

SOME RESULTS ON CONCIRCULAR VECTOR FIELDS AND THEIR APPLICATIONS TO RICCI SOLITONS

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ABSTRACT. A vector field on a Riemannian manifold (M, g) is called concircular if it satisfies $\nabla_X v = \mu X$ for any vector X tangent to M , where ∇ is the Levi-Civita connection and μ is a non-trivial function on M . A smooth vector field ξ on a Riemannian manifold (M, g) is said to define a *Ricci soliton* if it satisfies the following Ricci soliton equation:

$$\frac{1}{2}\mathcal{L}_\xi g + Ric = \lambda g,$$

where $\mathcal{L}_\xi g$ is the Lie-derivative of the metric tensor g with respect to ξ , Ric is the Ricci tensor of (M, g) and λ is a constant. A Ricci soliton (M, g, ξ, λ) on a Riemannian manifold (M, g) is said to have concircular potential field if its potential field ξ is a concircular vector field.

In the first part of this paper we determine Riemannian manifolds which admit a concircular vector field. In the second part we classify Ricci solitons with concircular potential field. In the last part we prove some important properties of Ricci solitons on submanifolds of a Riemannian manifold equipped with a concircular vector field.

1. Introduction

A. Fialkow introduced in [13] the notion of *concircular vector fields* on a Riemannian manifold M as vector fields which satisfy

$$(1) \quad \nabla_X v = \mu X, \quad X \in TM,$$

where ∇ denotes the Levi-Civita connection, TM the tangent bundle of M and μ a non-trivial function on M . Concircular vector fields also known as geodesic fields in literature since integral curves of such vector fields are geodesics (see e.g. [18]). A concircular vector field v is called a *concurrent* vector field if the function μ in Eq. (1) is one (see, for instance, [2, 8, 9]). The notion of concircular vector fields can be extended naturally to concircular vector fields in pseudo-Riemannian manifolds.

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Concircular vector fields appeared in the study of concircular mappings, i.e., conformal mappings preserving geodesic circles [20]. Such vector fields also play an important role in the theory of projective and conformal transformations. Concircular vector fields have interesting applications in physics as well, in particular in general relativity, e.g., trajectories of time-like concircular fields in the de Sitter model determine the world lines of receding or colliding galaxies satisfying the Weyl hypothesis (cf. [19]). Also it was proved recently by the author in [4] that a Lorentzian manifold is a generalized Robertson-Walker spacetime if and only if it admits a timelike concircular vector field.

A smooth vector field ξ on a Riemannian manifold (M, g) is said to define a *Ricci soliton* if it satisfies the following Ricci soliton equation:

$$(2) \quad \frac{1}{2}\mathcal{L}_\xi g + Ric = \lambda g,$$

where $\mathcal{L}_\xi g$ is the Lie-derivative of the metric tensor g with respect to ξ , Ric is the Ricci tensor of (M, g) and λ is a constant. We shall denote a Ricci soliton by (M, g, ξ, λ) . We call the vector field ξ the *potential field*. A Ricci soliton (M, g, ξ, λ) is called *shrinking*, *steady* or *expanding* according to $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. A *trivial Ricci soliton* is one for which ξ is zero or Killing, in which case the metric is Einsteinian (see, for instance, [6, 7, 10, 14, 16, 17]).

Compact Ricci solitons are the fixed points of the Ricci flow:

$$\frac{\partial}{\partial t}g(t) = -2Ric(g(t))$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. Further, Ricci solitons model the formation of singularities in the Ricci flow and they correspond to self-similar solutions (cf. e.g. [16]).

A Ricci soliton (M, g, ξ, λ) is called *gradient* if its potential field ξ is the gradient of some function f on M . We denote such a gradient Ricci soliton by (M, g, f, λ) and call the smooth function f the *potential function*. For a gradient Ricci soliton (M, g, f, λ) , the soliton equation can be expressed as

$$Ric_f = \lambda g,$$

where $Ric_f := Ric + Hess(f)$ is known as the *Bakry-Emery curvature*. Hence a gradient Ricci soliton has constant Bakry-Emery curvature; a similar role as an Einstein manifold.

A gradient Ricci soliton (M, g, f, λ) is called *trivial* if its potential function f is a constant. It follows from Eq.(2) that trivial gradient Ricci solitons are trivial Ricci solitons automatically. G. Perelman proved in [17] that the Ricci solitons on compact simply connected Riemannian manifolds are gradient Ricci solitons as solutions of Ricci flow.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. It has become more important

after G. Perelman applied Ricci solitons to solve the long standing Poincaré conjecture posed in 1904.

In the first part of this paper we determine Riemannian manifolds which admit a concircular vector field. In the second part we classify Ricci solitons with concircular potential fields. In the last part we prove some important properties of Ricci solitons on submanifolds of a Riemannian manifold equipped with a concircular vector field.

2. Preliminaries

2.1. Basic formulas and definitions for submanifolds

For general references on Riemannian submanifolds, we refer to [2, 3, 5].

Let (N^m, \tilde{g}) denote an m -dimensional Riemannian manifold and let $\phi : M^n \rightarrow N^m$ be an isometric immersion from an n -dimensional Riemannian manifold (M^n, g) into (N^m, \tilde{g}) . Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on (M^n, g) and (N^m, \tilde{g}) , respectively.

For vector fields X, Y tangent to M^n and η normal to M^n , the formula of Gauss and the formula of Weingarten are given respectively by

$$(3) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(4) \quad \tilde{\nabla}_X \eta = -A_\eta X + D_X \eta,$$

where $\nabla_X Y$ and $h(X, Y)$ are the tangential and the normal components of $\tilde{\nabla}_X Y$. Similarly, $-A_\eta X$ and $D_X \eta$ are the tangential and normal components of $\tilde{\nabla}_X \eta$. These two formulas define the second fundamental form h , the shape operator A , and the normal connection D of M^n in the ambient space N^m .

For a normal vector $\eta \in T_p^\perp M$ at $p \in M$, A_η is a self-adjoint endomorphism of the tangent space $T_p M$. The shape operator and the second fundamental form are related by

$$(5) \quad \tilde{g}(h(X, Y), \eta) = g(A_\eta X, Y).$$

The mean curvature vector H of M^n in N^m is defined by

$$(6) \quad H = \left(\frac{1}{n}\right) \text{trace } h.$$

The equations of Gauss and Codazzi are given respectively by

$$(7) \quad g(R(X, Y)Z, W) = \tilde{g}(\tilde{R}(X, Y)Z, W) + \tilde{g}(h(X, W), h(Y, Z)) - \tilde{g}(h(X, Z), h(Y, W)),$$

$$(8) \quad (\tilde{R}(X, Y)Z)^\perp = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z),$$

for vectors X, Y, Z, W tangent to M and ζ, η normal to M , where $(\tilde{R}(X, Y)Z)^\perp$ is the normal component of $\tilde{R}(X, Y)Z$ and $\tilde{\nabla} h$ is defined by

$$(9) \quad (\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

For a function f on a Riemannian manifold M , we denote by ∇f and H^f the gradient of f and the Hessian of f , respectively. Thus we have

$$(10) \quad g(\nabla f, X) = Xf,$$

$$(11) \quad H^f(X, Y) = XYf - (\nabla_X Y)f.$$

Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame on a Riemannian n -manifold M^n . Denote by $\{\omega^1, \dots, \omega^n\}$ the dual frame of 1-forms of $\{e_1, \dots, e_n\}$. The connection forms ω_i^j ($i, j = 1, \dots, n$) are defined by

$$(12) \quad \nabla_X e_i = \sum_{j=1}^n \omega_i^j(X) e_j, \quad i = 1, \dots, n.$$

From Cartan's structure equations, we have

$$(13) \quad d\omega^i = - \sum_{j=1}^n \omega_j^i \wedge \omega^j, \quad i = 1, \dots, n.$$

A *foliation* \mathcal{D} on a manifold M is an integrable distribution, i.e., \mathcal{D} is a vector subbundle of the tangent bundle TM such that, for any vector fields X, Y in \mathcal{D} , the Lie bracket $[X, Y]$ takes values in \mathcal{D} as well. A foliation \mathcal{D} on a Riemannian manifold M is called *totally umbilical*, if every leaf of \mathcal{D} is a totally umbilical submanifold of M . If, in addition, the mean curvature vector of every leaf is parallel in the normal bundle, then \mathcal{D} is called a *spherical foliation*. In this case, leaves of \mathcal{D} are *extrinsic spheres* of M . If leaves of a foliation \mathcal{D} are totally geodesic submanifolds, \mathcal{D} is called a *totally geodesic foliation* (cf. e.g., [3, 5]).

3. Riemannian manifolds with concircular vector fields

First, we give the following examples of Riemannian manifolds endowed with concircular vector fields.

Example 3.1. Let I be an open interval of the real line \mathbf{R} and let $\varphi(s)$, $s \in I$, be a function on I which is nowhere zero. Consider a warped product manifold of the form:

$$(14) \quad I \times_{\varphi(s)} F,$$

where F is a Riemannian manifold. The metric tensor g of $I \times_{\varphi(s)} F$ is given by $g = ds^2 + \varphi^2(s)g_F$, where g_F is the metric tensor of the second factor F . Consider the vector field given by

$$(15) \quad v = \varphi(s) \frac{\partial}{\partial s}.$$

It follows from Proposition 4.1 of [3, page 79] that the vector field v satisfies Eq. (1) with $\mu = \varphi'(s)$. Thus v is a concircular vector field.

The next result implies that Example 3.1 provides all Riemannian manifolds which admit a nowhere zero concircular vector field.

Theorem 3.1. *If M is a Riemannian n -manifold which admits a nowhere zero concircular vector field, then M is locally a warped product $I \times_{\varphi(s)} F$, where $\varphi(s)$ is a nowhere vanishing function and F is a Riemannian $(n - 1)$ -manifold.*

Proof. Assume that v is a concircular vector field on a Riemannian manifold such that v is nowhere zero. Let us put

$$(16) \quad v = \varphi e_1,$$

where e_1 is a unit vector field in the direction of v . Extend e_1 to an orthonormal frame e_1, \dots, e_n on M . It follows from (1) and a direct computation that

$$(17) \quad \begin{aligned} R(e_i, v)v &= \nabla_{e_i} \nabla_v v - \nabla_v \nabla_{e_i} v - \nabla_{[e_i, v]} v \\ &= (e_i \mu)v - (v\mu)e_i, \quad i = 2, \dots, n, \end{aligned}$$

where μ is defined by (1). From (17) and $\langle R(e_i, v)v, v \rangle = 0$, we get

$$(18) \quad e_2 \mu = \dots = e_n \mu = 0.$$

Thus the gradient $\nabla \mu$ is parallel to v .

From (1) with $X = e_1$ and (16) we find

$$\mu e_1 = \nabla_{e_1}(\varphi e_1) = (e_1 \varphi)e_1 + \varphi \nabla_{e_1} e_1,$$

which gives

$$(19) \quad e_1 \varphi = \mu, \quad \nabla_{e_1} e_1 = 0.$$

From the second equation in (19) we know that the integral curves of e_1 are geodesics in M . Therefore the distribution $\mathcal{D}_1 = \text{Span}\{e_1\}$ is a totally geodesic foliation. Let us put $\mathcal{D}_2 = \text{Span}\{e_2, \dots, e_n\}$.

From (16) and (1) with $X = e_i$ with $i = 2, \dots, n$, we find

$$\mu e_i = \nabla_{e_i}(\varphi e_1) = (e_i \varphi)e_1 + \varphi \nabla_{e_i} e_1,$$

which implies

$$(20) \quad e_2 \varphi = \dots = e_n \varphi = 0,$$

$$(21) \quad \varphi \nabla_{e_i} e_1 = \mu e_i.$$

It follows from (12) and (21) that

$$(22) \quad \omega_i^1(e_j) = \frac{\mu}{\varphi} \delta_{ij}, \quad 2 \leq i, j \leq n.$$

From (22) we conclude that \mathcal{D}_2 is an integrable distribution whose leaves are totally umbilical in M . Moreover, the mean curvature of leaves of \mathcal{D}_2 are given by μ/φ . Since leaves of \mathcal{D}_2 are hypersurfaces, it follows from (18) and (20) that the mean curvature vector fields of leaves of \mathcal{D}_2 are parallel in the normal bundle in M . Thus \mathcal{D}_2 is a spherical foliation. Consequently, by a result of [15] (or Theorem 4.4 of [3, page 90]) we conclude that M is locally a warped product

$I \times_{f(s)} F$, where $f(s)$ is a function on I , $\partial/\partial s = e_1$, and F is a Riemannian $(n - 1)$ -manifold. Therefore the sectional curvature of M satisfies

$$(23) \quad K(e_1, X) = -\frac{f''(s)}{f(s)}$$

for any unit vector X orthogonal to e_1 .

On the other hand, it follows from (16) and (17) that we also have

$$(24) \quad \varphi K(e_1, X) = -v\mu = -\mu'(s)$$

for any unit vector X orthogonal to e_1 . Thus, after combining (24) with (19) and (23), we obtain

$$\frac{f''(s)}{f(s)} = \frac{\mu'(s)}{\varphi} = \frac{\varphi''(s)}{\varphi(s)}.$$

Thus if we choose $f(s) = \varphi(s)$, then M is locally a warped product $I \times_{\varphi(s)} F$. A direct computation yields $\nabla_X(\varphi(s)\frac{\partial}{\partial s}) = \varphi'(s)X$ for X tangent to $I \times_{f(s)} F$. \square

4. Concircular vector fields on real space forms

In this section we determine all concircular vector fields on Riemannian manifolds of constant curvature.

First we give the following necessary and sufficient condition for a gradient vector field on a Riemannian manifold to be concircular.

Lemma 4.1. *Let f be a function on a Riemannian manifold M . Then the gradient ∇f of f is a concircular vector field if and only if the Hessian H^f of f satisfies*

$$(25) \quad H^f(X, Y) = \mu g(X, Y)$$

for X, Y tangent to M , where μ is the function on M . Moreover, in such case the function μ satisfies Eq. (1) with $v = \nabla f$.

Proof. Let f be a function of a Riemannian manifold M . Assume that the Hessian of f satisfies (25). Then we have

$$(26) \quad \begin{aligned} g(X, \mu Y) &= XYf - \nabla_X Yf \\ &= X\langle Y, \nabla f \rangle - \langle \nabla_X Y, \nabla f \rangle \\ &= \langle Y, \nabla_X(\nabla f) \rangle \end{aligned}$$

for vector fields X, Y tangent to M . Thus we obtain $\nabla_X(\nabla f) = \mu X$, which implies that $v = \nabla f$ is a concircular vector field satisfying Eq. (1).

The converse can be verified in a similar way. \square

The following result classifies all concircular vector fields on \mathbb{E}^n .

Proposition 4.2. *Let v be a nonzero vector field on the Euclidean n -space \mathbb{E}^n . Then v is a concircular vector field if and only if $v = b\mathbf{x}$, where b is a nonzero constant and \mathbf{x} is a concurrent vector field.*

Proof. If v is a concircular vector field on a Riemannian manifold, then it follows from (1) and a direct computation that

$$(27) \quad \begin{aligned} R(X, v)v &= \nabla_X \nabla_v v - \nabla_v \nabla_X v - \nabla_{[X, v]}v \\ &= (X\mu)v - (v\mu)X, \end{aligned}$$

for any vector field X perpendicular to v , where μ is defined by (1). It follows from (27) that the gradient $\nabla\mu$ is parallel to v .

If M is the Euclidean n -space \mathbb{E}^n , we have $R = 0$. Hence we obtain from (27) that $Xv = v\mu = 0$, which implies that μ is a nonzero constant, say b . Consequently, $\mathbf{x} = v/b$ is a concurrent vector field.

The converse is trivial. □

The next result determines all concircular vector fields on the unit n -sphere.

Proposition 4.3. *Let $\{u_1, \dots, u_n\}$ be an isothermal coordinate system on $S^n(1)$ so that the metric tensor of $S^n(1)$ is given by*

$$(28) \quad g = \frac{4}{(1 + \sum_{j=1}^n u_j^2)^2} \sum_{i=1}^n du_i^2.$$

Then a vector field v on $S^n(1)$ is concircular if and only if, up to translations of u_1, \dots, u_n , v is a gradient vector field given by $v = -\nabla\mu$, where

$$(29) \quad \mu = \frac{1 - \sum_{i=1}^n u_i^2}{2 + 2 \sum_{j=1}^n u_j^2}.$$

Moreover, μ is exactly the function satisfying (1).

Proof. Let $\{u_1, \dots, u_n\}$ be the isothermal coordinates so that the metric tensor of $S^n(1)$ is given by Eq. (28). If v is a concircular vector field of $S^n(1)$, then (27) holds.

Since $S^n(1)$ is of constant curvature one, we also have

$$(30) \quad R(X, v)v = g(v, v)X - g(X, v)v$$

for X perpendicular to v . Thus we find from (27) and (30) that

$$(31) \quad X\mu = -g(X, v), \quad v\mu = -g(v, v).$$

Hence $v = \nabla f$ is a gradient field with $f = -\mu$.

On the other hand, it follows from a direct computation that the Levi-Civita connection of (28) satisfies

$$(32) \quad \begin{aligned} \nabla_{\partial_{u_i}} \partial_{u_i} &= \frac{-2}{1 + \sum_{i=1}^n u_i^2} \left\{ u_i \partial_{u_i} - \sum_{j \neq i} u_j \partial_{u_j} \right\}, \\ \nabla_{\partial_{u_i}} \partial_{u_j} &= \frac{-2}{1 + \sum_{i=1}^n u_i^2} \{ u_i \partial_{u_j} + u_j \partial_{u_i} \}, \quad 1 \leq i \neq j \leq n, \end{aligned}$$

where $\partial_{u_i} = \partial/\partial u_i$, $i = 1, \dots, n$.

From $f = -\mu$, Eqs. (11) and (32) we find

$$(33) \quad \begin{aligned} H^f(\partial_{u_i}, \partial_{u_i}) &= -\frac{\partial^2 \mu}{\partial^2 u_i} - \frac{2}{1 + \sum_{i=1}^n u_i^2} \left\{ u_i \frac{\partial \mu}{\partial u_i} - \sum_{j \neq i} u_j \frac{\partial \mu}{\partial u_j} \right\}, \\ H^f(\partial_{u_i}, \partial_{u_j}) &= -\frac{\partial^2 \mu}{\partial u_i \partial u_j} - \frac{2}{1 + \sum_{i=1}^n u_i^2} \left\{ u_j \frac{\partial \mu}{\partial u_i} + u_i \frac{\partial \mu}{\partial u_j} \right\} \end{aligned}$$

for $1 \leq i \neq j \leq n$. After combining Eq. (33) with Eq. (25) of Lemma 4.1, we get

$$(34) \quad \frac{\partial^2 \mu}{\partial u_i \partial u_j} = \frac{-2}{1 + \sum_{i=1}^n u_i^2} \left\{ u_j \frac{\partial \mu}{\partial u_i} + u_i \frac{\partial \mu}{\partial u_j} \right\},$$

$$(35) \quad \frac{\partial^2 \mu}{\partial^2 u_i} = \frac{-4\mu}{(1 + \sum_{i=1}^n u_i^2)^2} - \frac{2}{1 + \sum_{i=1}^n u_i^2} \left\{ u_i \frac{\partial \mu}{\partial u_i} - \sum_{j \neq i} u_j \frac{\partial \mu}{\partial u_j} \right\}$$

for $1 \leq i \neq j \leq n$.

After solving system (34)-(35) via long computation and by applying suitable translations on u_i , we obtain

$$(36) \quad \mu = \frac{1 - \sum_{i=1}^n u_i^2}{2 + 2 \sum_{j=1}^n u_j^2}.$$

Consequently, by Lemma 4.1, the concircular vector field v is given by

$$(37) \quad v = \frac{1}{2} \sum_{j=1}^n u_j \partial u_j.$$

Conversely, by applying (28), (32) and a straight-forward computation, we obtain $\nabla_V v = \mu V$ for V tangent to $S^n(1)$. □

Similarly, we have the following result for the hyperbolic n -space $H^n(-1)$ with constant curvature -1 .

Proposition 4.4. *Let $\{v_1, \dots, v_n\}$ be an isothermal coordinates on $H^n(-1)$ so that the metric tensor of $H^n(-1)$ is*

$$(38) \quad g = \frac{4}{(1 - \sum_{j=1}^n v_j^2)^2} \sum_{i=1}^n dv_i^2.$$

Then a vector field v on $H^n(-1)$ is concircular if and only if, up to translations of v_1, \dots, v_n , v is a gradient vector field given by $v = -\nabla \mu$, where

$$(39) \quad \mu = \frac{1 + \sum_{i=1}^n v_i^2}{2 - 2 \sum_{j=1}^n v_j^2}.$$

Moreover, μ is exactly the function satisfying (1).

Proof. This can be done in the same way as Proposition 4.3. □

5. Ricci solitons with concircular potential field

Next, we apply our previous results to classify Ricci solitons with concircular potential field.

Theorem 5.1. *A Ricci soliton (M, g, v, λ) on a Riemannian n -manifold (M, g) with $n \geq 3$ has concircular potential field v if and only if the following three conditions hold:*

- (a) *The function μ in (1) is a nonzero constant, say b ;*
- (b) *$\lambda = b$;*
- (c) *M is an open portion of a warped product manifold $I \times_{bs+c} F$, where I is an open interval with arc-length s , c is a constant, and F is an Einstein $(n - 1)$ -manifold whose Ricci tensor satisfies $Ric_F = (n - 2)b^2g_F$, g_F being the metric tensor of F .*

Proof. Assume that (M, g, v, λ) is a Ricci soliton on a Riemannian n -manifold with a concircular potential field v . Then Eq. (1) holds, which implies

$$(40) \quad (\mathcal{L}_v g)(X, Y) = g(\nabla_X v, Y) + g(\nabla_Y v, X) = 2\mu g(X, Y)$$

for any X, Y tangent to M . Combining (2) and (40) gives

$$(41) \quad Ric(X, Y) = (\lambda - \mu)g(X, Y),$$

which shows that M is an Einstein manifold. Since $n \geq 3$ and M is Einsteinian, M has constant scalar curvature. So $\lambda - \mu$ is constant. Thus the function μ is also a nonzero constant, say b . This gives statement (a). In particular, we have

$$(42) \quad \nabla_X v = bX, \quad X \in TM.$$

Since $\mu = b$ is constant, (27) implies $R(X, v)v = 0$ for X orthogonal to v . Hence $Ric(v, v) = 0$. Therefore M is Ricci-flat. Consequently, we obtain from (41) that $\lambda = b$. This gives statement (b).

It follows from Theorem 3.1 that M is locally a warped product $I \times_{\varphi(s)} F$ for some function $\varphi(s)$ and Riemannian $(n - 1)$ -manifold F . Moreover, from statement (a) and the proof of Theorem 3.1 we also have $\varphi'(s) = \mu = b$. Thus $\varphi = bs + c$ for some constant c . Hence M is an open part of a warped product manifold $I \times_{bs+c} F$.

Since M is Ricci-flat, (41) yields $\lambda = \mu = b$. Now, it follows from formula (9.109) of [1, page 267] that the second factor F of $I \times_{bs+c} F$ is an Einstein manifold satisfying $Ric_F = (n - 2)b^2g_F$.

The converse can be verified by direct computation. □

An easy consequence of Theorem 5.1 is the following.

Corollary 5.2. *The only Riemannian manifold of constant sectional curvature admitting a Ricci soliton with concircular potential field is a Euclidean space.*

Proof. Follows from Proposition 4.3, Proposition 4.4 and Theorem 5.1(a). □

Associated with the Ricci tensor on a Riemannian n -manifold, let Q denote the $(1, 1)$ -tensor defined by $\langle Q(X), Y \rangle = Ric(X, Y)$.

The Weyl conformal curvature tensor W is of type $(1, 3)$ is defined by

$$(43) \quad \begin{aligned} W(X, Y)Z &= R(X, Y)Z + \frac{1}{n}\{Ric(X, Z)Y - Ric(Y, Z)X + \langle X, Z \rangle QY \\ &\quad - \langle Y, Z \rangle QX\} - \frac{2\tau}{n(n+1)}\{\langle X, Z \rangle Y - \langle Y, Z \rangle X\}, \end{aligned}$$

where τ is the scalar curvature.

It is well-known that the Weyl conformal curvature tensor W vanishes identically when $\dim M = 3$. A Riemannian n -manifold with $n \geq 4$ is called *conformally flat* if its Weyl conformal curvature tensor vanishes identically.

Another easy consequence of Theorem 5.1 is the following.

Corollary 5.3. *The only conformally flat manifold of dimension ≥ 4 which admits a Ricci soliton with concircular potential field is the Euclidean space.*

Proof. It follows immediately from (43) that the only conformally flat, Einstein manifolds of dimension ≥ 4 are Riemannian manifolds of constant curvature. Hence we obtain this corollary from Corollary 5.2. \square

The concircular curvature tensor Z of type $(1, 3)$ on a Riemannian manifold is defined by

$$(44) \quad Z(X, Y)Z = R(X, Y)Z - \frac{2\tau}{n(n+1)}\{g(X, Z)Y - g(Y, Z)X\}.$$

A Riemannian manifold is called *concircularly flat* if its concircular curvature tensor Z vanishes identically.

Similarly, Theorem 5.1 also implies the following.

Corollary 5.4. *The only concircularly flat manifold which admits a Ricci soliton with concircular potential field is the Euclidean space.*

6. Ricci solitons on Riemannian submanifolds arisen from concircular vector fields

Finally, in this last section we prove some properties of Ricci solitons on submanifolds of Riemannian manifolds equipped with a concircular vector field. These extend some results of [8].

We make the following.

Assumption. (N^m, \tilde{g}) is a Riemannian m -manifold endowed with a concircular vector field v .

For a Riemannian submanifold (M^n, g) of (N^m, \tilde{g}) , we denote by v^T and v^\perp the tangential and normal components on M^n of the concircular vector field v , respectively. As before, let h, A and D denote the second fundamental form, the shape operator and the normal connection of the submanifold, respectively.

Theorem 6.1. *A submanifold M^n in N^m admits a Ricci soliton (M^n, g, v^T, λ) if and only if the Ricci tensor of M^n satisfies*

$$(45) \quad Ric(X, Y) = (\lambda - \mu)g(X, Y) - \tilde{g}(h(X, Y), v^\perp)$$

for any X, Y tangent to M^n .

Proof. Let $\phi : M^n \rightarrow N^m$ denote the isometric immersion. We have

$$(46) \quad v = v^T + v^\perp.$$

Since v is a concircular vector field on N^m , it follows from (1), (46) and formulas of Gauss and Weingarten that

$$(47) \quad \mu X = \nabla_X v^T + h(X, v^T) - A_{v^\perp} X + D_X v^\perp$$

for any X tangent to M^n . By comparing the tangential and normal components from (47) we obtain

$$(48) \quad \nabla_X v^T = A_{v^\perp} X + \mu X.$$

From the definition of Lie derivative and (48) we find

$$(49) \quad (\mathcal{L}_v \tau g)(X, Y) = 2\mu g(X, Y) + 2\tilde{g}(h(X, Y), v^\perp)$$

for X, Y tangent to M^n . Consequently, by applying (2) and (49), we conclude that (M^n, g, v^T, λ) is a Ricci soliton if and only if

$$(50) \quad Ric(X, Y) + \mu g(X, Y) + \tilde{g}(h(X, Y), v^\perp) = \lambda g(X, Y),$$

which is nothing but (45). □

A Riemannian submanifold M^n is called η -umbilical with respect to a normal vector field η if its shape operator satisfies $A_\eta = \varphi I$, where φ is a function on M^n and I is the identity map.

The following two results are immediate consequences of Theorem 6.1.

Corollary 6.2. *A Ricci soliton (M^n, g, v^T, λ) ($n \geq 3$) on a submanifold M^n in N^m is trivial if and only if M^n is v^\perp -umbilical.*

Corollary 6.3. *Every Ricci soliton (M^n, g, v^T, λ) on a totally umbilical submanifold M^n of N^m is a trivial Ricci soliton.*

Following [3], the scalar curvature τ of (M^n, g) is defined to be

$$(51) \quad \tau = \sum_{1 \leq i < j \leq n} K(e_i, e_j),$$

where $\{e_1, \dots, e_n\}$ is an orthonormal frame of M^n .

Another easy application of Theorem 6.1 is the following.

Corollary 6.4. *If (M^n, g, v^T, λ) is a Ricci soliton on a minimal submanifold M^n in N^m , then the scalar curvature τ of M^n is given by $n(\lambda - \mu)/2$.*

Proof. Assume that (M^n, g, v^T, λ) is a Ricci soliton on a submanifold M^n in N^m . Then it follows from Theorem 6.1 that the Ricci tensor satisfies

$$(52) \quad Ric(X, Y) = (\lambda - \mu)g(X, Y) - \tilde{g}(h(X, Y), v^\perp)$$

for X, Y tangent to M^n . If M^n is minimal, then $\tilde{g}(H, v^\perp) = 0$. Hence (52) gives $\sum_{i=1}^n Ric(e_i, e_i) = n(\lambda - \mu)$. Thus M^n has constant scalar curvature $n(\lambda - \mu)/2$. \square

Corollary 6.5. *Let $R^{n+1}(c)$ be a Riemannian manifold of constant curvature c equipped with a concircular vector field v . If (M^n, g, v^T, λ) is a Ricci soliton on a hypersurface of M^n of $R^{n+1}(c)$, then M^n has at most two distinct principal curvatures given by*

$$(53) \quad \kappa_1, \kappa_2 = \frac{n\alpha + \rho \pm \sqrt{(n\alpha + \rho)^2 - 4(\lambda - \mu - (n - 1)c)}}{2},$$

where α is the mean curvature, i.e., $H = \alpha N$ and $\rho = \langle N, v \rangle$ with N being a unit normal vector field.

Proof. Under the hypothesis, assume that (M^n, g, v^T, λ) is a Ricci soliton on a hypersurface of M^n of $R^{n+1}(c)$, where v^T denotes the tangential component of the concircular vector field v . Let $\{e_1, \dots, e_n\}$ be an orthonormal frame on M^n such that e_1, \dots, e_n are eigenvectors of the shape operator A_N . Then we have

$$(54) \quad A_N e_i = \kappa_i e_i, \quad i = 1, \dots, n.$$

From equation (7) of Gauss we obtain

$$(55) \quad Ric(X, Y) = (n - 1)cg(X, Y) + n\tilde{g}(h(X, Y), H) - \sum_{i=1}^n \tilde{g}(h(X, e_i), h(Y, e_i)),$$

where \tilde{g} denotes the metric of $R^{n+1}(c)$.

It follows from (54), (55) and Theorem 6.1 that (M^n, g, v^T, λ) is a Ricci soliton if and only if we have

$$(56) \quad \{(n - 1)c + (n\alpha - \kappa_j)\kappa_i\}\delta_{ij} = (\lambda - \mu)\delta_{ij} - \rho\kappa_i\delta_{ij},$$

where δ_{ij} is the Kronecker delta. Equation (56) is equivalent to

$$\kappa_i^2 - (n\alpha + \rho)\kappa_i + \lambda - \mu - (n - 1)c = 0, \quad i = 1, \dots, n,$$

which implies the proposition \square

Remark 6.6. Results in this section extend some results obtained in [8].

Remark 6.7. Ricci solitons on Riemannian submanifolds have also been studied in [10, 11, 12] by J. T. Cho and M. Kimura from a different viewpoint. They proved several interesting results on Ricci solitons on submanifolds; however their potential fields of the Ricci solitons are quite different from ours.

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References

- [1] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.
- [2] B.-Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.
- [3] ———, *Pseudo-Riemannian Geometry, δ -invariants and Applications*, World Scientific, Hackensack, NJ, 2011.
- [4] ———, *A simple characterization of generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation **46** (2014), no. 12, Art. 1833, 5 pp.
- [5] ———, *Total Mean Curvature and Submanifolds of Finite Type, 2nd Edition*, World Scientific, Hackensack, NJ, 2015.
- [6] B.-Y. Chen and S. Deshmukh, *Geometry of compact shrinking Ricci solitons*, Balkan J. Geom. Appl. **19** (2014), no. 1, 13–21.
- [7] ———, *Classification of Ricci solitons on Euclidean hypersurfaces*, Internat. J. Math. **25** (2014), no. 11, 1450104, 22 pp.
- [8] ———, *Ricci solitons and concurrent vector fields*, Balkan J. Geom. Appl. **20** (2015), no. 1, 14–25.
- [9] B.-Y. Chen and K. Yano, *On submanifolds of submanifolds of a Riemannian manifold*, J. Math. Soc. Japan **23** (1971), no. 3, 548–554.
- [10] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tohoku Math. J. (2) **61** (2009), no. 2, 205–212.
- [11] ———, *Ricci solitons of compact real hypersurfaces in Kähler manifolds*, Math. Nachr. **284** (2011), no. 11-12, 1385–1393.
- [12] ———, *Ricci solitons on locally conformally flat hypersurfaces in space forms*, J. Geom. Phys. **62** (2012), no. 8, 1882–1891.
- [13] A. Fialkow, *Conformals geodesics*, Trans. Amer. Math. Soc. **45** (1939), no. 3, 443–473.
- [14] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.
- [15] S. Hiepko, *Eine innere Kennzeichnung der verzerrten Produkte*, Math. Ann. **241** (1979), no. 3, 209–215.
- [16] J. Morgan and G. Tian, *Ricci Flow and the Poincaré Conjecture*, Clay Mathematics Monographs, **5**, Cambridge, MA, 2014.
- [17] G. Perelman, *The Entropy Formula For The Ricci Flow And Its Geometric Applications*, arXiv math/0211159.
- [18] Ya. L. Šapiro, *Geodesic fields of directions and projective path systems*, Mat. Sb. N.S. **36** (78) (1955), 125–148.
- [19] H. Takeno, *Concircular scalar field in spherically symmetric space-times I*, Tensor **20** (1967), no. 2, 167–176.
- [20] K. Yano, *Concircular geometry. I. Concircular transformations*, Proc. Imp. Acad. Tokyo **16** (1940), 195–200.

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