GENERALIZED LUCAS NUMBERS
OF THE FORM $5kx^2$ AND $7kx^2$

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Abstract. Generalized Fibonacci and Lucas sequences $(U_n)$ and $(V_n)$ are defined by the recurrence relations $U_{n+1} = PU_n + QU_{n-1}$ and $V_{n+1} = PV_n + QV_{n-1}$, $n \geq 1$, with initial conditions $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = P$. This paper deals with Fibonacci and Lucas numbers of the form $U_n(P, Q)$ and $V_n(P, Q)$ with the special consideration that $P \geq 3$ is odd and $Q = -1$. Under these consideration, we solve the equations $V_n = 5kx^2$, $V_n = 7kx^2$, $V_n = 5kx^2 \pm 1$, and $V_n = 7kx^2 \pm 1$ when $k \mid P$ with $k > 1$. Moreover, we solve the equations $V_n = 5x^2 \pm 1$ and $V_n = 7x^2 \pm 1$.

1. Introduction

Let $P$ and $Q$ be nonzero integers such that $P^2 + 4Q \neq 0$. Generalized Fibonacci sequence $(U_n(P, Q))$ and Lucas sequence $(V_n(P, Q))$ are defined as follows:

$U_0(P, Q) = 0, U_1(P, Q) = 1, U_{n+1}(P, Q) = PU_n(P, Q) + QU_{n-1}(P, Q)$ ($n \geq 1$)

and

$V_0(P, Q) = 2, V_1(P, Q) = P, V_{n+1}(P, Q) = PV_n(P, Q) + QV_{n-1}(P, Q)$ ($n \geq 1$).

The numbers $U_n = U_n(P, Q)$ and $V_n = V_n(P, Q)$ are called the $n$-th generalized Fibonacci and Lucas numbers, respectively. Furthermore, generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$U_{-n} = U_n(P, Q)/(-Q)^n$

and

$V_{-n} = V_n(P, Q)/(-Q)^n$

for $n \geq 1$. If $\alpha = \frac{P + \sqrt{P^2 + 4Q}}{2}$ and $\beta = \frac{P - \sqrt{P^2 + 4Q}}{2}$ are the roots of the equation $x^2 - Px - Q = 0$, then we have the following well known expressions named
Binet’s formulas

\[ U_n = U_n(P, Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = V_n(P, Q) = \alpha^n + \beta^n \]

for all \( n \in \mathbb{Z} \). Since \( U_n = U_n(-P, Q) = (-1)^n U_n(P, Q) \) and \( V_n = V_n(-P, Q) = (-1)^n V_n(P, Q) \), it will be assumed that \( P \geq 1 \). Moreover, we assume that \( P^2 + 4Q > 0 \). Special cases of the sequences \((U_n)\) and \((V_n)\) are known. If \( P = Q = 1 \), then \((U_n(1,1))\) is the familiar Fibonacci sequence \((F_n)\) and the sequence \((V_n(1,1))\) is the familiar Lucas sequence \((L_n)\). If \( P = 2 \) and \( Q = 1 \), we have the well known Pell sequence \((P_n)\) and Pell–Lucas sequence \((Q_n)\). For more information about generalized Fibonacci and Lucas sequences, the reader can follow [3, 8, 9, 10].

Generalized Fibonacci and Lucas numbers of the form \( kx^2 \) have been investigated by many authors and progress in determining the square or \( k \) times a square terms of \( U_n \) and \( V_n \) has been made in certain special cases. Interested readers can consult [5] or [14] for a brief history of this subject. In [11], the authors, applying only congruence properties of sequences, determined all indices \( n \) such that \( U_n = x^2, U_n = 2x^2, V_n = x^2, \) and \( V_n = 2x^2 \) for all odd relatively prime values of \( P \) and \( Q \). Furthermore, the same authors [12] solved \( V_n = kx^2 \) under some assumptions on \( k \). In [1], when \( P \) is odd, Cohn solved \( V_n = Px^2 \) and \( V_n = 2Px^2 \) with \( Q = \pm 1 \). In [14], the authors determined, assuming \( Q = 1 \), all indices \( n \) such that \( V_n(P,1) = kx^2 \) when \( k \mid P \) and \( P \) is odd, where \( k \) is a square-free positive divisor of \( P \). The values of \( n \) have been found for which \( V_n(P,1) = kx^2 \) is of the form \( 2kx^2, \ 2kx^2, \ kx^2 \pm 1, \) and \( 2kx^2 \pm 1 \) with \( k \mid P \) and \( k > 1 \) [4]. Moreover, the values of \( n \) have been found for which \( V_n(P,-1) \) is of the form \( 2x^2 \pm 1, \ 3x^2 - 1, \) and \( 6x^2 \pm 1 \) [4] and the author give all integer solutions of the preceding equations. Our results in this paper add to the above list the values of \( n \) for which \( V_n(P,-1) \) is of the form \( 5kx^2, \ 5kx^2 \pm 1, \ 7kx^2, \) and \( 7kx^2 \pm 1 \) when \( k \mid P \) and \( k > 1 \). Furthermore, we determine all indices \( n \) such that \( V_n(P,-1) = 5x^2 \pm 1 \) and \( V_n(P,-1) = 7x^2 \pm 1 \) and then give all integer solutions of these equations. We need to state here that our method is elementary and used by Cohn, Ribenboim and McDaniel in [1] and [12], respectively.

2. Preliminaries

In this section, we present some theorems, lemmas, and identities, which will be needed during the proof of the main theorems. Instead of \( U_n(P, -1) \) and \( V_n(P, -1) \), we sometimes write \( U_n \) and \( V_n \). Throughout the paper \( (\frac{a}{b}) \) denotes the Jacobi symbol.

Now we can give the following lemma without proof since its proof can be done by induction.
Lemma 1. Let $n$ be a positive integer. Then
\[ V_n \equiv \begin{cases} \pm 2 \pmod{P^2} & \text{if } n \text{ is even,} \\ \pm nP \pmod{P^2} & \text{if } n \text{ is odd.} \end{cases} \]

We omit the proof of the following lemma due to Keskin and Demirtürk [2].

Lemma 2. All positive integer solutions of the equation $x^2 - 5y^2 = 1$ are given by $(x, y) = (L_{3z}/2, F_{3z}/2)$ with $z$ even natural number.

The following theorem is well known (see [6] or [7]).

Theorem 1. All positive integer solutions of the equation $x^2 - (P^2 - 4)y^2 = 4$ are given by $(x, y) = (V_n, U_n)$ with $n \geq 1$.

The proofs of the following two theorems can be found in [13].

Theorem 2. Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$ and $m$ be a nonzero integer. Then
\[ U_{2mn+r} \equiv U_r \pmod{U_m} \]
and
\[ V_{2mn+r} \equiv V_r \pmod{U_m}. \]

Theorem 3. Let $n \in \mathbb{N} \cup \{0\}$ and $m, r \in \mathbb{Z}$. Then
\[ U_{2mn+r} \equiv (-1)^nU_r \pmod{V_m} \]
and
\[ V_{2mn+r} \equiv (-1)^nV_r \pmod{V_m}. \]

The following identities are well known.
\[ U_{2n} = U_nV_n, \]
\[ V_n^2 - (P^2 - 4)U_n^2 = 4, \]
\[ V_{2n} = V_n^2 - 2. \]

From Lemma 1 and identity (2.6), we have
\[ 5 \mid V_n \text{ if and only if } 5 \mid P \text{ and } n \text{ is odd} \]
and
\[ 7 \mid V_n \text{ if and only if } 7 \mid P \text{ and } n \text{ is odd or } P^2 \equiv 2 \pmod{7} \text{ and } n \equiv 2 \pmod{4}. \]

If $r \geq 1$, then by (2.7),
\[ V_{2r} \equiv \pm 2 \pmod{P}. \]
If $r \geq 2$, then by (2.7),
\[ V_{2r} \equiv 2 \pmod{P}. \]

It is obvious that
\[ 5 \mid U_5 \text{ if } P^2 \equiv -1 \pmod{5}, \]
(2.13) \[ 7 \mid U_4 \text{ if } P^2 \equiv 2 \pmod{7}. \]

From now on, unless otherwise stated, we assume that \( P \geq 3 \) is odd and \( Q = -1 \).

By using (2.2) together with the fact \( 8 \mid U_3 \), we get

\[ (2.14) \quad V_{0q+r} \equiv V_r \pmod{8}. \]

Moreover, by using induction, it can be seen that

\[ V_{2r} \equiv 7 \pmod{8} \]

and thus

\[ (2.15) \quad \left( \frac{2}{V_{2r}} \right) = 1 \]

and

\[ (2.16) \quad \left( \frac{-1}{V_{2r}} \right) = -1 \]

for \( r \geq 1 \).

If \( r \geq 1 \), then

\[ (2.17) \quad \left( \frac{5}{V_{2r}} \right) = \begin{cases} -1 & \text{if } 5 \mid P, \\ 1 & \text{if } P^2 \equiv 1 \pmod{5}, \\ -1 & \text{if } P^2 \equiv -1 \pmod{5}. \end{cases} \]

Moreover,

\[ (2.18) \quad \left( \frac{7}{V_{2r}} \right) = \pm 1 \text{ if } 7 \mid P \text{ and } r \geq 1, \]

\[ (2.19) \quad \left( \frac{7}{V_{2r}} \right) = -1 \text{ if } 7 \mid P \text{ and } r \geq 2, \]

and

\[ (2.20) \quad \left( \frac{7}{V_{2r}} \right) = 1 \text{ if } P^2 \equiv 1 \pmod{7} \text{ and } r \geq 1. \]

If \( r \geq 2 \), then \( V_{2r} \equiv -1 \left( \mod \frac{P^2 - 3}{2} \right) \) and thus

\[ (2.21) \quad \left( \frac{(P^2 - 3)/2}{V_{2r}} \right) = \left( \frac{P^2 - 3}{V_{2r}} \right) = 1. \]

If \( 3 \nmid P \), then \( V_{2r} \equiv -1 \pmod{3} \) for \( r \geq 1 \). Therefore we have

\[ (2.22) \quad \left( \frac{3}{V_{2r}} \right) = 1. \]

If \( 3 \mid P \), then \( V_{2r} \equiv -1 \pmod{3} \) for \( r \geq 2 \) and therefore

\[ (2.23) \quad \left( \frac{3}{V_{2r}} \right) = 1. \]
Besides, we have
\[(2.24) \quad \frac{P - 1}{V_{2r}} = \frac{P + 1}{V_{2r}} = 1.\]

3. Main theorems

We assume from this point on that \(n\) is a positive integer.

**Theorem 4.** If \(V_n = 5kx^2\) for some integer \(x\), then \(n = 1\).

*Proof.* Assume that \(V_n = 5kx^2\) for some integer \(x\). Then by (2.8), it follows that \(5 \mid P\) and \(n\) is odd. Let \(n = 6q + r\) with \(r \in \{1, 3, 5\}\). Then by (2.14), we obtain \(V_n = V_{6q+r} \equiv V_r \pmod{8}\). Hence, we have \(V_n \equiv V_1, V_3, V_5 \pmod{8}\). This implies that \(V_n = 5kx^2 \equiv P, 6P \pmod{8}\) and therefore \(kx^2 \equiv 5P, 6P \pmod{8}\).

On the other hand, using the fact that \(k \mid P\), we may write \(P = kM\) with odd \(M\). Thus, we get \(kMx^2 \equiv 5PM, 6PM \pmod{8}\), implying that \(Px^2 \equiv 5PM, 6PM \pmod{8}\). Since \(P\) is odd, then the preceding congruence gives \(x^2 \equiv 5M, 6M \pmod{8}\). This shows that \(M \equiv 5 \pmod{8}\) since \(M\) is odd. Now suppose that \(n > 1\). Then \(n = 4q + 1 = 2 \cdot 2^a + 1, 2 \nmid a\) and \(r \geq 1\). Substituting this value of \(n\) into \(V_n = 5kx^2\) and using (2.4) give
\[5kx^2 = V_n = V_{4q+1} = V_{2 \cdot 2^a + 1} = -V_1 = -P \pmod{V_{2r}}.\]

This shows that
\[5x^2 \equiv -M \pmod{V_{2r}}\]

since \((k, V_{2r}) = 1\). According to the above congruence, we have
\[\left(\frac{5}{V_{2r}}\right) = \left(-1\right) \left(\frac{M}{V_{2r}}\right).\]

By using the facts that \(M \equiv 5 \pmod{8}\), \(M \mid P\) and identity (2.10), we get
\[\left(\frac{M}{V_{2r}}\right) = \left(\frac{V_{2r}}{M}\right) = \left(\frac{\pm 2}{M}\right) = \left(\frac{2}{M}\right) = -1.\]

On the other hand, this is impossible since \(\left(\frac{5}{V_{2r}}\right) = -1\) by (2.17) and \(\left(\frac{1}{V_{2r}}\right) = -1\) by (2.16). Thus, \(n = 1\), as claimed. \(\square\)

By using Theorem 1, we can give the following corollary.

**Corollary 1.** The equation \(25P^2x^4 - (P^2 - 4)y^2 = 4\) has no integer solutions.

**Theorem 5.** Let \(k \mid P\) with \(k > 1\). Then there is no integer \(x\) such that \(V_n = 5kx^2 + 1\).

*Proof.* Assume that \(V_n = 5kx^2 + 1\) for some integer \(x\). Then by Lemma 1, we have \(n\) is even. Let \(n = 2m\) for some \(m > 0\). Thus, by (2.7), we get \(V_n = V_{2m} = V_m^2 - 2 = 5kx^2 + 1\), implying that \(V_m^2 \equiv 3 \pmod{5}\), a contradiction. \(\square\)

**Corollary 2.** The equation \((5Px^2 + 1)^2 - (P^2 - 4)y^2 = 4\) has no integer solutions.
Theorem 6. Let $k \mid P$ with $k > 1$. Then there is no integer $x$ such that $V_n = 5kx^2 - 1$.

Proof. Assume that $V_n = 5kx^2 - 1$ for some integer $x$. Then by Lemma 1, $n$ is even. We divide the proof into three cases.

Case 1: Assume that $5 \mid P$. Then by Lemma 1, it follows that $V_n \equiv \pm 2 \pmod{5}$, which contradicts the fact that $V_n \equiv 4 \pmod{5}$.

Case 2: Assume that $P^2 \equiv -1 \pmod{5}$. Then we immediately have from (2.12) that $5 \mid U_5$. Let

$$n = 10q + r$$

with $r \in \{0, 2, 4, 6, 8\}$. By (2.2), we get

\begin{align*}
V_n & = V_{10q+r} \equiv V_r \pmod{U_5}, \\
V_n & \equiv V_0, V_2, V_4, V_6, V_8 \pmod{5}.
\end{align*}

This shows that $V_n \equiv 2 \pmod{5}$, which contradicts the fact that $V_n \equiv 4 \pmod{5}$.

Case 3: Assume that $P^2 \equiv 1 \pmod{5}$. Since $n$ is even, $n = 2m$ for some $m > 0$. Hence, we get

$$5kx^2 - 1 = V_n = V_{2m} = V_m^2 - 2$$

by (2.7). If $m$ is odd, then $P \mid V_m$ by Lemma 1 and so we get $k \mid 1$, a contradiction. Thus, $m$ is even. Let $m = 2u$ for some $u > 0$. Then $n = 4u$ and therefore

$$5kx^2 - 1 = V_{4u} \equiv V_0 \equiv 2 \pmod{U_2}$$

by (2.2), which implies that

$$5kx^2 \equiv 3 \pmod{P}.$$

Since $k \mid P$, it follows that $k \mid 3$ and therefore $k = 3$. So, we conclude that $3 \mid P$.

In this case, we have $n = 4u$ and thus by (2.2), we have

$$15x^2 - 1 = V_n = V_{4u} \equiv V_0 \pmod{U_2},$$

implying that

$$15x^2 \equiv 3 \pmod{P}.$$

Since $3 \mid P$, it is seen that $5x^2 \equiv 1 \pmod{P/3}$. This shows that $\left(\frac{5}{P/3}\right) = 1$.

We are in the case that $P^2 \equiv 1 \pmod{5}$. So, $P \equiv 1, 4 \pmod{5}$. A simple computation shows that $P/3 \equiv \pm 2 \pmod{5}$. Hence, we have

\begin{align*}
1 & = \left(\frac{5}{P/3}\right) = \left(\frac{P/3}{5}\right) = \left(\frac{\pm 2}{5}\right) = -1,
\end{align*}

a contradiction.

\[\square\]

Corollary 3. The equation $(5Px^2 - 1)^2 - (P^2 - 4)y^2 = 4$ has no integer solutions.

Theorem 7. If $V_n = 5x^2 + 1$, then $n = 1$ and $V_1 = 5x^2 + 1$ where $x$ is even.

Proof. Assume that $V_n = 5x^2 + 1$ for some integer $x$. If $n$ is even, then $n = 2m$ and therefore $V_n = V_{2m} = V_m^2 - 2$ by (2.7). This implies that $V_m^2 \equiv 3 \pmod{5}$, a contradiction. Thus, $n$ is odd.

Case 1: Assume that $5 \mid P$. Since $n$ is odd, it follows from Lemma 1 that $5 \mid V_n$, implying that $5 \mid 1$, a contradiction.
Case 2: Assume that $P^2 \equiv 1 \pmod{5}$. If $n > 1$, then $n = 4q \pm 1 = 2 \cdot 2^r a \pm 1$, $2 \nmid a$ and $r \geq 1$. Thus,

$$5x^2 = V_n - 1 \equiv -V_1 - 1 \equiv -(P + 1) \pmod{V_{2r}}$$

by (2.4). By using (2.16), (2.17), and (2.24), it is seen that

$$1 = \left(\frac{-1}{V_{2r}}\right) \left(\frac{5}{V_{2r}}\right) \left(\frac{P + 1}{V_{2r}}\right) = -1,$$

a contradiction. So, $n = 1$. This implies that $P = 5x^2 + 1$ with $x$ even.

Case 3: Assume that $P^2 \equiv -1 \pmod{5}$. If $P \equiv 1 \pmod{4}$, since $n$ is odd, we can write $n = 4q \pm 1$. Thus, $5x^2 + 1 = V_n \equiv V_1 \equiv P \pmod{U_2}$ by (2.2). This implies that

$$5x^2 \equiv -1 \pmod{P}.$$ 

Since $P^2 \equiv -1 \pmod{5}$, it follows that $P \equiv \pm 2 \pmod{5}$. Hence, we have

$$\left(\frac{5}{P}\right) = \left(\frac{P}{5}\right) = \left(\frac{\pm 2}{5}\right) = -1.$$

On the other hand, using the fact that $P \equiv 1 \pmod{4}$, we get

$$\left(\frac{-1}{P}\right) = (-1)^{\frac{p-1}{2}} = 1.$$

And so, (3.1) is impossible. Now if $P \equiv 3 \pmod{4}$, let $n = 6q + r$ with $r \in \{1, 3, 5\}$. Hence, by (2.14), we have

$$5x^2 + 1 = V_n = V_{6q+r} \equiv V_r \equiv V_1, V_3, V_5 \, \pmod{8},$$

implying that

$$5x^2 + 1 \equiv P, 6P \, \pmod{8}.$$ 

If $x$ is even, then $P, 6P \equiv 1 \pmod{4}$, which is impossible since $P \equiv 3 \pmod{4}$. If $x$ is odd, then $P, 6P \equiv 6 \pmod{8}$, which is impossible since $P \equiv 3, 7 \pmod{8}$. This completes the proof. \qed

**Corollary 4.** The equation $(5x^2 + 1)^2 - (P^2 - 4)y^2 = 4$ has integer solutions only when $P = 5x^2 + 1$ with $x$ even.

**Theorem 8.** If $V_n = 5x^2 - 1$, then $n = 1$ and $V_1 = 5x^2 - 1$ with $x$ is even, or $n = 2$ and $P = L_{3z}/2$ and $x = F_{3z}/2$ where $z$ is even.

**Proof.** Assume that $V_n = 5x^2 - 1$ for some integer $x$. Dividing the proof into three cases, we have

- **Case 1:** Assume that $5 \mid P$. Lemma 1 implies that $V_n \equiv 0, \pm 2 \pmod{5}$, which contradicts $V_n = 5x^2 - 1$.
- **Case 2:** Assume that $P^2 \equiv 1 \pmod{5}$. If $n > 1$ is odd, then $n = 4q \pm 1 = 2 \cdot 2^r a \pm 1$, $2 \nmid a$ and $r \geq 1$. Thus,

$$5x^2 - 1 = V_n \equiv -V_1 \equiv -P \pmod{V_{2r}},$$

implying that
\[ 5x^2 \equiv -(P - 1) \pmod{V_{2r}}. \]
By using (2.16), (2.17), and (2.24), it is seen that
\[ 1 = \left( \frac{-1}{V_{2r}} \right) \left( \frac{5}{V_{2r}} \right) \left( \frac{P - 1}{V_{2r}} \right) = -1, \]
which is impossible. So \( n = 1 \) and \( P = 5x^2 - 1 \) with \( x \) even. Now if \( n \) is even, then \( n = 2m \) for some \( m > 0 \). If \( m > 1 \) is odd, then \( n = 2(4q \pm 1) = 2 \cdot 2^r a \pm 2, \) \( 2 \nmid a \) and \( r \geq 2 \). Thus,
\[ 5x^2 - 1 = V_n \equiv -V_2 \equiv -(P^2 - 2) \pmod{V_{2r}}, \]
implying that
\[ 5x^2 \equiv -(P^2 - 3) \pmod{V_{2r}}. \]
By using (2.16), (2.17), and (2.21), it is seen that
\[ 1 = \left( \frac{-1}{V_{2r}} \right) \left( \frac{5}{V_{2r}} \right) \left( \frac{P^2 - 3}{V_{2r}} \right) = -1, \]
a contradiction. Hence, we have \( m = 1 \) and therefore \( n = 2 \). Substituting this value of \( n \) into \( V_n = 5x^2 - 1 \) gives \( V_2 = P^2 - 2 = 5x^2 - 1 \), i.e., \( P^2 - 5x^2 = 1 \).
By Lemma 2, we get \( P = L_{3z}/2 \) and \( x = F_{3z}/2 \) with \( z \) even natural number. If \( m \) is even, then \( m = 2u \) for some \( u > 0 \) and therefore \( n = 4u = 2 \cdot 2^r a, \) \( 2 \nmid a \) and \( r \geq 1 \). Thus, by (2.4)
\[ 5x^2 - 1 = V_n = V_{4u} \equiv -V_0 \equiv -2 \pmod{V_{2r}}, \]
implying that
\[ 5x^2 \equiv -1 \pmod{V_{2r}}. \]
But this is impossible since \( \left( \frac{5}{V_{2r}} \right) = 1 \) by (2.17) and \( \left( \frac{-1}{V_{2r}} \right) = -1 \) by (2.16).

Case 3 : Assume that \( P^2 \equiv -1 \pmod{5} \). So, \( P \equiv \pm 2 \pmod{5} \). If \( n \) is odd, then we can write \( n = 4q \pm 1 \). Thus, \( 5x^2 - 1 = V_n \equiv V_1 \pmod{U_2} \) by (2.2). This implies that \( 5x^2 \equiv 1 \pmod{P} \). But this is impossible since
\[ 1 = \left( \frac{5}{P} \right) = \left( \frac{P}{5} \right) = \left( \frac{\pm 2}{5} \right) = -1. \]
If \( n \equiv 2 \pmod{4} \), where \( n \geq 6 \), then \( n = 2(4q) \pm 2 \). This shows that \( 5x^2 - 1 = V_n \equiv V_2 \pmod{V_2} \) by (2.4), implying that \( 5x^2 \equiv 1 \pmod{P^2 - 2} \). But this is impossible since
\[ 1 = \left( \frac{5}{P^2 - 2} \right) = \left( \frac{P^2 - 2}{5} \right) = \left( \frac{-3}{5} \right) = -1. \]
And so \( n = 2 \). Substituting this value of \( n \) into \( V_n = 5x^2 - 1 \) gives \( P^2 - 2 = 5x^2 - 1 \), implying that \( P^2 \equiv 1 \pmod{5} \), which is impossible since \( P^2 \equiv -1 \pmod{5} \). Now if \( n \equiv 0 \pmod{4} \), then \( n = 4u \) for some \( u \). By (2.7), we have \( 5x^2 - 1 = V_n = V_{4u} = V_{2u}^2 - 2 \), which implies that \( V_{2u}^2 - 1 = 5x^2 \). That is,
(\(V_{2n} - 1\))(\(V_{2n} + 1\)) = 5x^2. Clearly, \(d = (V_{2n} - 1, V_{2n} + 1) = 1\) or 2. If \(d = 1\), then either
(3.2)  
\[V_{2n} - 1 = a^2, \quad V_{2n} + 1 = 5b^2\]
or
(3.3)  
\[V_{2n} - 1 = 5a^2, \quad V_{2n} + 1 = b^2\]
for some integers \(a\) and \(b\). It can be easily seen that (3.2) and (3.3) are impossible. If \(d = 2\), then either
(3.4)  
\[V_{2n} - 1 = 2a^2, \quad V_{2n} + 1 = 10b^2\]
or
(3.5)  
\[V_{2n} - 1 = 10a^2, \quad V_{2n} + 1 = 2b^2\]
for some integers \(a\) and \(b\). Obviously, by (2.7), we get (3.5) is impossible since \(V_u^2 \equiv 3 \pmod{5}\) in this case. Suppose (3.4) is satisfied. Then by (2.7), we have
(3.6)  
\[V_u^2 - 3 = 2a^2.\]
If \(3 \nmid a\), then \(a^2 \equiv 1 \pmod{3}\) and therefore we get \(V_u^2 \equiv 2 \pmod{3}\), which is impossible. Hence, we have \(3 \mid a\). This implies that \(3 \mid V_u\). For the case when \(3 \mid a\) and \(3 \mid V_u\), we easily see from (3.6) that \(9 \mid 3\), a contradiction. This completes the proof. \(\square\)

**Corollary 5.** The equation \((5x^2 - 1)^2 - (P^2 - 4)y^2 = 4\) has integer solutions only when \(P = 5x^2 - 1\) with \(x\) even or \(P = L_{3z}/2\) with \(z\) even.

**Theorem 9.** If \(V_n = 7kx^2\) with \(k \mid P\) and \(k > 1\), then \(n = 1\).

**Proof.** Suppose that \(V_n = 7kx^2\) for some integer \(x\). Then by Lemma 1, we see that \(n\) is odd. And since \(7 \mid V_n\) and \(n\) is odd, it follows from (2.9) that \(7 \mid P\). Then by (2.14), we have \(V_n \equiv V_1, V_3, V_5 \equiv P, 6P \pmod{8}\). This implies that \(7kx^2 \equiv P, 6P \pmod{8}\), i.e., \(kx^2 \equiv 2P, 7P \pmod{8}\). Since \(k \mid P\), we may write \(P = kM\) with odd \(M\). Thus, we get \(kMx^2 \equiv 2PM, 7PM \pmod{8}\). And so \(x^2 \equiv 2M, 7M \pmod{8}\). This shows that \(M \equiv 7 \pmod{8}\) since \(M\) is odd. Now suppose \(n > 1\). Then \(n = 4q \pm 1 = 2 \cdot 2^r a \pm 1, 2 \nmid a\) and \(r \geq 1\). Substituting this value of \(n\) into \(V_n = 7kx^2\) and using (2.4) give

\[7kx^2 = V_n \equiv -V_1 \equiv -P \pmod{V_{2r}}.\]

This shows that

\[7x^2 = -M \pmod{V_{2r}}\]

since \((k, V_{2r}) = 1\). Thus, we have

\[1 = \left(\frac{7}{V_{2r}}\right) \left(\frac{-1}{V_{2r}}\right) \left(\frac{M}{V_{2r}}\right).\]

If \(r \geq 2\), then by (2.16), it is seen that \(\left(\frac{7}{V_{2r}}\right) = -1\). On the other hand, \(\left(\frac{7}{V_{2r}}\right) = -1\) by (2.19). Besides, using \(M \mid P\) and identity (2.11), we get \(\left(\frac{M}{V_{2r}}\right) = -1\) by (2.19). Therefore, \(\left(\frac{7}{V_{2r}}\right) = -1\). This completes the proof. \(\square\)
This shows that \((-1) \left( \frac{k}{49} \right) = -1\). This means that \(1 = -1\), a contradiction. Thus, \(r = 1\). But in this case, since \(\left( \frac{1}{49} \right) = -1\) by (2.16), \(\left( \frac{7}{49} \right) = 1\) by (2.18), and \(\left( \frac{49}{49} \right) = 1\) by (2.10), we again get a contradiction. So, we conclude that \(n = 1\). \(\square\)

**Corollary 6.** The equation \(49P^2x^4 - (P^2 - 4)y^2 = 4\) has no integer solutions.

**Theorem 10.** Let \(k \mid P\) with \(k > 1\). Then the equation \(V_n = 7kx^2 + 1\) has no integer solutions.

**Proof.** Suppose that \(V_n = 7kx^2 + 1\) for some integer \(x\). Then by Lemma 1, we have \(n\) is even and therefore \(n = 2m\) for some \(m > 0\). By (2.7), we get \(V_n = V_{2m} = V_m^2 - 2 = 7kx^2 + 1\), implying that \(V_m^2 \equiv 3\) (mod 7), a contradiction. \(\square\)

**Corollary 7.** The equation \((7Px^2 + 1)^2 - (P^2 - 4)y^2 = 4\) has no integer solutions.

**Theorem 11.** Let \(k \mid P\) with \(k > 1\). Then the equation \(V_n = 7kx^2 - 1\) has no integer solutions.

**Proof.** Suppose that \(V_n = 7kx^2 - 1\) for some integer \(x\). Then by Lemma 1, it is seen that \(n\) is even. Dividing the proof into four cases, we have,

Case 1 : Assume that \(7 \mid P\). Then by Lemma 1, it follows that \(V_n \equiv \pm 2\) (mod 7), which contradicts the fact that \(V_n \equiv 6\) (mod 7).

Case 2 : Assume that \(P^2 \equiv 2\) (mod 7). Then it is easily seen from (2.13) that \(7 \mid U_4\). Since \(n\) is even, \(n = 8q + r\) with \(r \in \{0, 2, 4, 6\}\), so by (2.2), \(V_n \equiv V_0, V_2, V_4, V_6\) (mod \(U_4\)), implying that \(V_n \equiv 0, 2\) (mod 7), which is impossible.

Case 3 : Assume that \(P^2 \equiv 4\) (mod 7). Then \(P \equiv 2, 5\) (mod 7). On the other hand, it can be easily seen that \(V_n \equiv 2, 5\) (mod 7) in this case. But this contradicts the fact that \(V_n \equiv 6\) (mod 7).

Case 4 : Assume that \(P^2 \equiv 1\) (mod 7). Since \(n = 2m\), we get \(V_n = V_{2m} = V_m^2 - 2 = 7kx^2 - 1\), i.e., \(V_m^2 = 7kx^2 + 1\). If \(m\) is odd, then \(P \mid V_m\) by Lemma 1 and therefore we obtain \(k \mid 1\), a contradiction. Thus, \(m\) is even. Let \(m = 2u\) for some \(u > 0\). Then \(n = 4u\) and therefore by (2.2), we have

\[7kx^2 - 1 = V_{4u} \equiv V_0 \equiv 2\) (mod \(U_2\)),

which implies that

\[7kx^2 \equiv 3\) (mod \(P\)).

Since \(k \mid P\), it follows that \(k \mid 3\) and this means that \(k = 3\). Hence, we find that \(3 \mid P\). In this case, we have \(n = 4u = 2 \cdot 2^u a, 2 \nmid a\) and \(r \geq 1\). And thus,

\[21x^2 - 1 = V_n \equiv -V_0 \equiv -2\) (mod \(V_2^r\)).

This shows that

\[\left( \frac{3}{V_2^r} \right) \left( \frac{7}{V_2^r} \right) = \left( \frac{-1}{V_2} \right) .\]

If \(r \geq 2\), then \(\left( \frac{3}{V_2^r} \right) = 1\) by (2.23) and \(\left( \frac{7}{V_2^r} \right) = 1\) by (2.20). But since \(\left( \frac{-1}{V_2} \right) = -1\) by (2.16), we get a contradiction. So, \(r = 1\). This means that
If $d = (V_{2u} - 1, V_{2u} + 1) = 1$ or 2. If $d = 1$, then either
\[ V_{2u} - 1 = a^2, \quad V_{2u} + 1 = 21b^2, \]
\[ V_{2u} - 1 = 3a^2, \quad V_{2u} + 1 = 7b^2, \]
\[ V_{2u} - 1 = 7a^2, \quad V_{2u} + 1 = 3b^2, \]
or
\[ V_{2u} - 1 = 21a^2, \quad V_{2u} + 1 = b^2. \]
A simple computation shows that all the above equalities are impossible. Now if $d = 2$, then
\begin{align*}
(3.7) & \quad V_{2u} - 1 = 2a^2, \quad V_{2u} + 1 = 42b^2, \\
(3.8) & \quad V_{2u} - 1 = 6a^2, \quad V_{2u} + 1 = 14b^2, \\
(3.9) & \quad V_{2u} - 1 = 14a^2, \quad V_{2u} + 1 = 6b^2, \\
or
(3.10) & \quad V_{2u} - 1 = 42a^2, \quad V_{2u} + 1 = 2b^2. 
\end{align*}
If we combine the two equations in (3.7), we get $a^2 \equiv 6 \pmod{7}$, a contradiction.
By using (2.7), we see that (3.9) and (3.10) are impossible since $V_u^2 \equiv 3 \pmod{7}$ in both cases. Now assume that (3.8) is satisfied. Let $u > 1$, and so $2u = 2 \cdot 2^r a \pm 2$, 2 \nmid a$ and $r \geq 2$. Thus, we obtain by (2.4) that
\[ 14b^2 - 1 = V_{2u} = V_{2 \cdot 2^r a \pm 2} \equiv -V_2 \equiv -(P^2 - 2) \pmod{V_2}, \]
implying that
\[ 7b^2 \equiv -(P^2 - 3)/2 \pmod{V_2}. \]
But this is impossible since $\left( \frac{7}{V_2} \right) = 1$ by (2.20), $\left( \frac{-1}{V_2} \right) = -1$ by (2.16), and $\left( \frac{(P^2 - 3)/2}{V_2} \right) = 1$ by (2.21). As a consequence, we get $u = 1$ and therefore $n = 4$. Substituting $n = 4$ into $V_n = 21x^2 - 1$ gives $V_4 = (P^2 - 2)^2 - 2 = 21x^2 - 1$, i.e., $(P^2 - 2)^2 - 21x^2 = 1$. Since all positive integer solutions of the equation $u^2 - 21x^2 = 1$ are given by $(u, v) = (V_s(110, -1)/2, 12U_s(110, -1))$ with $s \geq 1$, we get $P^2 - 2 = V_s(110, -1)/2$ for some $s \geq 0$. Thus, $V_s(110, -1) = 2P^2 - 4$. It can be shown that $s$ is odd. Taking $s = 4q \pm 1$ and using this into $V_s(110, -1)/2$ give
\[ 2P^2 - 4 = V_s \equiv V_1 \pmod{V_1}, \]
implying that
\[ 2P^2 \equiv 4 \pmod{5}. \]
Hence, we readily obtain that $P^2 \equiv 2 \pmod{5}$, which is impossible. This completes the proof. \hfill \square

**Corollary 8.** The equation $(7P^2 - 1)^2 - (P^2 - 4)y^2 = 4$ has no integer solutions.
Theorem 12. If \( V_n = 7x^2 + 1 \), then \( n = 1 \) and \( V_1 = 7x^2 + 1 \) where \( x \) is even.

Proof. Suppose that \( V_n = 7x^2 + 1 \) for some integer \( x \). If \( n \) is even, then \( n = 2m \) and therefore \( V_n = V_{2m} = V_m^2 + 2 \) by (2.7). This implies that \( V_m^2 \equiv 3 \mod 7 \), which is impossible. Thus, \( n \) is odd. Dividing the remainder of the proof into four cases, we have

Case 1: Assume that \( 7 \mid P \). Since \( n \) is odd, it follows from Lemma 1 that \( 7 \mid V_n \), implying that \( 7 \mid 1 \), a contradiction.

Case 2: Assume that \( P^2 \equiv 1 \mod 7 \). If \( n > 1 \), then \( n = 4q \pm 1 = 2 \cdot 2^r \cdot a \pm 1 \), \( 2 \nmid a \) and \( r \geq 1 \). Thus, \( 7x^2 = V_n - 1 \equiv -V_1 - 1 \equiv -(P + 1) \mod V_{2^r} \)

by (2.4). By using (2.20), (2.16), and (2.24), it is seen that

\[
1 = \left( \frac{7}{V_{2^r}} \right) \left( \frac{-1}{V_{2^r}} \right) \left( \frac{P + 1}{V_{2^r}} \right) = -1,
\]
a contradiction. So, \( n = 1 \). This implies that \( P = 7x^2 + 1 \) with \( x \) even.

Case 3: Assume that \( P^2 \equiv 2 \mod 7 \). Hence, \( P \equiv 3, 4 \mod 7 \). Moreover, it can be easily seen that \( 7 \mid V_2 \). Since \( n = 4q \pm 1 \), it follows from (2.4) that \( V_n = V_{4q \pm 1} \equiv \pm V_1 \mod V_2 \), implying that \( V_n \equiv 3, 4 \mod (7) \), which is impossible.

Case 4: Assume that \( P^2 \equiv 4 \mod 7 \). Hence, \( P \equiv 2, 5 \mod 7 \). In this case, it can be easily shown by induction that

\[
V_n \equiv \begin{cases} 
2 \mod (7) & \text{if } n \text{ is even,} \\
\quad P \mod (7) & \text{if } n \text{ is odd.}
\end{cases}
\]

This implies that \( V_n \equiv 2, 5 \mod (7) \), which is impossible. This completes the proof. \( \Box \)

Corollary 9. The equation \( (7x^2 + 1)^2 - (P^2 - 4)y^2 = 4 \) has integer solutions only when \( P = 7x^2 + 1 \) with even \( x \).

Theorem 13. If \( V_n = 7x^2 - 1 \), then \( n = 1 \) and \( V_1 = 7x^2 - 1 \) with \( x \) is even, or \( n = 2 \) and \( P = V_k(16, -1)/2 \) and \( x = 3U_k(16, -1) \) where \( k \) is even.

Proof. Suppose that \( V_n = 7x^2 - 1 \) for some \( x > 0 \).

Case 1: Assume that \( 7 \mid P \). Then by Lemma 1, \( V_n \equiv 0, \pm 2 \mod (7) \), and so \( V_n \neq 7x^2 - 1 \).

Case 2: Assume that \( P^2 \equiv 1 \mod (7) \). If \( n \) is odd, then we can write \( n = 4q \pm 1 = 2 \cdot 2^r \cdot a \pm 1 \), \( 2 \nmid a \) and \( r \geq 1 \). Hence, we get

\[
7x^2 - 1 = V_n \equiv -V_1 \equiv -P \mod V_{2^r},
\]

implying that \( 7x^2 \equiv -(P - 1) \mod V_{2^r} \).
By using (2.20), (2.16), and (2.24), we immediately have
\[ 1 = \left( \frac{7}{V_{2r}} \right) \left( \frac{-1}{V_{2r}} \right) \left( \frac{P - 1}{V_{2r}} \right) = -1, \]
which is impossible. If \( n \equiv 0 \pmod{4} \), then \( n = 4u \) for some \( u \). Hence, we get
\[ 7x^2 - 1 = V_n \equiv -V_0 \equiv -2 \pmod{V_{2r}}, \]
implying that
\[ 7x^2 \equiv -1 \pmod{V_{2r}}. \]
But this is impossible since \( \left( \frac{7}{V_{2r}} \right) = 1 \) by (2.20) and \( \left( \frac{-1}{V_{2r}} \right) = -1 \) by (2.16). If \( n \equiv 2 \pmod{4} \) with \( n \geq 6 \), then \( n = 2(4q \pm 1) = 2 \cdot 2^q a \pm 2, \) \( 2 \nmid a \) and \( r \geq 2 \).
Hence, we have
\[ 7x^2 - 1 = V_n \equiv -V_2 \equiv -(P^2 - 2) \pmod{V_{2r}}, \]
implying that
\[ 7x^2 \equiv -(P^2 - 3) \pmod{V_{2r}}. \]
By using (2.20), (2.16), and (2.21), we readily obtain
\[ 1 = \left( \frac{7}{V_{2r}} \right) \left( \frac{-1}{V_{2r}} \right) \left( \frac{P^2 - 3}{V_{2r}} \right) = -1, \]
a contradiction. So \( n = 2 \), and \( V_n = 7x^2 - 1 \) gives \( V_2 = P^2 - 2 = 7x^2 - 1 \), i.e., \( P^2 - 7x^2 = 1 \). Since all positive integer solutions of the equation \( u^2 - 7v^2 = 1 \) are given by \( (u, v) = (V_k(16, -1)/2, 3U_k(16, -1)) \) with \( k \geq 1 \), it follows that \( P = V_k(16, -1)/2 \) for positive even \( k \); since \( P \) is odd.

Case 3: Assume that \( P^2 \equiv 2 \pmod{7} \). And so \( 7 \mid V_2 \), and if we write \( n = 4q + r \) with \( r \in \{0, 1, 2, 3\} \), then
\[ V_n = V_{2q \pm r} \equiv \pm V_r \equiv \pm \{V_0, V_1, V_2, V_3\} \pmod{V_2}, \]
i.e.,
\[ V_n \equiv 0, 2, 3, 4, 5 \pmod{7}, \]
which is impossible.

Case 4: Assume that \( P^2 \equiv 4 \pmod{7} \). So, \( P \equiv 2, 5 \pmod{7} \). Using the fact that
\[ V_n \equiv \begin{cases} 
2 \pmod{7} & \text{if } n \text{ is even} \\
2 \pmod{7} & \text{if } n \text{ is odd}
\end{cases} \]
gives \( V_n \equiv 2, 5 \pmod{7} \), which is impossible. This completes the proof. \( \Box \)

**Corollary 10.** The equation \((7x^2 - 1)^2 - (P^2 - 4)y^2 = 4\) has integer solutions only when \( P = 7x^2 - 1 \), with \( x \) even or \( P = V_k(16, -1)/2 \), with \( k \) even.

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