H} = \begin{cases}
\alpha_1 \cos \theta + r' + \alpha_2 \beta \cos \theta - \alpha_1 \beta \sin \theta + \alpha_2 \sin \theta - r \beta^2 \\
+ \alpha_3 \kappa \cos \theta - r \kappa(\cos \theta)^2 - \alpha_4 \tau - r \tau^2(\sin \theta)^2 \\
+ \gamma \left( \alpha_2 \kappa - 2r' \kappa \cos \theta - \alpha_1 \kappa - r \kappa' \cos \theta - r \beta \kappa \sin \theta + \alpha_3 \eta + r \tau \eta \sin \theta + \alpha_5 \nu \\
+ \alpha_4 + r \tau \tau \sin \theta + \alpha_2 \tau + r \tau' \sin \theta \\
+ \rho \left( \alpha_5 + r \kappa \nu \cos \theta - \alpha_3 \nu - \alpha_4 \mu - r \tau \mu \sin \theta \\
\sum_{i=1}^{5} \alpha_i^2 + r'^2 + r^2 \beta^2 + 2r' \alpha_1 \cos \theta + 2r' \alpha_2 \sin \theta \\
+ r^2(\kappa^2 - \tau^2)(\cos \theta)^2 + 2r \alpha_4 \tau \sin \theta + r^2 \tau^2 \\
- 2(r \beta + r \tau \kappa \sin \theta + r \tau \cos \theta)(-r \alpha_1 \sin \theta + r \alpha_2 \cos \theta - r^2 \beta) \end{cases}
\]
Since $\gamma$, $\delta$ and $\rho$ satisfy (5), we first let $\gamma = -(\alpha_1 \cos \theta + \alpha_2 \sin \theta + r')/(\alpha_3 - r \kappa \cos \theta)$, $\delta = \rho = 0$ and $\mathcal{S} := \mathcal{H}(\alpha_3 - r \kappa \cos \theta)$. Then the coefficients of $\cos(3\theta)$ and $\sin(3\theta)$ of $\mathcal{S}$ are $r^2 \kappa (r^2 (\kappa^2 - \tau^2) + \alpha_1^2 - \alpha_5^2)/2$ and $r^2 \kappa \alpha_1 \alpha_2$ respectively. Since these should be equal to zero and $\kappa^2 \geq \tau^2$, we necessarily have $\alpha_1 = 0$.

Let $\delta = -(\alpha_2 \sin \theta + r')/(\alpha_4 + r \tau \sin \theta)$, $\gamma = \rho = 0$ and $\mathcal{T} := \mathcal{H}(\alpha_4 + r \tau \cos \theta)$. The coefficients of $\cos(2\theta)$ of $\mathcal{S}$ and $\sin(2\theta)$ of $\mathcal{T}$ are $r^3 (-5 \alpha_3 \kappa^2 + 2 \alpha_3 \tau^2 + \alpha_2 \rho \nu)$ and $(3 \alpha_3 \kappa \tau - r^3 \alpha_2 \rho \nu)/2$ respectively, which are equal to zero. Hence we have either $\kappa^2 - \tau^2 = 0$ or $\alpha_3 = 0$. On the other hand, the coefficient of $\cos(3\theta)$ of $\mathcal{T}$ is $-r^4 \tau (\kappa^2 - \tau^2)$, which implies that $\kappa^2 = \tau^2$. Then we have $\alpha_2 = 0$, and since we assumed that $\tau^2 > 0$, we also have $\alpha_3 = 0$. Substituting these into $\mathcal{S}$, we have $\alpha_4 = 0$ from the coefficient of $\sin(2\theta)$.

Suppose that $\alpha_5 \neq 0$, and let $\gamma = \delta = 0$ and $\rho = -r' / \alpha_5$. Then $\mathcal{H}$ becomes

$$r^2 [r^2 \kappa - 2 r^2 (\cos \theta)^2 - 2 r^2 \tau (\sin \theta)^2 - \frac{r'}{\alpha_5} \rho (\kappa + \tau \rho \sin \theta)] - r (\alpha_5^2 + r' / \alpha_5^2).$$

Therefore we have $\kappa = \tau = 0$, which contradicts the assumption $\kappa, \tau \neq 0$. From (5), (6) and $\alpha_5 = 0$, it follows that $\mu = \nu = 0$. This completes the proofs of i) and ii).

From the coefficients of $\sin \theta$ of $\mathcal{S}$ and $\cos \theta$ of $\mathcal{T}$, we have

(7) \quad $\beta \kappa = \tau \eta$

and

(8) \quad $\beta \tau = \kappa \eta$.

Since $\mu = \nu = 0$ and $\alpha_i = 0$ for all $i = 1, \ldots, 5$, the surface $X(t, \theta)$ lies in a 4-dimensional Affine subspace. \hfill $\Box$

**Remark 1.** When $n \geq 6$ and $X(t, \theta)$ is minimal, it is easy to see in the above proof that $\alpha_k = 0$ for $k \geq 5$ and $\gamma_5 = 0$ and $\beta \xi = 0$. Hence $X(t, \theta)$ should lie in a 4-dimensional Affine subspace.

**Lemma 2.** If the surface $X(t, \theta)$ defined by (2) is minimal with $\tau \equiv 0$, then the planes $\bar{P}$ lie in some 3-dimensional Affine subspace.

**Proof.** When $\tau \equiv 0$, we consider two cases: $\alpha_4 \equiv 0$ or $\alpha_4 \neq 0$. First of all, we have $\alpha_1 = 0$ as in the proof of the above lemma. If $\tau \equiv 0$ and $\alpha_4 \neq 0$, then we let $\gamma = \rho = 0$ and $\delta = -(\alpha_3 \sin \theta + r') / \alpha_4$. The coefficient of $\cos(2\theta)$ of $\mathcal{H}$ is $-2r^2 \kappa^2$. Since this must be 0, we have $\kappa \equiv 0$. Then $\eta, \nu$ and $\mu$ can be chosen to be zero, and $\bar{p}$ are parallel planes in a 3-dimensional Affine subspace.

If $\alpha_4 \equiv 0$, then $\mathcal{H}$ is independent of the choice of $\delta$. Hence we have $\eta \equiv 0$ and $\alpha_5 \mu \equiv 0$. If $\alpha_5 \equiv 0$, then we should have $\nu \equiv 0$. From (3), $e_4$ and $e_5$ are independent of $e_1$, $e_2$, and $e_3$, and $\bar{p}$ lie in a 3-dimensional Affine subspace. If $\alpha_5 \neq 0$, let $\gamma = \delta = 0$ and $\rho = -(\alpha_2 \sin \theta + r') / \alpha_5$. Then the coefficient of $\cos(2\theta)$ of $\alpha_3 \mathcal{H}$ is $-2r^2 \kappa$, which should be 0. Therefore $e_1$ and $e_2$ are independent of $e_3$, $e_4$, and $e_5$, and $\bar{p}$ lie in a 3-dimensional Affine subspace. \hfill $\Box$
Lemma 3. The circle-foliated minimal surfaces in $\mathbb{R}^4$ of Lemma 1 is the graph $G_c$ for some real $c$.

Proof. From (7), (8) and $\kappa^2 = \tau^2$, we see that $\beta^2 = \kappa^2 = \tau^2 = \eta^2$. Suppose that $\beta = \kappa$ and $\tau = \eta$ (the case $\beta = -\kappa$ and $\tau = -\eta$ can be dealt with in the same way). It follows that $(e_2 + e_3)' = 0$ and $(e_1 + e_4)' = 0$ (depending on the sign of $\kappa/\tau$). We may suppose that $e_1 + e_4 = (0, 0, 0, \sqrt{2})$ and $e_2 + e_3 = (0, 0, \sqrt{2}, 0)$. Then we have

\[
\begin{align*}
e_1 &= \frac{1}{\sqrt{2}}(\cos \psi(t), \sin \psi(t), 0, 1), \\
e_2 &= \frac{1}{\sqrt{2}}(\cos \phi(t), \sin \phi(t), 1, 0), \\
e_3 &= \frac{1}{\sqrt{2}}(-\cos \phi(t), -\sin \phi(t), 1, 0), \\
e_4 &= \frac{1}{\sqrt{2}}(-\cos \psi(t), -\sin \psi(t), 0, 1).
\end{align*}
\]

From $e_1' = -\beta e_2 + \beta e_3$, we see that $2\beta = \pm \psi'$. If $\psi' = 2\beta$, then we have $\psi = \pi/2 + \phi$. Moreover $e_3' = -\beta e_1 + \tau e_4$ implies that $\beta = \eta$. Similarly, when $\psi' = -2\beta$, we have $\kappa = \eta$. Therefore we may assume that $\beta = \kappa = \tau = \eta = 1$. Then direct computation shows that

\[
H = r \left( rr'' - 3r'^2 - 2r^2 \right)
\]

for the normals of $X(t, \theta)$ corresponding to the cases i) $\gamma = -r'/r \cos \theta, \delta = 0$ and ii) $\gamma = 0, \delta = r'/r \sin \theta$. Hence $r$ satisfies

\[
r r'' - 3r'^2 - 2r^2 = 0.
\]

The solution of (9) is $r = C_1 \cos(2t + C_2)\left(2t + C_2\right)^{-1/2}$, where $C_1$ and $C_2$ are constants.

We may let $C_1 = c$ and $C_2 = 0$ and $-\pi/4 < t < \pi/4$. Let $A$ be the $4 \times 4$ orthogonal matrix given by

\[
A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.
\]

Then $\tilde{X}(t, \theta) = A \circ X(t, \theta)$ represents the graph

\[
G_c = \left\{ \left( \zeta, \frac{c}{\zeta} \right) \mid \zeta \in \mathbb{C} - \{0\} \right\}.
\]

□

Remark 2. i) The parametrization of $G_c$ is given by

\[
\tilde{X}(t, \theta) = \left( r \cos \theta, r \sin \theta, \frac{c}{r} \cos \theta, -\frac{c}{r} \sin \theta \right).
\]

Clearly, $G_c$ has two ends that are asymptotic to the planes $\{(w, z) \mid w = 0\}$ and $\{(w, z) \mid z = 0\}$. 
Let $g_r = (r \cos \theta, r \sin \theta, \frac{c}{r} \cos \theta, -\frac{c}{r} \sin \theta)$ be the circle on $G_c$ for fixed $r$. The geodesic curvature of $g_r$ is

\[ \kappa_g = \frac{r|c^2 - r^4|}{(c^2 + r^4)^{3/2}}. \]

Hence we have

\[ \int_{g_r} \kappa_g ds = 2\pi |c^2 - r^4|. \]

Since $G_c$ is doubly-connected, Gauss-Bonnet theorem implies that

\[ \int_{G_c} K dA = -\lim_{r \to 0} \int_{g_r} \kappa_g ds - \lim_{r \to \infty} \int_{g_r} \kappa_g ds = -4\pi. \]

3. Circle-foliated minimal surfaces in $S^4$ and $H^4$

To construct a circle-foliated minimal surface in $S^4$, we consider $\mathbb{R}^4$ with the conformal metric $ds_s^2 = ds_0^2/((1 + \langle x, x \rangle)/2)^2$, where $ds_0^2$ is the Euclidean metric of $\mathbb{R}^4$ and $\langle , \rangle$ is the Euclidean inner product. Let $H_s$ and $H_0$ be the mean curvatures of a surface $M$ in $\mathbb{R}^4$ with respect to the metrics $ds_s^2$ and $ds_0^2$ respectively with respect to fixed Euclidean normal $N$ satisfying (4). We have

\[ H_s = \frac{1 + \langle x, x \rangle}{2|N|} H_0 + \left\langle x, \frac{N}{|N|} \right\rangle. \]

Similarly, to construct a circle-foliated minimal surfaces in $H^4$, we equip the conformal metric $ds_h^2 = ds_0^2/((1 - \langle x, x \rangle)/2)^2$ on the unit ball $B(O, 1)$ of $\mathbb{R}^4$. Then the mean curvature $H_h$ of a surface $M$ in $B(O, 1)$ with respect to $ds_h^2$ satisfies

\[ H_h = \frac{1 - \langle x, x \rangle}{2|N|} H_0 - \left\langle x, \frac{N}{|N|} \right\rangle. \]

Examples of circle-foliated minimal surfaces in $S^4$ and $H^4$. Let $e_1, e_2$ be defined as in the proof of Theorem 3:

\[ e_1 = \frac{1}{\sqrt{2}}(-\sin 2t, \cos 2t, 0, 1), \]

\[ e_2 = \frac{1}{\sqrt{2}}(\cos 2t, \sin 2t, 1, 0). \]

The mean curvature of the circle-foliated surface

\[ X(t, \theta) = r(t) (\cos \theta e_1 + \sin \theta e_2) \]

satisfies

\[ H_0 |N| = \frac{r \left( r r'' - 3r'^2 - 2r^2 \right)}{2r^2 (r^2 + r'^2)}. \]
Hence as a surface in $\mathbb{S}^4$, $X(t, \theta)$ has mean curvature

$$H_s |N| = \frac{1 + r^2}{2} \cdot \frac{r (r r'' - 3r'^2 - 2r^2)}{2r^2 (r^2 + r'^2)} + r$$

for all normal direction. Therefore the circles centered at the origin on the planes spanned by $e_1(t), e_2(t)$ with radius function $r(t)$ satisfying

$$\frac{1 + r^2}{2} \cdot \frac{r r'' - 3r'^2 - 2r^2}{2r^2 (r^2 + r'^2)} + 1 = 0$$

(12)

define a circle-foliated minimal surface in $\mathbb{S}^4$.

Lemma 4. Solution of (12) with the initial conditions

$$r(0) = a^2 > 0 \text{ and } r'(0) = 0$$

is periodic.

Proof. Note that if $r(t)$ is a solution of (12), then $r(-t)$ is also a solution of (12). Hence each solution $r$ of (12) is an even function. Moreover if $r'(t_1) = 0$, then $r(t_1 + t) = r(t_1 - t)$. Therefore it suffices to show that $r'(t_1) = 0$ for some $t_1 > 0$.

Suppose that $r'$ does not vanish except for $t = 0$, therefore, $r'(t) > 0$ for all $t > 0$. From (12), we have

$$(13) \quad r'' = \frac{(3 - r^2) r'^2 + 2r^2 (1 - r^2)}{r (1 + r^2)}.$$ 

We may assume that $a < 1$. Then we have $r''(0) > 0$ and $r'(t) > 0$ for $t$ close to 0. If $r$ is not bounded, then $r'' \to -\infty$ as $t \to \infty$ by (13). Then $r' \to -\infty$, which contradicts $r'(t) > 0$ for all $t > 0$.

If $r$ is bounded, then $r'' \to 0$ and $r' \to 0$ as $t \to \infty$. From (13) and the fact that $r'' \to 0$ as $t \to \infty$, it follows that $r \not\to 1$ as $t \to \infty$. From (13), we have $r''(t) > 0$ for all $t > 0$. On the other hand, since $r$ is bounded and increasing, we should have $r''(t) < 0$ for sufficiently large $t$. Hence we conclude that $r'(t_1) = 0$ for some $t_1$ and $r$ is periodic. □

To estimate the period of (13), we use the integrating factor to obtain a first integral

$$\frac{(1 + r^2)^4}{r^5} (r')^2 + \left( \frac{1}{r^4} + \frac{4}{r^3} + 4r^2 + r^4 \right) = C,$$

or

(14) \quad \left( r + \frac{1}{r} \right)^4 \left( \frac{r'}{r} \right)^2 + \left( r + \frac{1}{r} \right)^4 = C.
We suppose that \( r'(0) = 0 \) and the minimum value \( r_{\text{min}} \) is attained at \( t = 0 \), that is, \( r_{\text{min}} = r(0) \). Then

\[
C = \left( r_{\text{min}} + \frac{1}{r_{\text{min}}} \right)^4.
\]

Let \( r_{\text{max}} \) be the maximum value of \( r \) and \( r_{\text{max}} = r(t_{\text{max}}) \) so that the period of \( r \) is \( 2t_{\text{max}} \). Then have

\[
r_{\text{min}} = \frac{1}{r_{\text{max}}}.
\]

Since \( r \) is strictly increasing on \((0, t_{\text{max}})\), we consider the inverse function \( t = t(r) \) of \( r(t) \). From (14), we have

\[
t'^2 = \frac{1}{r^2} \frac{(r^2 + 1)^4}{Cr^4 - (r^2 + 1)^4}.
\]

Then

\[
t_{\text{max}} = \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{1}{r} \frac{(r^2 + 1)^2}{\sqrt{Cr^4 - (r^2 + 1)^4}} \, dr.
\]

Since \( 1/r \) also satisfies (12), we have

\[
t_{\text{max}} = 2 \int_{r_{\text{min}}}^{1} \frac{1}{r} \frac{(r^2 + 1)^2}{\sqrt{Cr^4 - (r^2 + 1)^4}} \, dr.
\]

Substituting \( R = r^2 \), we get

\[
(15) \quad t_{\text{max}} = 2 \int_{r_{\text{min}}}^{1} \frac{1}{2R} \frac{(R + 1)^2}{\sqrt{CR^2 - (R + 1)^4}} \, dR.
\]

For convenience, we let \( c^2 = C \) with \( c \geq 4 \), \( p = \sqrt{(c + 4)/c} \) and \( k^2 = (c + 4)/(c - 4) \).

Substituting \( R = (\rho - p)/(\rho + p) \), we have

\[
dR = \frac{2p \, d\rho}{\sqrt{c(c - 4)(\rho^2 - 1)((\rho^2 - 1)\rho^2 - (c + 4))}} = \frac{2}{\sqrt{c(c - 4)}} \frac{d\rho}{\sqrt{(\rho^2 - 1)(\rho^2 - k^2)}}.
\]

Then (15) becomes

\[
t_{\text{max}} = \frac{8}{\sqrt{c(c - 4)}} \int_{\rho_0}^{\infty} \frac{\rho^2 \, d\rho}{(\rho^2 - p^2) \sqrt{(\rho^2 - 1)(\rho^2 - k^2)}},
\]

where \( \rho_0 = p(1 + r_{\text{min}}^2)/(1 - r_{\text{min}}^2) = k \).
Substituting $\tau = k/\rho$, we get
\begin{equation}
t_{\text{max}} = \frac{8}{\sqrt{c(c + 4)}} \int_0^1 \left(1 - \left(\frac{k}{\rho}\right)^2 \tau^2\right) \sqrt{(1 - \tau^2) \left(1 - \frac{\tau^2}{c}\right)} d\tau
= \frac{8}{\sqrt{c(c + 4)}} \Pi \left(\frac{c - 4}{c}, \frac{c - 4}{c + 4}\right),
\end{equation}
where $\Pi \left((c - 4)/c, (c - 4)/(c + 4)\right)$ is the complete elliptic integral of the third kind. For the following lemma, we introduce the elliptic integral of the 1st kind $F(\phi, \alpha)$ and the elliptic integral of the 2nd kind $E(\phi, \alpha)$:
\begin{align*}
F(\phi, \alpha) &= \int_0^\phi d\theta \sqrt{1 - \sin^2 \alpha \sin^2 \theta}, \\
E(\phi, \alpha) &= \int_0^\phi \sqrt{1 - \sin^2 \alpha \sin^2 \theta} d\theta.
\end{align*}
Moreover, $K(\alpha) = F(\pi/2, \alpha)$ and $E(\alpha) = E(\pi/2, \alpha)$ are the complete elliptic integrals of the first and second kinds respectively. Letting $k = \sin \alpha$, we also let $E(k) = E(\pi/2; k), K(k) = F(\pi/2; k)$.

Note that $c^2 = (r_{\text{min}} + 1/r_{\text{min}})^2$. If $c \to 4$ or $r_{\text{min}} \to 1$, then $t_{\text{max}} \to \pi/\sqrt{2}$.

In this case, we have $r \equiv 1$ and the resulting minimal surface is a torus.

**Lemma 5.** As a function of $c \geq 4$, $t_{\text{max}}$ is decreasing and satisfies
\[ \lim_{c \to \infty} t_{\text{max}} = \frac{\pi}{2}. \]
Hence the period of the solution of (12) is between $\pi$ and $\sqrt{2}\pi$.

**Proof.** Straightforward computation shows that
\[ \frac{d}{dc} \left( \frac{8}{\sqrt{c(c + 4)}} \Pi \left(\frac{c - 4}{c}, \frac{c - 4}{c + 4}\right) \right) = \frac{E \left(\frac{c - 4}{c + 4}\right) - K \left(\frac{c - 4}{c + 4}\right)}{2(c - 4)\sqrt{c(c + 4)}}. \]
Since
\[ E \left(\frac{c - 4}{c + 4}\right) - K \left(\frac{c - 4}{c + 4}\right) < 0, \]
t$_{\text{max}}$ is a decreasing function of $c$.

Let $\alpha = \sin^{-1} \sqrt{(c - 4)/(c + 4)}$ with $0 < \alpha < \pi/2$ and let $\nu = (c - 4)/c$. According to [1], the integral $\Pi \left((c - 4)/c, \pi/2, (c - 4)/(c + 4)\right) = \Pi (\nu; \pi/2, \alpha)$ belongs to the circular case with $\sin^2 \alpha < \nu < 1$, and
\[ \Pi (\nu; \pi/2, \alpha) = K(\alpha) + \frac{\pi}{2} \delta_2 (1 - \Lambda_0(\phi, \alpha)), \]
where $\Lambda_0$ is the Heuman’s Lambda function satisfying
\[ \Lambda_0(\phi, \alpha) = \frac{2}{\pi} [K(\alpha) (E(\phi, \pi/2 - \alpha) - F(\phi, \pi/2 - \alpha)) + E(\alpha) F(\phi, \pi/2 - \alpha)]. \]
\[
\begin{align*}
\delta_2 &= \sqrt{\nu/(1 - \nu)(\nu - \sin^2 \alpha)} = \sqrt{c(c + 4)/4}, \\
\phi &= \sin^{-1}(1 - \nu)/(\cos^2 \alpha) = \sin^{-1}(c + 4)/2c.
\end{align*}
\]

Therefore \( \phi \to \pi/4 \) and \( \alpha \to \pi/2 \) as \( c \to \infty \).

Clearly,

\[
\lim_{c \to \infty} \left( E(\phi, \pi/2 - \alpha) - F(\phi, \pi/2 - \alpha) \right) = -\lim_{c \to \infty} \int_0^\phi \frac{\cos^2 \alpha \sin^2 \theta}{\sqrt{1 - \cos^2 \alpha \sin^2 \theta}} d\theta.
\]

We note that \( \cos \alpha = \sqrt{8/(c + 4)} \) and that \( \lim_{\alpha \to \pi/2} \cos K(\alpha) = 0 \) (cf. Lemma 8 of [5]). Then

\[
\lim_{c \to \infty} t_{\max} = \lim_{c \to \infty} \frac{8}{\sqrt{c(c + 4)}} \Pi(\nu; \pi/2, \alpha) = \frac{\pi}{2}.
\]

**Theorem 2.** The circle-foliated surface given by (11) with \( e_1, e_2 \) satisfying (10) and \( r \) satisfying (12) defines a one-parameter family of circle-foliated minimal surfaces in \( S^4 \). Moreover, the radius function \( r \) is periodic with the period between \( \pi \) and \( \sqrt{2\pi} \). Hence there are infinitely many circle-foliated immersed minimal tori in \( S^4 \).

In \( H^4 \), we let \( e_1, e_2 \) and \( X(t, \theta) \) be as in (10) and (11). Then the mean curvature of \( X(t, \theta) \) with respect to \( ds_h^2 \) satisfies

\[
H_h |N| = \frac{1 - r^2}{2} \cdot \frac{rr'' - 3r'^2 - 2r^2}{2r^2 (r^2 + r'^2)} - r
\]

with \( N \) satisfying (4). Hence if \( X(t, \theta) \) is minimal, then \( r \) satisfies

\[
\frac{1 - r^2}{2} \cdot \frac{rr'' - 3r'^2 - 2r^2}{2r^2 (r^2 + r'^2)} - 1 = 0.
\]

We note that \( r'' \) blows up as \( r \to 1 \), when \( X(t, \theta) \) approaches the ideal boundary of \( H^4 \). For each initial condition \( r(0) = b^2 < 1, r'(0) = 0 \) of (17), \( X(t, \theta) \) gives a complete circle-foliated minimal surface in \( H^4 \).

**Theorem 3.** The parametrization (11) with \( e_1, e_2 \) satisfying (10) and \( r \) satisfying (17) gives a one-parameter family of circle-foliated minimal surfaces in \( H^4 \).

**References**


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