REDUCING SUBSPACES FOR A CLASS OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE OF THE BIDISK

MOHAMMED ALBASEER, YUFENG LU, AND YANYUE SHI

ABSTRACT. In this paper, we completely characterize the nontrivial reducing subspaces of the Toeplitz operator $T_{z^N\bar{z}^M}$ on the Bergman space $A^2(D^2)$, where $N$ and $M$ are positive integers.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For $-1 < \alpha < \infty$, let $L^2(\mathbb{D}, dA_\alpha)$ be the Hilbert space of square integrable functions on $\mathbb{D}$ with the inner product

$$\langle f, g \rangle_\alpha = \int_{\mathbb{D}} f(z) \overline{g(z)} dA_\alpha(z), \quad f, g \in A^2_\alpha(\mathbb{D}),$$

where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z),$$

and $dA$ is the normalized area measure on $\mathbb{D}$.

The weighted Bergman space $A^2_\alpha(\mathbb{D})$ is the subspace of $L^2(\mathbb{D}, dA_\alpha)$ consisting of all the analytic functions in $\mathbb{D}$. We denote

$$\gamma_n = \|z^n\|_\alpha = \sqrt{\frac{n!\Gamma(2 + \alpha)}{\Gamma(n + \alpha + 2)}}$$

for $n = 0, 1, 2, \ldots$. Therefore,

$$\|f\|_\alpha^2 = \sum_{n=0}^{+\infty} \gamma_n^2 |a_n|^2 < \infty,$$

where $f(z) = \sum_{n=0}^{+\infty} a_n z^n \in A^2_\alpha(\mathbb{D})$. Especially when $\alpha = 0$, we write $A^2(\mathbb{D}) = A^2_0(\mathbb{D})$. In this case, $\gamma_n = \sqrt{\frac{1}{n+1}}$.

Received November 3, 2014; Revised February 19, 2015.
2010 Mathematics Subject Classification. Primary 47B35, 46E22.
Key words and phrases. reducing subspace, Toeplitz operator, polydisk.
This research is supported by NSFC (11271059,11201438), Research Fund for the Doctoral Program of Higher Education of China and Shandong Province Young Scientist Research Award Fund (BS2012SF031).
Denote by $D^2 = \mathbb{D} \times \mathbb{D}$ the bidisk. The Bergman space $A^2(D^2)$ is the space of all holomorphic functions in $L^2(D^2, d\mu)$ where $d\mu(z) = dA(z_1)A(z_2)$. For multi-index $\beta = (\beta_1, \beta_2)$, denote $z^\beta = z_1^{\beta_1}z_2^{\beta_2}$ and $e_\beta = \frac{z^\beta}{\gamma_{\beta_1}\gamma_{\beta_2}}$.

Then $\{e_\beta\}_{\beta \geq 0}$ ($\beta \geq 0$ means that $\beta_1 \geq 0$ and $\beta_2 \geq 0$) is an orthogonal basis in $A^2(D^2)$.

For a bounded measurable function $f \in L^\infty(D^2)$, the Toeplitz operator with symbol $f$ is defined by $T_fh = P(fh)$ for every $h \in A^2(D^2)$, where $P$ is the Bergman orthogonal projection from $L^2(D^2, d\mu)$ onto $A^2(D^2)$.

Recall that for a bounded linear operator $T$ on a Hilbert space $H$, a closed subspace $M$ is called a reducing subspace of the operator $T$ if $T(M) \subset M$ and $T^*(M) \subset M$. A reducing subspace $M$ is said to be minimal if there is no nonzero reducing subspace $N$ such that $N$ properly contains in $M$.

On the Bergman space over $\mathbb{D}$, it is proved that $T_B$ has just two non-trivial reducing subspaces [13, 16], where $B$ is the product of two Blaschke factors. In [12], M. Stessin and K. Zhu gave a complete description of the reducing subspaces of weighted unilateral shift operators of finite multiplicity. In particular, $T_{z^n}$ has $n$ distinct minimal reducing subspaces. If $B$ is a finite Blaschke product (order $n \geq 2$), the number of nontrivial minimal reducing subspaces of $T_B$ equals the number of connected components of the Riemann surface of $B^{-1} \circ B$ over $\mathbb{D}$ (see [2, 3, 4, 8, 9, 14] for details). Further, if $B$ is an infinite Blaschke product or a covering map, the relative research can be found in [5, 6, 7].

On the Bergman space of bidisk, Y. Lu and X. Zhou [10] characterized the reducing subspaces of $T_{z_1^{N_1}z_2^{N_2}}$, $T_{z_1^{N_1}}$ and $T_{z_2^{N_2}}$, respectively. The reducing subspaces of $T_{z_1^{N_1}z_2^{M}}$ on the weighted Bergman space $A^2_\alpha(D^2)$ have been completely described in [11]. For $p = \alpha z^k + \beta w^l$, the minimal reducing subspaces of $T_p$ on $A^2(D^2)$ and the commutant algebra $V^*(p) = \{T_p, T_p^*\}$ was described in [1, 15].

In this paper, we mainly consider the reducing subspaces for the Toeplitz operator $T_{z_1^{N_1}z_2^{M}}$ on the Bergman space $A^2(D^2)$, where $N$ and $M$ are positive integers.

2. Main results

In this section, we will give a complete characterization of the reducing subspaces of $T_{z_1^{N_1}z_2^{M}}$. To state our results, we need some notations and lemmas. Throughout this paper, denote $T = T_{z_1^{N_1}z_2^{M}}$, where $N$ and $M$ are positive integers. Denote by $[f]$ the reducing subspace of $T$ generated by $f \in A^2(D^2)$. Let $\mathbb{N}$ be the set of all the nonnegative integers.
By direct calculation, we know that
\[
T^h(z_1^k z_2^l) = \begin{cases}
\frac{\gamma_1^{k+hN} z_2^l}{\gamma_{1-hM}} z_1^k, & \text{if } l \geq hM \\
0, & \text{if } l < hM
\end{cases}
\]
\[
T^{-h}(z_1^k z_2^l) = \begin{cases}
\frac{\gamma_1^{k-hN} z_2^l}{\gamma_{1-hM}} z_1^k, & \text{if } k \geq hN \\
0, & \text{if } k < hN
\end{cases}
\]
for \(k, l, h \in \mathbb{N}\). Set
\[
E_0 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : 0 \leq k < N, 0 \leq l < M\},
E_1 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : k \geq 2N\},
E_2 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : l \geq 2M, 0 \leq k < 2N\},
E_3 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : N \leq k < 2N, M \leq l < 2M\},
E_4 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : 0 \leq k < N, M \leq l < 2M\},
E_5 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : 0 \leq l < M, N \leq k < 2N\}.
\]
Clearly,
\[
A^2(\mathbb{D}^2) = \bigoplus_{i=0}^5 \text{span}\{z_1^p z_2^q : (p, q) \in E_i\}.
\]
Notice that \(M_0 = \text{span}\{z_1^p z_2^q : (p, q) \in E_0\}\) is a reducing subspace of \(T\). To find other reducing subspaces, we first study the orthogonal decomposition of \(z_1^p z_2^q\) with respect to \(M\).

**Lemma 2.1.** Suppose \(M \subset M_0^\perp\) is a reducing subspace of \(T\). Let \(P_M\) be the orthogonal projection from \(A^2(\mathbb{D}^2)\) onto \(M\).

(i) If \((k, l) \in E_1 \cup E_2 \cup E_3\), then \(P_M z_1^k z_2^l = \lambda z_1^k z_2^l\) with some \(\lambda \in \mathbb{C}\).

(ii) If \((k, l) \in E_4\), then
\[
P_M z_1^k z_2^l \in \text{span}\{z_1^n z_2^m : (n, m) \in E_4\}.
\]

(iii) If \((k, l) \in E_5\), then
\[
P_M z_1^k z_2^l \in \text{span}\{z_1^n z_2^m : (n, m) \in E_5\}.
\]

**Proof.** Let \(k, l \in \mathbb{N}\). Since \(M \perp M_0\), \(\langle P_M(z_1^k z_2^l), z_1^n z_2^m \rangle = 0\) for \((p, q) \in E_0\).

In the following, we consider the inner product \(\langle P_M(z_1^k z_2^l), z_1^n z_2^m \rangle\) for \((p, q) \in \bigcup_{i=1}^5 E_i\).

For every nonnegative integer \(h\) satisfying \(l \geq hM\),
\[
T^{h \ast T^h}(z_1^k z_2^l) = \frac{\gamma_1^{2 k \ast hN}}{\gamma_{l-hM} \gamma_{l}^k} z_1^k z_2^l.
\]

By computation,
\[
\frac{\gamma_1^{2 k \ast hN}}{\gamma_{l-hM} \gamma_{l}^k} \langle P_M(z_1^k z_2^l), z_1^n z_2^m \rangle = \langle P_M T^{h \ast T^h}(z_1^k z_2^l), z_1^n z_2^m \rangle.
\]
\[ (P_M(z_1^k z_2^l), T^{h-h^*}(z_1^p z_2^q)) \]
\[ = \left\{ \begin{array}{ll}
\frac{\gamma_k^2 \gamma_k^{2+hM}}{\gamma_{k-hN}^l} (P_M(z_1^k z_2^l), z_1^p z_2^q), & q \geq hM \\
0, & q < hM.
\end{array} \right. \]

Recall that \([s] = \max\{n \in \mathbb{Z} : n \leq s\}\) for real number \(s\). By above equality, we get that if \([P_M(z_1^k z_2^l), z_1^p z_2^q] \neq 0\), then
\[
\frac{\gamma_k^2 \gamma_k^{2+hM}}{\gamma_{k-hN}^l} \frac{\gamma_{k+hN}^2}{\gamma_{k-M}^l - hM} = \frac{\gamma_q^2 \gamma_q^{2+hN}}{\gamma_{q-hM}^l}
\]
for \(0 \leq h \leq \left[ \frac{k}{N} \right], q \geq \left[ \frac{l}{M} \right] M\).

Equivalently,
\[
\frac{(k+1)(q+1)}{(p+1)(l+1)} = \frac{(k+1+hN)(q+1-hM)}{(p+1+hN)(l+1-hM)}
\]
for \(0 \leq h \leq \left[ \frac{k}{N} \right], q \geq \left[ \frac{l}{M} \right] M\).

(i) If \((k, l) \in E_1 \cup E_2 \cup E_3\), we will show that the equality (2) holds if and only if \(p = k\) and \(q = l\).

Case one: \(l \geq 2M\). Let \(g_1(\lambda) = (k+1)(q+1)(p+1+\lambda N)(l+1-\lambda M)\), \(g_2(\lambda) = (p+1)(l+1)(k+1+\lambda N)(q+1-\lambda M)\) and \(g(\lambda) = g_1(\lambda) - g_2(\lambda)\).

Since \(l \geq 2M\), we have \(g(0) = g(1) = g(2) = 0\). Considering \(g(\lambda)\) is a quadratic polynomial, we have \(g(\lambda) \equiv 0\) on \(C\). Therefore, \(g_1\) and \(g_2\) have the same zeros, i.e.,
\[
\left\{ \begin{array}{l}
(k+1)(q+1)NM = (p+1)(l+1)NM \\
(k+1)(q+1)M = (p+1)(l+1)M
\end{array} \right. = \left( \begin{array}{ll}
k+1 & l+1 \\
q+1 & p+1
\end{array} \right)
\]

It follows that \(p = k\) and \(q = l\).

Case two: \(k \geq 2N\). Replacing \(T^*T\) by \(TT^*\) in Case one, we can get the desire result. The details are listed as follows.

Since
\[ T^{h-h^*}(z_1^k z_2^l) = \frac{\gamma_k^2 \gamma_k^{2+hM}}{\gamma_{k-hN}^l} z_1^k z_2^l, \forall 0 \leq h \leq \left[ \frac{k}{N} \right], \]
we know that
\[
\frac{\gamma_k^2 \gamma_k^{2+hM}}{\gamma_{k-hN}^l} (P_M(z_1^k z_2^l), z_1^p z_2^q) = \langle P_M(T^{h-h^*}(z_1^k z_2^l), z_1^p z_2^q) \rangle
\]
\[ = \langle P_M(z_1^k z_2^l), T^{h-h^*}(z_1^p z_2^q) \rangle \]
\[ = \left\{ \begin{array}{ll}
\frac{\gamma_k^2 \gamma_k^{2+hM}}{\gamma_{k-hN}^l} (P_M(z_1^k z_2^l), z_1^p z_2^q) & \text{if } p \geq hN \\
0 & \text{if } p < hN.
\end{array} \right. \]

Therefore, \([P_M(z_1^k z_2^l), z_1^p z_2^q] \neq 0\) will give that
\[
\frac{\gamma_k^2 \gamma_k^{2+hM}}{\gamma_{k-hN}^l} = \frac{\gamma_q^2 \gamma_q^{2+hN}}{\gamma_{q-hM}^l}
\]
for $0 \leq h \leq \left[ \frac{k}{N} \right]$ and $p \geq \left[ \frac{k}{N} \right] N$. Equivalently,

$$\frac{(k + 1)(q + 1)}{(p + 1)(l + 1)} = \frac{(k + 1 - hN)(q + 1 + hM)}{(p + 1 - hN)(l + 1 + hM)}$$

for $0 \leq h \leq \left[ \frac{k}{N} \right]$ and $p \geq \left[ \frac{k}{N} \right] N$. So when $k \geq 2N$, the above equality follows for $h = 0, 1, 2$. In this case we will get $p = k$ and $q = l$ by the same arguments as the case $l \geq 2M$ has done.

Case three: $(k, l) \in E_3 = \{(n, m) \in \mathbb{N}^2 : N \leq n < 2N, M \leq m < 2M\}$. In this case, $\left[ \frac{k}{N} \right] \geq 1$ and $\left[ \frac{N}{M} \right] \geq 1$. Then equalities (3) and (5) hold for $h = 0, 1$. Recall that $g(\lambda) = g_1(\lambda) - g_2(\lambda)$, where $g_1(\lambda) = (k + 1)(q + 1)(p + 1 + \lambda N)(l + 1 - \lambda M)$ and $g_2(\lambda) = (p + 1)(l + 1)(k + 1 + \lambda N)(q + 1 - \lambda M)$. We get $g(0) = g(1) = g(-1) = 0$. Therefore, we obtain that $p = k$ and $q = l$.

(ii) Suppose that $(k, l) \in E_4$. We need only prove that

$$\text{span} \{z_1^n z_2^m : (n, m) \in \left( \bigcup_{i=1}^3 E_i \right) \cup E_5 \}.$$ 

If $(n, m) \in E_1 \cup E_2 \cup E_3$, the conclusion (i) implies that $P_M z_1^n z_2^m = \lambda z_1^n z_2^m$ for some $\lambda \in \mathbb{C}$. Thus

$$\langle P_M z_1^n z_2^m , z_1^n z_2^m \rangle = \langle z_1^n z_2^m , P_M z_1^n z_2^m \rangle = \langle z_1^n z_2^m , z_1^n z_2^m \rangle = 0.$$

That is, $P_M z_1^n z_2^m \perp \text{span} \{z_1^n z_2^m : (p, q) \in E_1 \cup E_2 \cup E_3 \}$.

If $(n, m) \in E_5 = \{(k, l) \in \mathbb{N} \times \mathbb{N} : 0 \leq l < M, N \leq k < 2N\}$,

$$\langle P_M z_1^n z_2^m , z_1^n z_2^m \rangle = \frac{\gamma_2^2 - M^2 \gamma_2^2}{\gamma_2^2 - k + N} \langle P_M z_1^n z_2^m , z_1^n z_2^m \rangle$$

$$= \frac{\gamma_2^2 - M^2 \gamma_2^2}{\gamma_2^2 - k + N} \langle T z_1^n z_2^m , z_1^n z_2^m \rangle = 0,$$

where the last equality comes from $\text{span} \{z_1^n z_2^m : (p, q) \in E_5 \} \subseteq \text{Ker} T$. Thus $P_M z_1^n z_2^m \perp \text{span} \{z_1^n z_2^m : (p, q) \in E_5 \}$.

(iii) Replacing $T^*T$ by $TT^*$ in (ii), we get the desired result. \qed

**Remark 2.1.** Let $\mathcal{M} \subseteq \mathcal{M}_d^+$ is a nonzero reducing subspace of $T$. In (i) of Lemma 2.1, we indeed get that $\lambda = 0$ or $1$, that is $z_1 z_2 \in \mathcal{M}$ or $z_1^2 z_2 \in \mathcal{M}$ for each $(k, l) \in E_1 \cup E_2 \cup E_3$.

If $z_1^2 z_2 \in \mathcal{M}$, then

$$[z_1^2 z_2] = \text{span} \{z_1^{k-hN} z_2^{l+hM} : k - hN \geq 0, l + hM \geq 0, h \in \mathbb{Z} \}$$

is a minimal reducing subspace of $T$, containing in $\mathcal{M}$. Moreover, if $z_1^2 z_2, z_1^2 z_2 \in \mathcal{M}$ and $(k, l), (p, q) \in E_1 \cup E_2 \cup E_3$, then it’s clear that either $[z_1^2 z_2] = [z_1^{2} z_2]$ or $[z_1^2 z_2] = [z_1^{2} z_2]$. So for any non-zero function $f(z) = \sum_{(k, l) \in E_1 \cup E_2 \cup E_3} a_{k,l} z_1^k z_2^l$, $[f]$ is the direct sum of some minimal reducing subspace as (6).

We define two equivalences on $E_4$ and $E_5$ respectively by:
(i) for \((p, q), (k, l) \in E_4, (p, q) \sim_1 (k, l)\) \(\Leftrightarrow \frac{(k+1)(q+1)}{(p+1)(l+1)} = \frac{(k+1+N)(q+1-M)}{(p+1)(l+1)}\).

(ii) for \((p, q), (k, l) \in E_5, (p, q) \sim_2 (k, l)\) \(\Leftrightarrow \frac{(k+1)(q+1)}{(p+1)(l+1)} = \frac{(k+1-N)(q+1+M)}{(p+1)(l+1)}\).

It is easy to check that

(i) \((p, q) \in E_4 \Leftrightarrow (p + N, q - M) \in E_5; \)

(ii) for \((p, q), (k, l) \in E_4, (p, q) \sim_1 (k, l) \Leftrightarrow (p + N, q - M) \sim_2 (k + N, l - M); \)

(iii) for \((p, q), (k, l) \in E_5, (p, q) \sim_2 (k, l) \Leftrightarrow (p - N, q + M) \sim_1 (k - N, l + M). \)

For \((n, m) \in E_4\) and \((k, l) \in E_5\), let

\[ P_{n,m} : A^2(\mathbb{D}^2) \to \text{span}\{z_1^{n}z_2^{m} : (p, q) \in (n, m), (p, q) \in E_4\}, \]

\[ Q_{k,l} : A^2(\mathbb{D}^2) \to \text{span}\{z_1^{k}z_2^{l} : (p, q) \sim_2 (k, l), (p, q) \in E_5\} \]

be two orthogonal projections. For \(f \in A^2(\mathbb{D}^2)\) and \(P_{n,m}f \neq 0\), we have

\[ [P_{n,m}f] = \text{span}\{P_{n,m}f, TP_{n,m}f\}. \]

since \(T^*P_{n,m}f = 0, T^2P_{n,m}f = 0\) and \(T^*TP_{n,m}f = \frac{\gamma_2\gamma_2 + M}{\gamma_2 - mM}P_{n,m}f\). Similarly, if \(f \in \mathcal{M}\) and \(Q_{k,l}f \neq 0\), then

\[ [Q_{k,l}f] = \text{span}\{Q_{k,l}f, T^*Q_{k,l}f\}. \]

\textbf{Lemma 2.2.} Let \(\mathcal{M} \subset \mathcal{M}^\perp_0\) be a reducing subspace of \(T\) and \((n, m) \in E_4\). Then the following statements hold.

(a) If \(f \in \mathcal{M}\), then \([P_{n,m}f] \subset \mathcal{M}\) and \([Q_{n+N,n-M}f] \subset \mathcal{M}\).

(b) If \(f_1, f_2 \in P_{n,m}\mathcal{M}\) and \(f_1 \perp f_2\), then \([f_1] \perp [f_2]\).

(c) \(P_{n,m}T^*f = T^*Q_{n+N,n-M}f\) and \(P_{n,m}f = Q_{n+N,n-M}Tf\) \(\forall f \in \mathcal{M}\).

(d) If \(f \in \mathcal{M}\), then \([P_{n,m}f] = [Q_{n+N,n-M}Tf]\) and \([Q_{n+N,n-M}f] = [P_{n,m}T^*f]\).

(e) \(P_{n,m}\mathcal{M} \oplus Q_{n+N,n-M}\mathcal{M} \subset \mathcal{M}\) is a reducing subspace of \(T\).

\textbf{Proof.} (a) For every \(f \in \mathcal{M}\), we know that \(P_{\mathcal{M}}P_{n,m}f = P_{n,m}f\), since \(P_{\mathcal{M}}P_{n,m} = P_{n,m}P_{\mathcal{M}}, \) which obtained by the following simple facts:

(i) if \((k, l) \in E_4, \) then \(P_{\mathcal{M}}z_1^{k}z_2^{l} \in \text{span}\{z_1^{n}z_2^{m} : (p, q) \in E_4\}; \)

(ii) if \((k, l) \notin E_4, \) then \(P_{\mathcal{M}}z_1^{k}z_2^{l} \notin \text{span}\{z_1^{n}z_2^{m} : (p, q) \in E_4\}. \)

So \(P_{n,m}f \in \mathcal{M}, \) which implies that \([P_{n,m}f] \subset \mathcal{M}. \)

Similarly, we have \(P_{\mathcal{M}}Q_{n+N,n-M}f = Q_{n+N,n-M}f, \) which shows that \(Q_{n+N,n-M}f \in \mathcal{M}. \) Thus \([Q_{n+N,n-M}f] \subset \mathcal{M}. \)

(b) It is clear that \(Tf_1, Tf_2 \in \text{span}\{z_1^{k}z_2^{l} : (k, l) \in E_5\}\) and

\[ (Tf_1, Tf_2) = (T^*Tf_1, f_2) = \frac{\gamma_2^2 \gamma_2^2 \gamma_2^2 m}{\gamma_2^2 \gamma_2^2 - M} (f_1, f_2) = 0. \]

Equality (7) shows that

\[ [f_1] = \text{span}\{f_1, Tf_1\}. \]

\[ [f_2] = \text{span}\{f_2, Tf_2\}. \]

So \([f_1] \perp [f_2]. \)
(c) For every \((n, m) \in E_4\), let

\[ M_{n,m} = \text{span}\{z_k^1 z_l^2 : (k, l) \sim_1 (n, m), (k, l) \in E_4\}, \]

\[ M_{n+N, m-M} = \text{span}\{z_k^1 z_l^2 : (k, l) \sim_2 (n + N, m - M), (k, l) \in E_5\}. \]

Then \(M_{n,m}\) and \(M_{n+N, m-M}\) are finite dimension, and the following statements hold:

(i) \(TM_{n,m} = M_{n+N, m-M}\) and \(T^*M_{n+N, m-M} = M_{n,m}\);

(ii) \(T(M_{n,m}) \subset M_{n+N, m-M}\) and \(T^*(M_{n+N, m-M}) \subset M_{n,m}\).

Therefore, \(TP_{n,m}f = Q_{n+N, m-M}f\) and \(P_{n,m}T^*f = T^*Q_{n+N, m-M}f\) for any \(f \in \mathcal{M}\).

(d) By equality (7), conclusion (c) and

(9) \[ T^*TP_{n,m}f = \frac{\gamma_2^2}{\gamma_1^2 n^2} P_{n,m}f, \]

we have

\[ [Q_{n+N, m-M}f] = \text{span}\{Q_{n+N, m-M}f, T^*Q_{n+N, m-M}f\} \]

\[ = \text{span}\{TP_{n,m}f\} \oplus \text{span}\{T^*TP_{n,m}f\} \]

\[ = \text{span}\{TP_{n,m}f\} \oplus \text{span}\{P_{n,m}f\} \]

\[ = [P_{n,m}f]. \]

Similarly, \([Q_{n+N, m-M}f] = [P_{n,m}T^*f]\) comes from equality (8), conclusion (c) and

(10) \[ TT^*Q_{n+N, m-M}f = \frac{\gamma_2^2}{\gamma_1^2 n^2} Q_{n+N, m-M}f. \]

(e) By equalities (9), (10) and conclusion (c), we have

(11) \[ Q_{n+N, m-M}M = TT^*(Q_{n+N, m-M}M) = TP_{n,m}T^*M, \]

\[ P_{n,m}M = T^*T(P_{n,m}M) = T^*Q_{n+N, m-M}TM. \]

Therefore, we only need to show that \(P_{n,m}M \oplus Q_{n+N, m-M}M\) is an invariant subspace of \(T\) and \(T^*\). In fact,

\[ T(P_{n,m}M \oplus Q_{n+N, m-M}M) = TP_{n,m}M = Q_{n+N, m-M}M, \]

where the last equality comes from \(TP_{n,m}f = Q_{n+N, m-M}f \in Q_{n+N, m-M}M\) and \(Q_{n+N, m-M}f \in TP_{n,m}M \subset TP_{n,m}M\) for all \(f \in \mathcal{M}\). Therefore,

\[ T(P_{n,m}M \oplus Q_{n+N, m-M}M) \subset P_{n,m}M \oplus Q_{n+N, m-M}M. \]

Similarly, we can prove that

\[ T^*(P_{n,m}M \oplus Q_{n+N, m-M}M) = T^*Q_{n+N, m-M}M = P_{n,m}M. \]

So we finish the proof. \(\Box\)
Remark 2.2. In the prove of (e), we also get that
\[ [P_{n,m}M] = P_{n,m}M \oplus Q_{n+N,m-M}M = [Q_{n+N,m-M}M], \]
where \([P_{n,m}M]\) and \([Q_{n+N,m-M}M]\) are the reducing subspaces generated by \(P_{n,m}M\) and \(Q_{n+N,m-M}M\), respectively.

**Theorem 2.1.** Let \(M \subset M_2^1\) be a non-zero reducing subspace of \(T\) on the bidisk. Then \(M = M_1 \oplus M_2\), where

(i) \(M_1\) is a direct sum of minimal reducing subspace \([z_1^{p} z_2^{q}]\) with \(z_1^{p} z_2^{q} \in M\) for some \((p,q) \in E_1 \cup E_2 \cup E_3\);  
(ii) \(M_2\) is a direct sum of minimal reducing subspace \([f]\) with \(f \in P_{n,m}M\) for some \((n,m) \in E_4\).

**Proof.** Firstly, we prove that
\[ (12) \quad M = M_1 \bigoplus \bigoplus_{(n,m) \in E} (P_{n,m}M \bigoplus Q_{n+N,m-M}M), \]
where \(M_1 = \bigoplus_{(p,q) \in \Lambda} [z_1^{p} z_2^{q}]\) with \(\Lambda = \{(p,q) \in E_1 \cup E_2 \cup E_3 : z_1^{p} z_2^{q} \in M\}\), and \(E\) is the partition of \(E_3\) by the equivalence \(\sim_1\). Set \(H_{n,m} = P_{n,m}M \bigoplus Q_{n+N,m-M}M\).  

On the one hand, \(M_1 \bigoplus \bigoplus_{(n,m) \in E} H_{n,m} \subset M\), since \(M_1 \subset M\) is a reducing subspace of \(T\), and conclusion (e) in Lemma 2.2 implies that \(\bigoplus_{(n,m) \in E} H_{n,m} \subset M\). On the other hand, for \(g = g_1 + g_2 \in M\) with
\[ (13) \quad g_1(z) = \sum_{(p,q) \in E_1 \cup E_2 \cup E_3} a_{p,q} z_1^{p} z_2^{q}, \quad g_2(z) = \sum_{(p,q) \in E_4 \cup E_5} a_{p,q} z_1^{p} z_2^{q}, \]
Remark 2.1 shows that \(g_1 \in M_1 \subset M\), which implies that \(g_2 = g - g_1 \in M\). Therefore, \(g_2 = \sum_{(n,m) \in E} (P_{n,m}g_2 + Q_{n+N,m-M}g_2) \in \bigoplus_{(n,m) \in E} H_{n,m}\). It follows that \(M\) is in the direct sum of \(M_1\) and \(\{H_{n,m}\}\) with \((n,m) \in E\). So we have equality (12) holds.

Secondly, for each \((n,m) \in E_4\), we prove that \(H_{n,m}\) is the direct sum of minimal reducing subspaces as \([f] = \text{span}\{f, Tf\}\) with \(f \in P_{n,m}M\). There are some steps in the proof.

**Step 1.** Take \(0 \neq f_1 \in P_{n,m}M\). Then \([f_1] = \text{span}\{f_1, Tf_1\} \subset H_{n,m}\).

**Step 2.** If \(P_{n,m}M \neq \mathbb{C} f_1\), take \(0 \neq f_2 \in P_{n,m}M \cap \mathbb{C} f_1\). Then \([f_2] = \text{span}\{f_2, Tf_2\} \subset H_{n,m} \ominus [f_1]\).

**Step 3.** If \(P_{n,m}M \neq \text{span}\{f_1, f_2\}\), take \(0 \neq f_3 \in P_{n,m}M \ominus \text{span}\{f_1, f_2\}\). Then \([f_3] = \text{span}\{f_3, Tf_3\} \subset H_{n,m} \ominus [f_1] \ominus [f_2]\).

If \(P_{n,m}M \neq \text{span}\{f_1, f_2, f_3\}\), continue this process. This process will stop in finite steps, since the dimension of \(H_{n,m}\) is finite. Thus, we finish the proof. \(\square\)
Remark 2.3. In particular, if \( M \) is a reducing subspace generated by \( g = g_1 + g_2 \in \mathcal{A}(\mathbb{D}^2) \) as in (13), then \([g] = [g_1] \oplus [g_2]\) and
\[
[g_2] = \bigoplus_{(n,m) \in E} [P_{n,m}g, Q_{n,N,m-M}g],
\]
where \([P_{n,m}g, Q_{n,N,m-M}g]\) is the reducing subspace generated by \(P_{n,m}g\) and \(Q_{n,N,m-M}g\). By conclusions (a) and (d) in Lemma 2.2 and equalities in (11), we get \([P_{n,m}g, Q_{n,N,m-M}g] = [P_{n,m}g, P_{n,m}T^*g] = \text{span}\{P_{n,m}g, P_{n,m}T^*g\} \oplus \text{span}\{Q_{n,N,m-M}g, Q_{n,N,m-M}Tg\}\).

Notice that \(\text{span}\{P_{n,m}g, P_{n,m}T^*g\}\) has an orthonormal basis \(\{e_1, \ldots, e_k\}\), since the dimension of \(\text{span}\{P_{n,m}g, P_{n,m}T^*g\}\) is finite. Conclusion (b) in Lemma 2.2 shows that \([e_i] \perp [e_j]\) for \(i \neq j\). Then we get
\[
[P_{n,m}g, P_{n,m}T^*g] = \bigoplus_{j=1}^{k} [e_j] = \bigoplus_{j=1}^{k} \text{span}\{e_j, Te_j\}.
\]

Similarly, we can prove that
\[
[g_2] = \bigoplus_{(n,m) \in E} [Q_{n+N,m-M}g, Q_{n+N,m-M}Tg],
\]
and
\[
[Q_{n+N,m-M}g, Q_{n+N,m-M}Tg] = \bigoplus_{j=1}^{l} [h_j] = \bigoplus_{j=1}^{l} \text{span}\{h_j, T^*h_j\},
\]
where \(\{h_1, \ldots, h_l\}\) is an orthonormal basis of \(\text{span}\{Q_{n+N,m-M}g, Q_{n+N,m-M}Tg\}\).

In the last part of this paper, we give some examples of the reducing subspaces of \(T_{z^i\zbar z^j}\) for the case that \(N = M\) and \(N \neq M\), respectively.

Example 2.1. Fix \(a, b, c, d, e \in \mathbb{C}\) with \(e \neq 0\). Let
\[
 f(z_1, z_2) = a z_1^9 z_2^{14} + b z_1^7 z_2^{15} + c z_1^5 z_2^{17} + d z_1^4 z_2^{19} + e z_1^{11} z_2^{12},
\]
and \([f]\) be the reducing subspace of \(T_{z_1^9, z_2^{14}}\) generated by \(f\). Then
\[
 [f] = \text{span}\{f_1, f_2\} \oplus \text{span}\{z_1^{11+10h} z_2^{12-10h} : h = -1, 0, 1\},
\]
where
\[
 f_1(z_1, z_2) = a z_1^9 z_2^{14} + b z_1^7 z_2^{15} + c z_1^5 z_2^{17} + d z_1^4 z_2^{19},
 f_2(z_1, z_2) = \frac{a}{z_1^9} z_2^{14} + \frac{b}{z_1^7} z_2^{15} + \frac{c}{z_1^5} z_2^{17} + \frac{d}{z_1^4} z_2^{19}.
\]

Proof. Notice that \((11, 12) \in \mathbb{E}_3\) and \((9, 14) \in \mathbb{E}_4\). A direct computation shows that \((9, 14) \sim (7, 15) \sim (5, 17) \sim (4, 19)\). Remark 2.1 implies that \(f_1 = P_{4,19} f\) and \(z_1^{11} z_2^{12}\) are in \(M\). As in Remark 2.3, there is \(\text{span}\{P_{4,19} f, P_{4,19} T^* f\} = [f_1] = \text{span}\{f_1, f_2\}\). Therefore we get the desired result. \(\square\)
Example 2.2. Let $f(z_1, z_2) = z_1^4z_2^2 + z_1^2z_2^2 + z_1^2z_2^2$ and $[f]$ be the reducing subspace of $T_{z_1z_2}$ generated by $f$. Then

$$[f] = \text{span}\{z_1^4z_2^2 + z_1^2z_2^2 + \frac{2}{9}z_1^2z_2^2 \oplus \text{span}\{z_1^2z_2^2, z_1z_2^3\}\}.$$

Proof. Notice that $(7, 7) \in E_5$, $(4, 14), (3, 15) \in E_4$ and $(4, 14) \sim_1 (3, 15)$. Let $f_1 = P_{4,14}f = z_1^4z_2^2 + z_1^2z_2^2$ and $f_2 = Q_{7,7}f = z_1z_2^2$. Then $[P_{4,14}f, P_{4,14}T^*f] = [f_1] = \text{span}\{z_1^4z_2^2 + z_1^2z_2^2, \frac{2}{9}z_1^2z_2^2 + \frac{2}{9}z_1^2z_2^2\}$, $[P_{2,17}f, P_{2,17}T^*f] = [Q_{7,7}f, Q_{7,7}T^*f] = [f_2] = \text{span}\{z_1z_2^2, z_1z_2^3\}$. Then we finish the proof. □

Example 2.3. Let $f(z_1, z_2) = z_1^2z_2^2 + z_1^2z_2^2$, and $[f]$ be the reducing subspace of $T_{z_1z_2}$ generated by $f$. Then

$$[f] = \text{span}\{z_1^2z_2^2, z_1^2z_2^2\}.$$

Proof. Notice that $(3, 8) \in E_4$, $(3, 3) \in E_5$. It is easy to check that $T_{z_1z_2}z_1^2z_2^2 = \frac{1}{2}z_1z_2^2$ and $T_{z_1z_2}z_1^2z_2^2 = \frac{1}{2}z_1z_2^2$. So $[z_1^2z_2^2] = [z_1z_2^2] = \text{span}\{z_1^2z_2^2, z_1z_2^3\}$. It means that $[f] = \text{span}\{z_1^2z_2^2, z_1z_2^3\}$. □

Example 2.4. Let $f(z_1, z_2) = z_1^2z_2^2 + z_1^2z_2^2 + z_1^2z_2^2 + z_1^2z_2^2 + z_1^2z_2^2$ and $[f]$ be the reducing subspace of $T_{z_1z_2}$ generated by $f$. Then

$$[f] = [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus [z_1^2z_2^2].$$

Proof. Notice that $(2, 17), (4, 14), (3, 15) \in E_4$, $(9, 4), (8, 5) \in E_5$ and $(4, 14) \sim_1 (3, 15), (9, 4) \sim_2 (8, 5).

(i) Since $P_{4,14}T^*f = T^*(z_1^2z_2^2 + z_1^2z_2^2) = \frac{1}{2}z_1z_2^2 + \frac{1}{2}z_1^2z_2^2$, we have

$$\text{span}\{P_{4,14}f, P_{4,14}T^*f\} = \text{span}\{z_1^2z_2^2, z_1^2z_2^2\}.$$

Therefore,

$$[f] = [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus \text{span}\{z_1^2z_2^2, z_1^2z_2^2\} \oplus \text{span}\{z_1^2z_2^2, z_1^2z_2^2\} \oplus \text{span}\{z_1^2z_2^2, z_1^2z_2^2\} \oplus \text{span}\{z_1^2z_2^2, z_1^2z_2^2\}.$$

(ii) It is easy to check that $(z_1^2z_2^2 - \frac{14}{25}z_1^3z_2^2, z_1^2z_2^2 + z_1^2z_2^2) = 0$ and

$$\text{span}\{P_{4,14}f, P_{4,14}T^*f\} = \text{span}\{z_1^2z_2^2, z_1^2z_2^2, z_1^2z_2^2, z_1^2z_2^2\}.$$

So $[f] = [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus [z_1^2z_2^2]$.

(iii) Notice that

$$\text{span}\{Q_{9,4}f, Q_{9,4}T^*f\} = \text{span}\{z_1^2z_2^2, z_1^2z_2^2, \frac{2}{9}z_1^2z_2^2 + z_1^2z_2^2\} \oplus \text{span}\{z_1^2z_2^2, z_1^2z_2^2, \frac{2}{9}z_1^2z_2^2 + z_1^2z_2^2\},$$

where $z_1^2z_2^2 - \frac{27}{25}z_1^2z_2^2 \perp Q_{9,4}f$. Then

$$[f] = [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus [z_1^2z_2^2] \oplus [z_1^2z_2^2] - \frac{27}{25}z_1^2z_2^2.$$
Remark 2.4. In Example 2.4, since $T^*(z_1^2 z_2^4 + z_1^8 z_2^5) = \frac{1}{2} z_1^4 z_2^4 + \frac{4}{5} z_1^4 z_2^{15}$ and $T^*(z_1^2 z_2^4 - \frac{27}{5} z_1^8 z_2^5) = \frac{1}{2} z_1^4 z_2^4 - \frac{12}{5} z_1^4 z_2^{15}$, conclusion (d) in Lemma 2.2 implies that $[f] = [z_1^4 z_2^4] = [\frac{1}{2} z_1^4 z_2^4 + \frac{4}{5} z_1^4 z_2^{15}]$.

Moreover, let $T = T_{z_1^2 z_2^4}$ and $g = z_1^4 z_2^4 + z_1^8 z_2^5 + z_1^{14} z_2$, then $[g] = [g + a z_1^8 z_2^5] = [z_1^4 z_2^4 + z_1^8 z_2^5]$ for $a \neq \frac{9}{5}$. In fact, span$\{P_{4,14}(g + a z_1^8 z_2^5)\}$, $P_{4,14} T^*(g + a z_1^8 z_2^5)$, span$\{z_1^4 z_2^4, z_1^8 z_2^5\}$, since $T^*(z_1^4 z_2^4 + a z_1^8 z_2^5)$ and $z_1^4 z_2^4 + z_1^8 z_2^5$ are linearly independent.

For the case that $a = \frac{9}{5}$, we have $\{g + \frac{9}{5} z_1^8 z_2^5\} = \text{span} \{z_1^4 z_2^4 + z_1^8 z_2^5, z_1^4 z_2^4 + \frac{9}{5} z_1^8 z_2^5\}$, span$\{z_1^4 z_2^4, z_1^8 z_2^5\}$, since $T^*(z_1^4 z_2^4 + \frac{9}{5} z_1^8 z_2^5)$ and $z_1^4 z_2^4 + \frac{9}{5} z_1^8 z_2^5$ are linearly independent.

Acknowledgments. The authors thank the reviewer very much for his helpful suggestions which led to the present version of this paper.

References


Mohammed Albaseer  
School of Mathematical Sciences  
Dalian University of Technology  
Dalian 116024, P. R. China  
E-mail address: amn@mail.net.sa

Yufeng Lu  
School of Mathematical Sciences  
Dalian University of Technology  
Dalian 116024, P. R. China  
E-mail address: lyfdlut@dlut.edu.cn

Yanyue Shi  
School of Mathematical Sciences  
Ocean University of China  
Qingdao 266100, P. R. China  
E-mail address: shiyanyue@163.com, shiyanyue@gmail.com