

NONLINEAR BIHARMONIC EQUATION WITH POLYNOMIAL GROWTH NONLINEAR TERM

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ABSTRACT. We investigate the existence of solutions of the nonlinear biharmonic equation with variable coefficient polynomial growth nonlinear term and Dirichlet boundary condition. We get a theorem which shows that there exists a bounded solution and a large norm solution depending on the variable coefficient. We obtain this result by variational method, generalized mountain pass geometry and critical point theory.

1. Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. Let Δ be the elliptic operator and Δ^2 be the biharmonic operator. Choi and Jung [3] showed that the problem

$$\begin{aligned}\Delta^2 u + c\Delta u &= bu^+ + s && \text{in } \Omega, \\ u &= 0, \quad \Delta u = 0 && \text{on } \partial\Omega\end{aligned}\tag{1.1}$$

has at least two nontrivial solutions when $(c < \lambda_1, \lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$) or $(\lambda_1 < c < \lambda_2, b < \lambda_1(\lambda_1 - c)$ and $s > 0$).

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We obtained these results by using variational reduction method. Jung and Choi [5] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, (1.1) has at least three nontrivial solutions by using degree theory. Tarantello [10] also studied

$$\begin{aligned} \Delta^2 u + c\Delta u &= b((u + 1)^+ - 1), \\ u = 0, \quad \Delta u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1.2)$$

She showed that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then (1.4) has a negative solution. She obtained this result by degree theory. Micheletti and Pistoia [8] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$ then (1.2) has at least four solutions by variational linking theorem and Leray-Schauder degree theory.

In this paper we consider the following nonlinear biharmonic equation with Dirichlet boundary condition

$$\begin{aligned} \Delta^2 u + c\Delta u &= a(x)g(u) \quad \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1.3)$$

where we assume that $c \in R$ is not an eigenvalue of $-\Delta$ and that $a : \overline{\Omega} \rightarrow R$ is a continuous function which changes sign in Ω .

We assume that g satisfies the following conditions:

(g1) $g \in C(R, R)$,

(g2) there are constants $a_1, a_2 \geq 0$ such that

$$|g(u)| \leq a_1 + a_2|u|^{\mu-1},$$

where $2 < \mu < \frac{2n}{n-2}$ if $n \geq 3$.

(g3) there exists a constant $r_0 \geq 0$ such that

$$0 < \mu G(\xi) = \mu \int_0^\xi g(t)dt \leq \xi g(\xi) \quad \text{for } |\xi| \geq r_0.$$

(g4) $g(u) = o(|u|)$ as $u \rightarrow 0$.

We note that (g3) implies the existence of the positive constants a_3, a_4, a_5 such that

$$\frac{1}{\mu}(\xi g(\xi) + a_3) \geq G(\xi) + a_4 \geq a_5|\xi|^\mu \quad \text{for } \xi \in R. \quad (1.4)$$

Khanfir and Lassoued [6] showed the existence of at least one solution for the nonlinear elliptic boundary problem when g is locally Hölder continuous on R_+ .

We are trying to find the weak solutions of (1.3), that is,

$$\int_{\Omega} ((\Delta^2 u + c\Delta u - a(x)g(u))v) dx = 0 \quad \text{for } v \in H,$$

where the space H is introduced in section 2. Let us set

$$\Omega^+ = \{x \in \Omega | a(x) > 0\}, \quad \Omega^- = \{x \in \Omega | a(x) < 0\}$$

and let

$$a^+ = a \cdot \chi_{\Omega^+}, a^- = -a \cdot \chi_{\Omega^-}.$$

Since $a(x)$ changes sign, the open subsets Ω^+ and Ω^- are nonempty. Now we can write $a = a^+ - a^-$. Our main results are as follows:

THEOREM A . *Assume that $\lambda_k < c < \lambda_{k+1}$, g satisfies (g1)-(g4) and $g(u)u - \mu G(u)$ is bounded. Then (1.3) has at least one bounded solution.*

THEOREM B. *Assume that $\lambda_k < c < \lambda_{k+1}$, g satisfies (g1)-(g4), $g(u)u - \mu G(u)$ is not bounded and there exists a small $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x) < \epsilon$. Then (1.3) has at least two solutions, (i) one of which is bounded and (ii) the other solution of which is large norm such that*

$$\max_{x \in \Omega} |u(x)| > M \quad \text{for some } M > 0.$$

In Section 2, we prove that $I(u)$ is continuous and *Fréchet* differentiable and satisfies the (P.S.) condition. In Section 3, we prove Theorem **A** . In Section 4, we prove Theorem **B** by variational method, generalized mountain pass geometry and critical point theory.

2. Eigenspaces and Palais-Smale condition

The eigenvalue problem with Dirichlet boundary condition

$$\begin{aligned} \Delta u + \lambda u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega \end{aligned}$$

has infinitely many eigenvalues λ_k , $k \geq 1$ and corresponding eigenfunctions ϕ_k , $k \geq 1$, the suitably normalized with respect to $L^2(\Omega)$ inner product, where each eigenvalue λ_k is repeated as often as its multiplicity. The eigenvalue problem

$$\begin{aligned} \Delta^2 u + c\Delta u &= \Lambda u & \text{in } \Omega, \\ u = 0, \quad \Delta u &= 0 & \text{on } \partial\Omega \end{aligned}$$

has also infinitely many eigenvalues $\lambda_k(\lambda_k - c)$, $k \geq 1$ and corresponding eigenfunctions ϕ_k , $k \geq 1$. We note that $\lambda_1(\lambda_1 - c) \leq \lambda_2(\lambda_2 - c) \leq \dots \rightarrow +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

Let $L^2(\Omega)$ be a square integrable function space defined on Ω . Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with} \quad \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows

$$H = \{u \in L^2(\Omega) \mid \sum |\lambda_k(\lambda_k - c)| < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\| = \left[\sum |\lambda_k(\lambda_k - c)| h_k^2 \right]^{\frac{1}{2}}.$$

Since $\lambda_k \rightarrow +\infty$ and c is fixed, we have

- (i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
 - (ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
 - (iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,
- which is proved in [2].

Let

$$H_+ = \{u \in H \mid h_k = 0 \text{ if } \lambda_k(\lambda_k - c) < 0\},$$

$$H_- = \{u \in H \mid h_k = 0 \text{ if } \lambda_k(\lambda_k - c) > 0\}.$$

Then $H = H_- \oplus H_+$, for $u \in H$, $u = u^- + u^+ \in H_- \oplus H_+$. Let P_+ be the orthogonal projection on H_+ and P_- be the orthogonal projection on H_- . We can write $P_+u = u^+$, $P_-u = u^-$, for $u \in H$.

We are looking for the weak solutions of (1.1). The weak solutions of (1.1) coincide with the critical points of the associated functional

$$I(u) \in C^1(H, R),$$

$$\begin{aligned} I(u) &= \int_{\Omega} \left[\frac{1}{2} |\Delta u|^2 - \frac{c}{2} |\nabla u|^2 \right] dx - \int_{\Omega} a(x) G(u) dx \quad (2.1) \\ &= \frac{1}{2} (\|P_+u\|^2 - \|P_-u\|^2) - \int_{\Omega} a(x) G(u) dx. \end{aligned}$$

By (g1) and (g2), I is well defined. By the following Proposition 2.1, $I \in C^1(H, R)$ and I is *Fréchet* differentiable in H :

PROPOSITION 2.1. *Assume that $\lambda_k < c < \lambda_{k+1}$, $k \geq 1$, and g satisfies (g1) – (g4). Then $I(u)$ is continuous and Fréchet differentiable in H with Fréchet derivative*

$$\nabla I(u)h = \int_{\Omega} [\Delta u \cdot \Delta h - c \nabla u \cdot \nabla h - a(x)g(u)h] dx. \quad (2.2)$$

If we set

$$K(u) = \int_{\Omega} a(x)G(u)dx,$$

then $K'(u)$ is continuous with respect to weak convergence, $K'(u)$ is compact, and

$$K'(u)h = \int_{\Omega} a(x)g(u)h dx \quad \text{for all } h \in H.$$

This implies that $I \in C^1(H, R)$ and $K(u)$ is weakly continuous.

The proof of Proposition 2.1 has the same process as that of the proof in Appendix B in [9].

PROPOSITION 2.2. *(Palais-Smale condition)*
 Assume that $\lambda_k < c < \lambda_{k+1}$, $k \geq 1$, g satisfies (g1) – (g4) and $f \in L^2(\Omega)$. We also assume that $g(u)u - \mu G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. Then $I(u)$ satisfies the Palais-Smale condition.

Proof. We assume that $g(u)u - \mu G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. Suppose that (u_m) is a sequence with $I(u_m) \leq M$ and $I'(u_m) \rightarrow 0$ as $m \rightarrow \infty$. Then by (g2), (g3), and Hölder inequality and Sobolev Embedding Theorem, for large m and $\mu > 2$ with

$u = u_m$, we have

$$\begin{aligned}
 M + \frac{1}{2}\|u\| &\geq I(u) - \frac{1}{2}I'(u)u = \int_{\Omega} \left[\frac{1}{2}a(x)g(u)u - a(x)G(u) \right] dx \\
 &= \int_{\Omega} a^+(x) \left[\frac{1}{2}g(u)u - G(u) \right] dx - \int_{\Omega} a^-(x) \left[\frac{1}{2}g(u)u - G(u) \right] dx \\
 &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \mu \int_{\Omega} a^+(x) \cdot G(u) dx \\
 &\quad - \max_{\Omega} \left| \frac{1}{2}g(u)u - G(u) \right| \int_{\Omega^-} a^-(x) dx \\
 &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \mu \int_{\Omega} a^+(x) \cdot (a_3|u|^{\mu} - a_4) dx \\
 &\quad - \max_{\Omega} \left| \frac{1}{2}g(u)u - G(u) \right| \int_{\Omega^-} a^-(x) dx.
 \end{aligned}$$

Thus if $\frac{1}{2}g(u)u - G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x) < \epsilon$, then we have

$$1 + \|u\| \geq M_1 \int_{\Omega} |u|^{\mu} \geq M_2 \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}\mu}. \quad (2.3)$$

Moreover since

$$|I'(u_m)\varphi| \leq \|\varphi\| \quad (2.4)$$

for large m and all $\varphi \in H$, choosing $\varphi = u_m^+ \in H_+$ gives

$$\begin{aligned}
 \|u_m^+\|^2 &= \int_{\Omega} (\Delta^2 u_m + c\Delta u_m) \cdot u_m^+ \\
 &= \int_{\Omega} a(x)g(u_m)u_m^+ \\
 &\leq \int_{\Omega} |a(x)||g(u_m)||u_m| \\
 &\leq \|a\|_{\infty} \int_{\Omega} (a_1|u_m|^{\mu} + a_2|u_m|) \\
 &\leq C_1 \int_{\Omega} |u_m|^{\mu} + C_2 \|u_m\|_{L^2(\Omega)} \\
 &\leq C_1 \int_{\Omega} |u_m|^{\mu} + C'_2 \|u_m\|.
 \end{aligned}$$

Taking $\varphi = -u_m^-$ in (2.4) yields

$$\begin{aligned} \|u_m^-\|^2 &= \int_{\Omega} (\Delta^2 u_m + c\Delta u_m) \cdot (-u_m^-) \\ &= \int_{\Omega} a(x)g(u_m) \cdot (-u_m^-) \\ &\leq \int_{\Omega} |a(x)||g(u_m)||u_m| \\ &\leq \|a\|_{\infty} \int_{\Omega} (a_1|u_m|^{\mu} + a_2|u_m|) \\ &\leq C_3 \int_{\Omega} |u_m|^{\mu} + C_4 \|u_m\|_{L^2(\Omega)} \\ &\leq C_3 \int_{\Omega} |u_m|^{\mu} + C'_4 \|u_m\|. \end{aligned}$$

Thus, by (2.3), we have

$$\begin{aligned} \|u_m\|^2 = \|u_m^+\|^2 + \|u_m^-\|^2 &\leq M_3 \int_{\Omega} |u_m|^{\mu} + M_4 \|u_m\| \\ &\leq M_5 (1 + \|u_m\|) + M_4 \|u_m\| \leq M_6 (1 + \|u_m\|), \end{aligned}$$

from which the boundedness of (u_m) follows. Thus (u_m) converges weakly in H . Since $P_{\pm}I'(u_m) = \pm P_{\pm}u_m + P_{\pm}\tilde{\mathcal{P}}(u_m)$ with $\tilde{\mathcal{P}}$ compact and the weak convergence of $P_{\pm}u_m$ imply the strong convergence of $P_{\pm}u_m$ and hence (PS) condition holds. \square

3. At least one bounded solution

We shall show that $I(u)$ satisfies generalized mountain pass geometrical assumptions.

We recall generalized mountain pass geometry:

Let $H = V \oplus X$, where H is a real Banach space and $V \neq \{0\}$ and is finite dimensional. Suppose that $I \in C^1(H, R)$, satisfies $(P.S.)$ condition, and

- (i) there are constants $\rho, \alpha > 0$ and a bounded neighborhood B_{ρ} of 0 such that $I|_{\partial B_{\rho} \cap X} \geq \alpha$,
- (ii) there is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q = (\bar{B}_R \cap V) \oplus \{re \mid 0 < r < R\}$, then $I|_{\partial Q} \leq 0$.

Then I possesses a critical value $b \geq \alpha$. Moreover b can be characterized as

$$b = \inf_{\gamma \in \Gamma} \max_{u \in Q} I(\gamma(u)),$$

where

$$\Gamma = \{\gamma \in C(\bar{Q}, H) \mid \gamma = id \text{ on } \partial Q\}.$$

Let $H_k = \text{span}\{\phi_1, \dots, \phi_k\}$. Then H_k is a subspace of H such that

$$H = \bigoplus_{k \in N} H_k \quad \text{and} \quad H = H_k \oplus H_k^\perp.$$

Let

$$B_r = \{u \in H \mid \|u\| \leq r\},$$

$$Q = (\bar{B}_R \cap H_k) \oplus \{re \mid 0 < r < R\}.$$

We have the following generalized mountain pass geometrical assumptions:

- LEMMA 3.1. Assume that $\lambda_k < c < \lambda_{k+1}$ and g satisfies (g1)–(g4). Then
- (i) there are constants $\rho > 0$, $\alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho \cap H_k^\perp} \geq \alpha$, and
 - (ii) there is an $e \in \partial B_1 \cap H_k^\perp$ and $R > \rho$ such that if $Q = (\bar{B}_R \cap H_k) \oplus \{re \mid 0 < r < R\}$, then $I|_{\partial Q} \leq 0$, and
 - (iii) there exists $u_0 \in H$ such that $\|u_0\| > \rho$ and $I(u_0) \leq 0$.

Proof. (i) Let $u \in H_k^\perp$. We note that

$$\text{if } u \in H_k^\perp, \int_{\Omega} (\Delta^2 u + c\Delta u)u dx \geq \lambda_{k+1}(\lambda_{k+1} - c)\|u\|_{L^2(\Omega)}^2 > 0.$$

Thus by (g3), (1.2) and the Hölder inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|P_+u\|^2 - \frac{1}{2}\|P_-u\|^2 - \int_{\Omega} a(x)G(u) \\ &\geq \frac{1}{2}\|P_+u\|^2 - \|a\|_{\infty} \int_{\Omega} C_1|u|^{\mu} \\ &\geq \frac{1}{2}\|P_+u\|^2 - \|a\|_{\infty} C'_1\|u\|^{\mu} \end{aligned}$$

for $C_1, C'_1 > 0$. Since $\mu > 2$, there exist $\rho > 0$ and $\alpha > 0$ such that if $u \in \partial B_\rho$, then $I(u) \geq \alpha$.

(ii) Let $u \in (\bar{B}_r \cap H_k) \oplus \{re \mid 0 < r\}$. Then $u = v + w$, $v \in B_r \cap H_k$, $w = re$. We note that

$$\text{if } v \in H_k, \int_{\Omega} (\Delta^2 v + c\Delta v)v dx \leq \lambda_k(\lambda_k - c)\|v\|_{L^2(\Omega)}^2 < 0.$$

Thus we have

$$\begin{aligned} I(u) &= \frac{1}{2}r^2 - \frac{1}{2}\|P_-v\|^2 - \int_{\Omega} a(x)G(v + re) \\ &\leq \frac{1}{2}r^2 + \frac{1}{2}(\lambda_k(\lambda_k - c))\|v\|_{L^2(\Omega)}^2 - \int_{\Omega^+} a(x)(a_5|v + re|^{\mu} - a_4) \end{aligned}$$

Since $\mu > 2$, there exists $R > 0$ such that if $u \in Q = (\bar{B}_R \cap H_k) \oplus \{re \mid 0 < r < R\}$, then $I(u) < 0$

(iii) If we choose $\psi \in H$ such that $\|\psi\| = 1$, $\psi \geq 0$ in Ω and $\text{supp}(\psi) \subset \Omega^+$, then we have

$$\begin{aligned} I(t\psi) &\leq \frac{1}{2}\|P_+(t\psi)\|^2 - \frac{1}{2}\|P_-(t\psi)\|^2 - \int_{\Omega^+} a(x)(a_3t^{\mu}\psi^{\mu} - a_4) \\ &\leq \frac{1}{2}\|t\psi\|^2 - \int_{\Omega^+} a(x)(a_3t^{\mu}\psi^{\mu} - a_4) \\ &= \frac{1}{2}t^2 - \int_{\Omega^+} a(x)(a_3t^{\mu}\psi^{\mu} - a_4) \end{aligned}$$

for all $t > 0$. Since $\mu > 2$, for t_0 great enough, $u_0 = t_0\psi$ is such that $\|u_0\| > \rho$ and $I(u_0) \leq 0$. □

THEOREM A . Assume that $\lambda_k < c < \lambda_{k+1}$, g satisfies (g1)-(g4) and $g(u)u - \mu G(u)$ is bounded. Then (1.3) has at least one bounded solution.

Proof. By Proposition 2.1 and Proposition 2.2, $I(u) \in C^1(H, \mathbb{R})$ and satisfies the Palais-Smale condition. By Lemma 3.1, there are constants $\rho > 0$, $\alpha > 0$ and a bounded neighborhood B_{ρ} of 0 such that $I|_{\partial B_{\rho} \cap H_m^{\perp}} \geq \alpha$, and there is an $e \in \partial B_1 \cap H_k^{\perp}$ and $R > \rho$ such that if $Q = (\bar{B}_R \cap H_k) \oplus \{re \mid 0 < r < R\}$, then $I|_{\partial Q} \leq 0$, and there exists $u_0 \in H$ such that $\|u_0\| > \rho$ and $I(u_0) \leq 0$. By the generalized mountain pass theorem, $I(u)$ has a critical value $b \geq \alpha$. Moreover b can be characterized as

$$b = \inf_{\gamma \in \Gamma} \max_{u \in Q} I(\gamma(u)),$$

where

$$\Gamma = \{\gamma \in C(\bar{Q}, H) \mid \gamma = id \text{ on } \partial Q\}.$$

We denote by \tilde{u} a critical point of I such that $I(\tilde{u}) = b$. We claim that there exists a constant $C > 0$ such that

$$\|a^+(x)^{\frac{1}{\mu}}\tilde{u}\|_{L^2(\Omega)} \leq C \left(1 + L \int_{\Omega^-} a^-(x)dx\right)^{\frac{1}{\mu}},$$

where $L = \max_{\Omega} |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|$.

In fact, we have

$$b \leq \max I(tu_0), \quad 0 \leq t \leq 1,$$

and

$$\begin{aligned} I(tu_0) &= t^2 \left(\frac{1}{2}\|P_+u_0\|^2 - \frac{1}{2}\|P_-u_0\|^2 \right) - \int_{\Omega} a(x)G(tu_0)dx \\ &\leq t^2\|u_0\|^2 - \int_{\Omega} a^+(x)G(tu_0)dx + \int_{\Omega} a^-(x)G(tu_0)dx \\ &\leq t^2\|u_0\|^2 - a_3t^\mu \int_{\Omega} a^+(x)u_0^\mu + a_4 \int_{\Omega} a^+(x) + a_5t^\mu \int_{\Omega} a^-(x)u_0^\mu \\ &= Ct^2 - Ct^\mu + C + C't^\mu. \end{aligned}$$

Since $0 \leq t \leq 1$, b is bounded: $b < \tilde{C}$.

We can write

$$\begin{aligned} b &= I(\tilde{u}) - \frac{1}{2}I'(\tilde{u})\tilde{u} \\ &= \int_{\Omega} a(x) \left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u}) \right) dx \\ &= \int_{\Omega} a^+(x) \left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u}) \right) dx - \int_{\Omega} a^-(x) \left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u}) \right) dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \int_{\Omega} a^+(x)g(\tilde{u})\tilde{u} - \max_{\Omega} |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})| \int_{\Omega^-} a^-(x)dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \mu \int_{\Omega} a^+(x) (a_3|\tilde{u}|^\mu - a_4) - L \int_{\Omega^-} a^-(x)dx, \end{aligned}$$

where $L = \max_{\Omega} |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|$. Thus we have

$$\begin{aligned} C \left(1 + L \int_{\Omega^-} a^-(x)dx\right) &\geq \int_{\Omega} a^+(x)|\tilde{u}|^\mu \\ &\geq \left[\int_{\Omega} \left(a^+(x)^{\frac{1}{\mu}}|\tilde{u}| \right)^2 \right]^{\frac{\mu}{2}}, \end{aligned} \tag{3.1}$$

from which we can conclude that \tilde{u} is bounded. In fact, suppose that \tilde{u} is not bounded. Then for any $R > 0$, $|\tilde{u}| \geq R$. Thus we have

$$\int_{\Omega} a^+(x)|\tilde{u}|^\mu \geq R^\mu \int_{\Omega} a^+(x)dx$$

for any R , which contradicts to the fact (3.1) and the proof of theorem is complete. \square

4. At least two solutions

THEOREM B. *Assume that $\lambda_k < c < \lambda_{k+1}$, g satisfies (g1)-(g4), $g(u)u - \mu G(u)$ is not bounded and there exists a small $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x) < \epsilon$. Then (1.3) has at least two solutions, (i) one of which is bounded and (ii) the other solution of which is large norm such that*

$$\max_{x \in \Omega} |u(x)| > M \quad \text{for some } M > 0.$$

Proof. Assume that $\frac{1}{2}g(u)u - G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x, t) < \epsilon$. By Proposition 2.1 and Proposition 2.2, $I \in C^1(H, \mathbb{R})$ and satisfies the Palais-Smale condition. By Lemma 3.1 and generalized mountain pass theorem, $I(u)$ has a critical value b with critical point \tilde{u} such that $I(\tilde{u}) = b$. If $\int_{\Omega^-} a^-(x)dx$ is sufficiently small, by (3.1), we have

$$\int_{\Omega} a^+(x)|\tilde{u}|^\mu \leq C$$

for $C > 0$, from which we can conclude that \tilde{u} is bounded and the proof of (i) is complete. \square

Next we shall prove (ii). We may assume that $R_n < R_{n+1}$ for all $n \in \mathbb{N}$. Let us set $D_n = B_{R_n} \cap H_n$, $\partial D_n = \partial B_{R_n} \cap H_n$.

LEMMA 4.1. *Assume that g satisfies (g1)-(g4). Then there exists an $R_n > 0$ such that*

$$I(u) \leq 0 \quad \text{for } u \in H_n \setminus B_{R_n}, \tag{4.1}$$

where $B_{R_n} = \{u \in H \mid \|u\| \leq R_n\}$.

Proof. Let us choose $\psi \in H$ such that $\|\psi\| = 1$, $\psi \geq 0$ in Ω and $\text{supp}(\psi) \subset \Omega^+$. Then, by (g3), (1.2) and the Hölder inequality, we have

$$\begin{aligned} I(t\psi) &= \frac{1}{2}\|P_+t\psi\|^2 - \frac{1}{2}\|P_-t\psi\|^2 - \int_{\Omega} a(x)G(t\psi) \\ &\leq \frac{1}{2}t^2 - \|a\|_{\infty} \int_{\Omega} C_1t^{\mu}\psi^{\mu} + \|a\|_{\infty}a_1t \\ &\leq \frac{1}{2}t^2 - t^{\mu}\|a\|_{\infty}C'_1\psi^{\mu} + \|a\|_{\infty}a_1t \end{aligned}$$

for $C_1, C'_1 > 0$. Since $\mu > 2$, there exist t_n great enough for each n and an $R_n > 0$ such that $u_n = t_n\psi$ and $I(u_n) < 0$ if $u_n \in H_n \setminus B_{R_n}$ and $\|u_n\| > R_n$, so the lemma is proved \square

Let us set

$$\Gamma_n = \{\gamma \in C([0, 1], H) \mid \gamma(0) = 0 \text{ and } \gamma(1) = u_n\}$$

and

$$b_n = \inf_{\gamma \in \Gamma_n} \max_{[0,1]} I(\gamma(u)) \quad n \in N.$$

Proof of THEOREM B (ii).

We assume that $g(u)u - \mu G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)dx < \epsilon$. By Proposition 2.1 and Proposition 2.2, $I \in C^1(H, R)$ and satisfies the Palais-Smale condition. By Lemma 4.1, there exists an $R_n > 0$ such that $I(u_m) \leq 0$ for $u_n \in H_n \setminus B_{R_n}$. We note that $I(0) = 0$. By Lemma 4.1 and the generalized mountain pass theorem, for n large enough $b_n > 0$ is a critical value of I and $\lim_{n \rightarrow \infty} b_n = +\infty$. Let \tilde{u}_n be a critical point of I such that $I(\tilde{u}_n) = b_n$. Then for each real number M , $\max_{\Omega} |\tilde{u}_n(x)| \geq M$. In fact, by contradiction, $\Delta^2 u + c\Delta u = a(x)g(u)$ and $\max_{\Omega} |\tilde{u}_n(x)| \leq K$ imply that

$$I(\tilde{u}_n) \leq \max_{|\tilde{u}_n| \leq K} \left(\frac{1}{2}g(\tilde{u}_n)\tilde{u}_n - G(\tilde{u}_n) \right) \int_{\Omega} |a(x)|,$$

which means that b_n is bounded. This is absurd to the fact that $\lim_{n \rightarrow \infty} b_n = +\infty$. Thus we complete the proof.

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