BARRELLEDNESS OF SOME SPACES OF VECTOR MEASURES AND BOUNDED LINEAR OPERATORS

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Abstract. In this paper we investigate the barrelledness of some spaces of $X$-valued measures, $X$ being a barrelled normed space, and provide examples of non barrelled spaces of bounded linear operators from a Banach space $X$ into a barrelled normed space $Y$, equipped with the uniform convergence topology.

1. Preliminaries

The barrelledness of certain spaces of vector-valued functions has been widely studied, see [7, Chapters 8-10] and references therein. If $K$ is a locally compact Hausdorff space, $(\Omega, \Sigma)$ a measurable space, $\mu \in ca^+(\Sigma)$ and $X$ a normed space over the field $\mathbb{K}$ of the real or complex numbers, the following are among the most beautiful results on this topic.

(1) The space $B(\Sigma, X)$ over $\mathbb{K}$ of all those functions $f : \Omega \to X$ that are the uniform limit of a sequence of $\Sigma$-simple $X$-valued functions, equipped with the supremum norm, is barrelled if and only if $X$ is barrelled, [12].

(2) The space $C(K, X)$ over $\mathbb{K}$ of all continuous functions $f : K \to X$ endowed with the compact-open topology is barrelled if and only if $C(K)$ and $X$ are barrelled, [13].

(3) If $\mu$ is atomless the space $L_p(\mu, X)$ over $\mathbb{K}$, with $1 \leq p \leq \infty$, of all (classes of) strongly measurable functions $f : \Omega \to X$ that are Bochner integrable if $1 \leq p < \infty$, or essentially bounded if $p = \infty$, equipped with the integral norm $\|f\|_p$ or with the essential supremum norm $\|f\|_\infty$, respectively, is barrelled ([2] and [3]), regardless $X$ is barrelled or not.
The space $\ell_\infty (\Sigma, X)$ over $\mathbb{K}$ of all bounded $\Sigma$-measurable functions $f : \Omega \to X$, equipped with the supremum norm, is barrelled if and only if $X$ is barrelled, [5].

If $X$ is a Banach space, the space $\mathcal{P}_1 (\mu, X)$ over $\mathbb{K}$ of all \textup{[classes of scalarly equivalent]} weakly $\mu$-measurable and Pettis integrable functions $f : \Omega \to X$, equipped with the so-called Pettis norm or semivariation norm, is barrelled, as well as the subspace $P_1 (\mu, X)$ of all \textup{[classes of] strongly measurable functions}, [3].

The space $\ell_\infty (\Omega, X)$ over $\mathbb{K}$ of all bounded functions $f : \Omega \to X$, equipped with the supremum norm, is barrelled whenever $X$ is barrelled and either $|\Omega|$ or $|X|$ is a nonmeasurable cardinal, [4].

If $K$ is (locally compact and) normal, the space $C_0 (K, X)$ over $\mathbb{K}$ of all continuous functions $f : K \to X$ vanishing at infinity, i.e., such that for each $\epsilon > 0$ there exists a compact set $K_{f, \epsilon} \subseteq K$ with the property that $\|f(\omega)\| < \epsilon$ for each $\omega \in K \setminus K_{f, \epsilon}$, provided with the supremum norm, is barrelled if and only if $X$ is barrelled, [6].

Let us point out that $B (\Sigma, X)$ coincides with the closure in $\ell_\infty (\Omega, X)$ of the subspace $\ell^\infty_0 (\Sigma, X) \mid \ell_\infty (\Omega, X)$ consisting of all $X$-valued $\Sigma$-simple functions. If $X$ is separable then $\ell_\infty (\Omega, X) = \ell_\infty (2^\Omega, X)$. In the sequel we shall write $\ell^\infty_0 (\Sigma)$ instead of $\ell^\infty_0 (\Sigma, \mathbb{K})$ and $\ell^\infty_0 (\Xi)$ instead of $\ell^\infty_0 (2^\Xi)$. Clearly, $\ell^\infty_0 (\Sigma)$ coincides with the dense subspace of $\ell_\infty$ of those sequences $(\xi_n)$ of finite range. Regardless of $\Sigma$, the space $\ell^\infty_0 (\Sigma)$ is always barrelled (see [7, Theorem 5.2.4]). If $\Gamma$ is a nonempty set, the linear space $c_0 (\Gamma, X)$ over $\mathbb{K}$ of all functions $f : \Gamma \to X$ such that for each $\epsilon > 0$ the set $\{\omega \in \Gamma : \|f(\omega)\| > \epsilon\}$ is finite, equipped with the supremum norm, coincides with $C_0 (\Gamma, X)$ for discrete $\Gamma$, so that $c_0 (\Gamma, X)$ is barrelled if and only if $X$ is barrelled. We shall frequently require the following result.

Theorem 1 (Freniche [8]). The space $\ell^\infty_0 (\Sigma, E)$ of $\Sigma$-simple functions with values in a Hausdorff locally convex space $E$, where $\Sigma$ is an infinite $\sigma$-algebra of subsets of a set $\Omega$, endowed with the uniform convergence topology is barrelled if and only if $\ell^\infty_0 (\Sigma)$ and $E$ are barrelled and $E$ is nuclear.

Yet there are several spaces of vector-valued measures and of bounded linear operators which have received less attention. Next we investigate the barrelledness of some of them. Along this paper $X$ will be a normed or a Banach space, $Y$ a normed space and $(\Omega, \Sigma)$ a nontrivial measurable space. If $X$ is a normed space, we denote by $bvca (\Sigma, X)$ the linear space over $\mathbb{K}$ of countably additive $X$-valued measures $\nu : \Sigma \to X$ of bounded variation equipped with the variation norm $|\nu| = |\nu| (\Omega)$, where $|\nu| (E) = \sup \sum_{A \in \pi} \|F (A)\|$ and the supremum runs over all finite partitions $\pi$ of $E \in \Sigma$ by elements of $\Sigma$. By $ca (\Sigma, X)$ we represent the space of all $X$-valued countably additive measures provided with the semivariation norm, and by $cca (\Sigma, X)$ the subspace of $ca (\Sigma, X)$ of those measures of relatively compact range. We denote by $L (X, Y)$ the linear space over $\mathbb{K}$ of all bounded linear operators from $X$
into $Y$ equipped with the uniform convergence topology, and by $K(X,Y)$ the subspace of $L(X,Y)$ of all those compact linear operators. By $L_{w^*}(X^*,Y)$ we denote the subspace of $L(X^*,Y)$ of all weak*-weakly continuous operators from $X^*$ into $Y$. The linear space of the weakly compact linear operators from $X$ into $Y$ is denoted by $W(X,Y)$. Recall that spaces of vector-valued measures and spaces of linear operators are close related, and sometimes they are representable by tensor products. For example, if $X$ is a normed space then $L^1(\Sigma, X)$ is linearly isomorphic to $c(\Sigma, X)$, whereas $L^1(\Sigma, X) = c(\Sigma) \hat{\otimes} X$. Naturally, if $K$ is compact then $C(K, X) = C(K) \hat{\otimes} X$. If $K = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of the discrete space $\mathbb{N}$ and $E$ is a linear space over $K$ of uncountable dimension provided with the strongest locally convex topology, then $C(K, E)$ is no longer barrelled, [17]. Research on barrelledness conditions is still active (see [10, 15, 16]).

2. Barrelledness of some spaces of vector measures

Let $X$ be a normed space. If $\mu \in \text{ca}^+ (\Sigma)$, we shall represent by $\text{bvca} (\Sigma, X)$ the linear subspace of $\text{ca} (\Sigma, X)$ consisting of all those vector measures that are $\mu$-continuous, whereas $L^1(\mu, X)$ will stand for the linear space over $\mathbb{K}$ of all (equivalence classes of) strongly measurable $X$-valued Bochner integrable functions defined on $\Omega$ endowed with the norm

$$\| f \|_1 = \int_{\Omega} \| f(\omega) \| \, d\mu(\omega).$$

The linear map $T : L^1(\mu, X) \to \text{bvca} (\Sigma, \mu, X)$ defined by

$$Tf(E) = \int_E f(\omega) \, d\mu(\omega)$$

for $E \in \Sigma$ is an isometry into since $|Tf| = \| f \|_1$. If $X$ is a Banach space, $T$ becomes an isometry onto the whole of $\text{bvca} (\Sigma, X)$ if and only if $X$ has the Radon-Nikodým property with respect to $\mu$.

Theorem 2. Assume that the completion $\hat{X}$ of $X$ has the Radon-Nikodým property with respect to each $\mu \in \text{ca}^+ (\Sigma)$. Then $\text{bvca} (\Sigma, X)$ is barrelled if and only if $X$ is barrelled.

Proof. If $X$ is barrelled and $\omega \in \Omega$, the standard map $P_\omega : \text{bvca} (\Sigma, X) \to \text{bvca} (\Sigma, X)$ defined by $P_\omega F = F(\Omega) \delta_\omega$ is a bounded linear projection from $\text{bvca} (\Sigma, X)$ onto the copy $\{x\delta_\omega : x \in X\}$ of $X$ within $\text{bvca} (\Sigma, X)$. Since $P_\omega$ is a quotient map, then $X$ is barrelled if $\text{bvca} (\Sigma, X)$ does [9, 11.3.1 Proposition (a)].

For the converse let us fix $\mu \in \text{ca}^+ (\Sigma)$. If $S_1(\mu)$ denotes the barrelled linear subspace of $L^1(\mu)$ of all (classes of) scalarly valued $\mu$-simple functions and $S_1(\mu, X)$ stands for the subspace of $L^1(\mu, X)$ consisting of the $X$-valued $\mu$-simple functions, the mapping $\varphi : S_1(\mu) \otimes X \to S_1(\mu, X)$ obtained by
linearizing the ansatz \( \varphi (\chi_E \otimes x) = \chi_E x \) with \( E \in \Sigma \) and \( x \in X \) is an isometry. This implies that the composition \( T \circ \varphi \) is a linear isometry from \( S_1 (\mu) \otimes_\pi X \) into a subspace of \( \text{bvca}(\Sigma, \mu, \tilde{X}) \). But if \( x_i \in X \) and \( E_i \in \Sigma \) for \( 1 \leq i \leq n \) then

\[
(T \circ \varphi) \left( \sum_{i=1}^{n} \chi_{E_i} \otimes x_i \right) (A) = \sum_{i=1}^{n} \int_A \chi_{E_i} (\omega) x_i d\mu (\omega) = \sum_{i=1}^{n} \mu (E_i \cap A) x_i \in X
\]

for every \( A \in \Sigma \), so that \( \text{Im} (T \circ \varphi) \subseteq X \). Hence actually \( T \circ \varphi \) is a linear isometry from \( S_1 (\mu) \otimes_\pi X \) into a subspace of \( \text{bvca}(\Sigma, \mu, X) \).

Denote by \( S \) the canonical map \((2.1)\) from \( L_1 (\mu, \tilde{X}) \) into \( \text{bvca}(\Sigma, \mu, \tilde{X}) \) and reserve the letter \( T \) for the restriction of \( S \) to the subspace \( L_1 (\mu, X) \). Since \( \tilde{X} \) is supposed to have the Radon-Nikodým property with respect to \( \mu \), then \( S \) maps isometrically \( L_1 (\mu, \tilde{X}) \) onto \( \text{bvca}(\Sigma, \mu, \tilde{X}) \). Given that \( S_1 (\mu, X) \) is a dense subspace of \( S_1 (\mu, \tilde{X}) \) and \( S_1 (\mu, \tilde{X}) \) is dense in \( L_1 (\mu, \tilde{X}) \), then \( S (S_1 (\mu, X)) = (T \circ \varphi) (S_1 (\mu) \otimes_\pi X) \) is a dense subspace of \( \text{bvca}(\Sigma, \mu, \tilde{X}) \) contained in \( \text{bvca}(\Sigma, \mu, X) \). So we conclude that \( S_1 (\mu) \otimes_\pi X \) is linearly isometric to a dense subspace of \( \text{bvca}(\Sigma, \mu, X) \).

On the other hand, since each \( F \in \text{bvca}(\Sigma, X) \) is \( |F| \)-continuous we have

\[
\text{bvca}(\Sigma, X) = \bigcup \{ \text{bvca}(\Sigma, \mu, X) : \mu \in \text{ca}^+(\Sigma) \}.
\]

Let us show that \( \text{bvca}(\Sigma, X) \) is the locally convex hull of \( \{ \text{bvca}(\Sigma, \mu, X) : \mu \in \text{ca}^+(\Sigma) \} \). Let \( U \) be an absolutely convex set of \( \text{bvca}(\Sigma, X) \) which meets each \( \text{bvca}(\Sigma, \mu, X) \) in a neighborhood of the origin in \( \text{bvca}(\Sigma, \mu, X) \). We claim that \( U \) is a neighborhood of the origin of \( \text{bvca}(\Sigma, X) \). Otherwise there exists a normalized sequence \( \{ F_n \}_{n=1}^\infty \) in \( \text{bvca}(\Sigma, X) \) such that \( F_n \notin nU \) for each \( n \in \mathbb{N} \). Since \( \{ F_n : n \in \mathbb{N} \} \) is bounded in \( \text{bvca}(\Sigma, X) \), then the scalar measure \( \nu := \sum_{n=1}^\infty 2^{-n} |F_n| \) belongs to \( \text{ca}^+(\Sigma) \) and, consequently, \( F_n \in \text{bvca}(\Sigma, \nu, X) \) for every \( n \in \mathbb{N} \). But since \( U \cap \text{bvca}(\Sigma, \nu, X) \) is a neighborhood of the origin in \( \text{bvca}(\Sigma, \nu, X) \), there must exist \( m \in \mathbb{N} \) such that \( F_m \in mU \), a contradiction.

Since \( S_1 (\mu) \) and \( X \) are barrelled normed spaces, we have that \( S_1 (\mu) \otimes_\pi X \) is barrelled too [7, Theorem 1.6.6], and since \( S_1 (\mu) \otimes_\pi X \) is linearly isometric to a dense subspace of \( \text{bvca}(\Sigma, \mu, X) \), then this latter subspace is also barrelled [11, 27.1.(2)]. Finally, the conclusion follows from the fact that the locally convex hull of a family of barrelled spaces is barrelled [11, 27.1.(3)].

\( \square \)

Remark 3. An alternative proof. The proof of the previous theorem solves Problem 6 of [7, Chapter 8]. Another approach may be the following. If \( \tilde{X} \) has the Radon-Nikodým property with respect to each \( \mu \in \text{ca}^+(\Sigma) \), it can be shown (cf. [14, Corollary 5.23]) that \( \text{ca}(\Sigma) \otimes_\pi X = \text{ca}(\Sigma) \otimes_\pi \tilde{X} = \text{bvca}(\Sigma, \tilde{X}) \) isometrically. But a careful reading of the proof of [14, Theorem 5.22] shows that (under the assumption that \( \tilde{X} \) has the Radon-Nikodým property with respect to each \( \mu \in \text{ca}^+(\Sigma) \)) even for normed spaces the projective product space \( \text{ca}(\Sigma) \otimes_\pi X \) is in fact linearly isometric to a dense subspace of \( \text{bvca}(\Sigma, X) \).
Since \( \text{ca}(\Sigma) \otimes \varepsilon X \) is barrelled if \( X \) is barrelled (cf. [7, Theorem 1.6.6]), it follows that \( \text{bca}(\Sigma, X) \) is barrelled if and only if \( X \) is barrelled.

**Corollary 4.** Let \( X \) be a normed space and suppose that each \( \mu \in \text{ca}^+(\Sigma) \) is purely atomic. Then \( \text{bca}(\Sigma, X) \) is barrelled if and only if \( X \) is barrelled.

**Proof.** Since each \( \mu \in \text{ca}^+(\Sigma) \) is purely atomic, the Banach space \( \hat{X} \) has the Radon-Nikodým property with respect to every \( \mu \in \text{ca}^+(\Sigma) \). So the previous theorem applies. \( \square \)

**Theorem 5.** Assume that the \( \sigma \)-algebra \( \Sigma \) is infinite. Then \( \text{ca}(\Sigma, \ell_0^\infty) = \text{cca}(\Sigma, \ell_0^\infty) \) and neither \( \text{ca}(\Sigma, \ell_0^\infty) \) nor \( \text{cca}(\Sigma, \ell_0^\infty) \) are barrelled, despite the fact that \( \ell_0^\infty \) is barrelled.

**Proof.** Let \( F \in \text{ca}(\Sigma, \ell_0^\infty) \). Let us see first that \( F(\Sigma) \) is contained in a finite-dimensional subspace of \( \ell_0^\infty \). Indeed, assume by contradiction that \( F(\Sigma) \) is infinite-dimensional. In this case there is a sequence \( \{E_n : n \in \mathbb{N} \} \subseteq \Sigma \) such that the linear space span \( \{F(E_n) : n \in \mathbb{N} \} \) is infinite-dimensional. Setting \( A_1 := E_1 \) and \( A_n := E_n \setminus \bigcup_{i=1}^{n-1} A_i \) for \( n \geq 2 \) as is frequently done, then \( \{A_n : n \in \mathbb{N} \} \) is a countable family of pairwise disjoint sets of \( \Sigma \) such that \( F(E_n) = \sum_{i=1}^{n} F(A_i) \). Thus we have span \( \{F(E_n) : n \in \mathbb{N} \} \subseteq \text{span} \{F(A_n) : n \in \mathbb{N} \} \). But the series \( \sum_{i=1}^{\infty} F(A_n) \) is subseries convergent in \( \ell_0^\infty \) as a consequence of the fact that \( \sum_{i=1}^{\infty} F(A_n) = F(\bigcup_{i=1}^{\infty} A_n) \in \ell_0^\infty \) for every increasing sequence \( \{n_i\}_{i=1}^{\infty} \) of positive integers. Thus, according to [1, Theorem 1(b)], the linear subspace span \( \{F(A_n) : n \in \mathbb{N} \} \) of \( \ell_0^\infty \) must be finite-dimensional, a contradiction.

Since \( F(\Sigma) \) is contained in a finite-dimensional subspace of \( \ell_0^\infty \) and (because of \( F \) is countably additive) the set \( F(\Sigma) \) is weakly compact, it follows that \( F(\Sigma) \) is relatively compact in \( \ell_0^\infty \), which ensures that \( \text{ca}(\Sigma, \ell_0^\infty) = \text{cca}(\Sigma, \ell_0^\infty) \).

On the other hand, the fact that the range \( F(\Sigma) \) of \( F \) is finite-dimensional also tells us that there is a finite family \( \{B_1, \ldots, B_p\} \) of pairwise disjoints elements of \( \Sigma \), which depends on \( F \), such that \( F(\Sigma) \subseteq \text{span} \{F(B_1), \ldots, F(B_p)\} \). Consequently, the vector measure \( F \) must be of the form

\[
F(E) = \sum_{i=1}^{p} \mu_i(E) F(B_i),
\]

where each \( \mu_i : \Sigma \to \mathbb{K} \) is clearly a countably additive scalar measure, i.e., \( \mu_i \in \text{ca}(\Sigma) \). Setting \( x_i := F(B_i) \) for \( 1 \leq i \leq p \), we see that we can represent the measure \( F \) as a tensor product of the form \( F = \sum_{i=1}^{p} \mu_i \otimes x_i \), so that clearly \( \text{ca}(\Sigma, \ell_0^\infty) = \text{cca}(\Sigma, \ell_0^\infty) \) can be represented as a (topological) subspace of \( \text{ca}(\Sigma) \otimes \varepsilon \ell_0^\infty \). Since \( \text{ca}(\Sigma) \otimes \varepsilon \ell_0^\infty \) embeds linearly into \( \text{cca}(\Sigma, \ell_0^\infty) \), it follows that

\[
\text{ca}(\Sigma, \ell_0^\infty) = \text{cca}(\Sigma, \ell_0^\infty) = \text{ca}(\Sigma) \otimes \varepsilon \ell_0^\infty = \ell_0^\infty \left( \mathbb{N}, \text{ca}(\Sigma) \right).
\]

Now, given that \( \text{ca}(\Sigma) \) is an infinite-dimensional normed space, and a normed space is nuclear if and only if is finite-dimensional, Theorem 1 assures that
\[ \ell_0^\infty \left( 2^\mathbb{N}, \text{ca}(\Sigma) \right) \] is not a barrelled space. So we conclude that neither \( \text{ca}(\Sigma, \ell_0^\infty) \) nor \( \text{cca}(\Sigma, \ell_0^\infty) \) are barrelled.

3. Barrelled and non-barrelled \( L(X, Y) \) spaces

If \( X \) is a Banach space and \( Y \) is a non complete barrelled normed space, it turns out that there are non barrelled spaces of bounded linear operators \( T : X \to Y \), as the following propositions shows.

**Proposition 6.** If \( X \) is an infinite-dimensional Banach space, the space \( L(X, \ell_0^\infty) \) equipped with the operator norm is not barrelled.

**Proof.** If \( T \in L(X, \ell_0^\infty) \) then, according to [1, Theorem 3(a)], the range of \( T \) is finite-dimensional. This forces to conclude that \( L(X, \ell_0^\infty) \) coincides with \( X^* \otimes \ell_0^\infty \). Indeed, on the one hand \( X^* \otimes \ell_0^\infty \) can be identified with the subspace of \( L(X, \ell_0^\infty) \) of all those linear operators \( T \) such that \( \text{Im} \ T \) is a finite-dimensional subspace of \( \ell_0^\infty \) and, on the other hand, given \( T \in L(X, \ell_0^\infty) \), since the range of \( T \) is finite-dimensional and the family \( \{ \chi_A : A \in 2^\mathbb{N} \} \) contains a Hamel basis of \( \ell_0^\infty \), even a discrete one, there is a finite partition \( \{ A_1, \ldots, A_p \} \) of \( \mathbb{N} \) such that \( \text{Im} \ T = \text{span} \{ \chi_{A_i} : 1 \leq i \leq p \} \), so that

\[
Tx = \sum_{i=1}^p \alpha_i(x) \chi_{A_i}
\]

for every \( x \in X \), where \( \alpha_i : X \to \mathbb{K} \) is a bounded linear form for \( 1 \leq i \leq p \). In fact \( \alpha_i \) is clearly linear and there is \( K > 0 \) such that

\[
|\alpha_i(x)| \leq \sup_{1 \leq j \leq p} |\alpha_j(x)| = \sup_{n \in \mathbb{N}} \left| \sum_{j=1}^p \alpha_j(x) \chi_{A_j}(n) \right| = \|Tx\| \leq K \|x\|.
\]

So we can write \( T = \sum_{i=1}^p x_i^* \otimes \chi_{A_i} \), with \( x_i^* \in X^* \) for \( 1 \leq i \leq p \), verifying that

\[
\|T\| = \max \{ \|x_1^*\|, \ldots, \|x_p^*\| \} = \left\| \sum_{i=1}^p x_i^* \otimes \chi_{A_i} \right\|.
\]

Thus we have the following linear isometries

\[ L(X, \ell_0^\infty) = X^* \otimes \ell_0^\infty = \ell_0^\infty \left( 2^\mathbb{N}, X^* \right). \]

Since \( X^* \) is an infinite-dimensional Banach space and the family \( 2^\mathbb{N} \) of all the subsets of \( \mathbb{N} \) is an infinite \( \sigma \)-algebra, it follows again from Theorem 1 that the space \( \ell_0^\infty \left( 2^\mathbb{N}, X^* \right) \) is not barrelled. Hence \( L(X, \ell_0^\infty) \) is a non barrelled operator space.

**Proposition 7.** If \( X \) is an infinite-dimensional Banach space, then the operator space \( L_{w^*} \left( X^*, \ell_0^\infty \right) \) is not barrelled.

**Proof.** Since each operator \( T \in L_{w^*} \left( X^*, \ell_0^\infty \right) \) is weak*-weakly continuous, standing for \( Q \) the closed unit ball of \( X^* \) then \( T(Q) \) is an absolutely convex weakly compact subset of \( \ell_0^\infty \) whence \( T(Q) \) is a Banach disk of \( \ell_0^\infty \).
\(\sigma(\ell_0^\infty, \text{ba}(2^\mathbb{N}))\), hence of \((\ell_0^\infty, \sigma(\ell_0^\infty, \mathbb{K}^{(N)}))\). Since the linear span of every Banach disk of \((\ell_0^\infty, \sigma(\ell_0^\infty, \mathbb{K}^{(N)}))\) is finite-dimensional [1, Theorem 3(b)] (see also \[7, \text{Corollary 6.2.5}\]), it follows that the range of \(T\) is a finite-dimensional subspace of \(\ell_0^\infty\). This implies that \(L_{w^*}(X^*, \ell_0^\infty) = X \otimes_\varepsilon \ell_0^\infty = \ell_0^\infty (\mathbb{N}, X)\). Indeed \(T \in L_{w^*}(X^*, \ell_0^\infty)\) if and only if there is a partition \(\{A_1, \ldots, A_p\}\) of \(\mathbb{N}\) such that \(T x^* = \sum_{i=1}^p \alpha_i (x^*) \chi_{A_i}\), each \(\alpha_i : X^* \text{ (weak*)} \to \mathbb{K}\) being linear and continuous. Hence we can write \(T = \sum_{i=1}^p x_i \otimes \chi_{A_i}\), with \(x_i \in X\) for \(1 \leq i \leq p\). Since \(X\) is infinite-dimensional, \(\ell_0^\infty (2^{\mathbb{N}}, X)\) cannot be barrelled. Therefore \(L_{w^*}(X^*, \ell_0^\infty)\) is not barrelled.

If \(T \in W(X, \ell_0^\infty)\) or \(T \in K(X, \ell_0^\infty)\), as before the range of \(T\) is a finite-dimensional subspace of \(\ell_0^\infty\), which implies that \(W(X, \ell_0^\infty) = K(X, \ell_0^\infty) = X^* \otimes_\varepsilon \ell_0^\infty = \ell_0^\infty (2^{\mathbb{N}}, X^*)\). So if \(X\) is infinite-dimensional, again \(\ell_0^\infty (2^{\mathbb{N}}, X^*)\) is not barrelled, whereas \(W(X, \ell_0^\infty)\) neither \(K(X, \ell_0^\infty)\) is barrelled. However, the following positive result holds.

**Theorem 8.** Let \(X\) be a Banach space such that \(X^*\) is an \(\mathcal{L}^\infty\)-space with the approximation property. If \(Y\) is the locally convex hull of a sequence of Banach subspaces (which cover it), then \(K(X, Y)\) is barrelled.

**Proof.** Assume that \(Y\) is the locally convex hull \(\text{ind}_{n \in \mathbb{N}} Y_n\) of a sequence \(\{Y_n : n \in \mathbb{N}\}\) of Banach subspaces of \(Y = \bigcup_{n=1}^\infty Y_n\). First observe that \(K(X, Y) = \bigcup_{n=1}^\infty K(X, Y_n)\).

Let \(T \in K(X, Y)\). If \(B_X\) denotes the unit ball of \(X\), then \(\overline{T(B_X)^Y}\) is an absolutely convex compact set of \(Y\), hence a Banach disk of \(Y\). Since \(\{Y_n : n \in \mathbb{N}\}\) is a countable covering of \(Y\) by closed sets, the Baire category theorem provides \(\varepsilon > 0\) and \(n_0 \in \mathbb{N}\) such that \(\overline{T(B_X)^Y} \subseteq Y_{n_0}\). Consequently \(T \in K(X, Y_{n_0})\), so that \(K(X, Y) = \bigcup_{n=1}^\infty K(X, Y_n)\).

Let us show that this implies that \(K(X, Y)\) is barrelled. Indeed, according to [9, 16.5 Proposition], the fact that \(X^*\) is a \(gDF\)-space ensures that \(X^* \otimes_\varepsilon (\bigoplus_{n=1}^\infty Y_n)\) is canonically isomorphic to \(\bigoplus_{n=1}^\infty (X^* \otimes_\varepsilon Y_n)\). So, since \(X^*\) is assumed to be an \(\mathcal{L}\)-space, this yields a topological isomorphism from \(\text{ind}_{n \in \mathbb{N}} (X^* \otimes_\varepsilon Y_n)\) onto \(X^* \otimes_\varepsilon Y\) in the canonical manner [9, 16.3.6 Remark]. Thus we have that

\[
(3.1) \quad \text{ind}_{n \in \mathbb{N}} (X^* \otimes_\varepsilon Y_n) = X^* \otimes_\varepsilon Y.
\]

But since \(X^*\) has the approximation property, then \(X^* \otimes_\varepsilon Y_n = K(X, Y_n)\) whereas \(X^* \otimes_\varepsilon Y\) is isometric to a dense linear subspace of \(K(X, Y)\). Let \(U\) be a barrel of \(K(X, Y)\), i.e., a closed absolutely convex and absorbing set. Clearly \(U\) meets each subspace \(K(X, Y_n)\) in a neighborhood of the origin in \(K(X, Y_n)\), consequently \(U\) meets each \(X^* \otimes_\varepsilon Y_n\) in a neighborhood of the origin in \(X^* \otimes_\varepsilon Y_n\). But due to (3.1) this implies that \(U\) meets \(X^* \otimes_\varepsilon Y\) in a neighborhood of the origin of \(X^* \otimes_\varepsilon Y\). Since \(X^* \otimes_\varepsilon Y\) is dense in \(K(X, Y)\) and \(U\) is closed in \(K(X, Y)\), it follows that \(U\) is a neighborhood of the origin in
$K(X,Y)$. In other words, since $\text{ind}_{n \in \mathbb{N}} K(X,Y_n)$ is an ultrabaroneological (hence barrelled) dense subspace of $K(X,Y)$, then $K(X,Y)$ is itself barrelled. $\square$

References

[13], Necessary and sufficient conditions for $C(X,E)$ to be barrelled or infrabarrelled, Simon Stevin 57 (1983), no. 1-2, 103–123.

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