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HARMONIC MAPPING RELATED WITH THE MINIMAL SURFACE GENERATED BY ANALYTIC FUNCTIONS

Sook Heui Jun

Abstract. In this paper we consider the meromorphic function $G(z)$ with a pole of order 1 at $-a$ and analytic function $F(z)$ with a zero $-a$ of order 2 in $\mathbb{D} = \{z : |z| < 1\}$, where $-1 < a < 1$. From these functions we obtain the regular simply-connected minimal surface $S = \{(u(z), v(z), H(z)) : z \in \mathbb{D}\}$ in E^3 and the harmonic function $f = u + iv$ defined on $\mathbb{D}$, and then we investigate properties of the minimal surface S and the harmonic function f.

1. Introduction

Let $G(z)$ be an arbitrary meromorphic function in $\mathbb{D} = \{z : |z| < 1\}$ and $F(z)$ be an analytic function in $\mathbb{D}$ having the property that at each point where $G(z)$ has a pole of order n, $F(z)$ has a zero of order at least $2n$. Then the functions

$$\phi_1 = \frac{1}{2}F(1 - G^2),\ \phi_2 = \frac{i}{2}F(1 + G^2),\ \phi_3 = FG$$

are analytic in $\mathbb{D}$ and satisfy

$${\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2 = 0.$$

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Every simply-connected minimal surface S in E^3 has the representation of the form

$$S = \{(u(z), v(z), H(z)) : z \in \mathbb{D}\}$$

where

$$\begin{aligned} u(z) &= Re\left\{ \int_0^z \phi_1(z)dz \right\} + c_1, \\ v(z) &= Re\left\{ \int_0^z \phi_2(z)dz \right\} + c_2, \\ H(z) &= Re\left\{ \int_0^z \phi_3(z)dz \right\} + c_3 \end{aligned}$$

are harmonic in $\mathbb{D} = \{z : |z| < 1\}$.

The coordinates $u(z)$ and $v(z)$ are real harmonic in $\mathbb{D}$, and therefore $f = u + iv$ is harmonic in $\mathbb{D}$. The integral is taken along an arbitrary path from the origin to the point z. The surface will be regular if F satisfies the further property that it vanishes only at the poles of G, and the order of its zero at such a point is exactly twice the order of the pole of G [7, Lemmas 8.1 and 8.2].

In this paper we consider the meromorphic function $G(z) = \frac{i(az+1)}{z+a}$ with a pole of order 1 at $-a$ and analytic function $F(z) = \frac{(z+a)^2}{(1-z)^4}$ with a zero $-a$ of order 2 in $\mathbb{D}$, where $-1 < a < 1$. From these functions we obtain the regular simply-connected minimal surface

$$S = \{(u(z), v(z), H(z)) : z \in \mathbb{D}\}$$

in E^3 where

$$\begin{aligned} u(z) =& Re\left\{ \frac{(1+a)^2 z^3}{3(1-z)^3} + \frac{(1+a^2+2az)z}{2(1-z)^2} \right\} + c_1, \\ v(z) =& Re\left\{ \frac{i(a^2-1)z}{2(1-z)^2} \right\} + c_2, \\ H(z) =& Re\left\{ \frac{i(1+a)^2 z^3}{3(1-z)^3} + \frac{i\left[(1+a^2)z + 2a\right] z}{2(1-z)^2} \right\} + c_3, \end{aligned}$$

and the harmonic function $f = u + iv$ defined on $\mathbb{D}$, and then we investigate properties of the minimal surface S and the harmonic function

$$f(z) = \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$$

where $a_k = \frac{1}{12}\left\{(1+a)^2k^2 + 3(1-a^2)k + 2(a^2-a+1)\right\}$ and $b_k = \frac{1}{12}\left\{(1+a)^2k^2 - 3(1-a^2)k + 2(a^2-a+1)\right\}$.

2. The minimal surface and harmonic function

Let us consider the meromorphic function $G(z) = \frac{i(az+1)}{z+a}$ with a pole of order 1 at $-a$ and analytic function $F(z) = \frac{(z+a)^2}{(1-z)^4}$ with a zero $-a$ of order 2 in $\mathbb{D}$, where $-1 < a < 1$. Then

$$\begin{aligned}
\phi_1 &= \frac{1}{2}F(1-G^2) = \frac{(z+a)^2 + (az+1)^2}{2(1-z)^4},\\
\phi_2 &= \frac{i}{2}F(1+G^2) = \frac{i(a^2-1)(z+1)}{2(1-z)^3},\\
\phi_3 &= FG = \frac{i(z+a)(az+1)}{(1-z)^4}
\end{aligned}$$

are analytic in $\mathbb{D}$. The relevant integrals are

$$\begin{aligned}
\int_0^z \phi_1(z)dz &= \int_0^z \frac{(z+a)^2 + (az+1)^2}{2(1-z)^4}dz = \frac{(1+a)^2z^3}{3(1-z)^3} + \frac{(1+a^2+2az)z}{2(1-z)^2},\\
\int_0^z \phi_2(z)dz &= \int_0^z \frac{i(a^2-1)(z+1)}{2(1-z)^3}dz = \frac{i(a^2-1)z}{2(1-z)^2},\\
\int_0^z \phi_3(z)dz &= \int_0^z \frac{i(z+a)(az+1)}{(1-z)^4}dz = \frac{i(1+a)^2z^3}{3(1-z)^3} + \frac{i\left[(1+a^2)z+2a\right]z}{2(1-z)^2}.
\end{aligned}$$

Thus the minimal surface S obtained by $F(z)$ and $G(z)$ has the representation of the form

$$S = \{(u(z), v(z), H(z)) : z \in \mathbb{D}\}$$

where

$$\begin{aligned}
u(z) =& Re\left\{\frac{(1+a)^2z^3}{3(1-z)^3} + \frac{(1+a^2+2az)z}{2(1-z)^2}\right\} + c_1,\\
v(z) =& Re\left\{\frac{i(a^2-1)z}{2(1-z)^2}\right\} + c_2,\\
H(z) =& Re\left\{\frac{i(1+a)^2z^3}{3(1-z)^3} + \frac{i\left[(1+a^2)z+2a\right]z}{2(1-z)^2}\right\} + c_3
\end{aligned}$$

are harmonic in $\mathbb{D}$ and the surface S is regular. In addition, this is a conformal parametrization. The first fundamental form for the Euclidean length on S is $ds^2 = \lambda^2 |dz|^2$ where $\lambda^2 = \frac{1}{2}\sum_{k=1}^{3} |\phi_k|^2$.

Let $c_1 = c_2 = c_3 = 0$. Then the harmonic function $f(z) = u(z) + iv(z)$ is

$$f(z) = Re\left\{\frac{(1+a)^2 z^3}{3(1-z)^3} + \frac{(1+a^2+2az)z}{2(1-z)^2}\right\} + iRe\left\{\frac{i(a^2-1)z}{2(1-z)^2}\right\}.$$

So f can be written in the form $f = h + \overline{g}$, where

$$\begin{aligned} h(z) &= \frac{(1+a)^2 z^3}{6(1-z)^3} + \frac{(1+az)z}{2(1-z)^2} \\ &= \sum_{k=1}^{\infty} \frac{1}{12}\left\{(1+a)^2 k^2 + 3(1-a^2)k + 2(a^2-a+1)\right\} z^k \end{aligned}$$

and

$$\begin{aligned} g(z) &= \frac{(1+a)^2 z^3}{6(1-z)^3} + \frac{a(a+z)z}{2(1-z)^2} \\ &= \sum_{k=1}^{\infty} \frac{1}{12}\left\{(1+a)^2 k^2 - 3(1-a^2)k + 2(a^2-a+1)\right\} z^k \end{aligned}$$

are analytic in $\mathbb{D}$. Since the Jacobian of f

$$J(z) = |h'(z)|^2 - |g'(z)|^2 = \left|\frac{(1+az)^2}{2(1-z)^4}\right|^2 - \left|\frac{(a+z)^2}{2(1-z)^4}\right|^2 > 0,$$

the harmonic mapping f is locally 1-1 and orientation-preserving, that is locally univalent in $\mathbb{D}$ [4, 6]. We will give the proof that this f is univalent in $\mathbb{D}$ in the following theorem.

A domain D is called convex in the direction of the real axis if it has a connected intersection with every line parallel to the real axis.

Theorem 1. [4, Theorem 5.3] *A harmonic $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis if and only if $h - g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis.*

Theorem 2. *The locally univalent harmonic function $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis.*

Proof. The Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots$$

is conformal univalent in $\mathbb{D}$ and maps the unit disk onto the entire complex plane minus the portion of the negative real axis from $-\infty$ to $-\frac{1}{4}$, that is a domain convex in the direction of the real axis.

The analytic function

$$h(z) - g(z) = \frac{(1-a^2)}{2} k(z)$$

is also a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis. Thus the locally univalent harmonic $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis by Theorem 1. □

THEOREM 3. *The regular minimal surfaces S obtained by $F(z)$ and $G(z)$ lie over $f(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -(a^2 - a + 1)/6]$.*

Proof. Let $w = \frac{1+z}{1-z} = c + di$, then $c > 0$. From this we get

$$\begin{aligned} f(z) &= u(z) + iv(z) \\ &= \frac{(1+a)^2c^3 - 3(1+a)^2cd^2 + 3(a-1)^2c - 4(a^2-a+1)}{24} + i\frac{(1-a^2)cd}{4}. \end{aligned}$$

If $v = 0$, then $d = 0$ and u varies from $-(a^2 - a + 1)/6$ to $+\infty$. On the horizontal line $v \neq 0$, the real part u of f varies from $-\infty$ to $+\infty$. □

Now we will express the basic geometric quantities associated with the minimal surface S in terms of the univalent harmonic orientation-preserving function $f = h + \overline{g}$ with $b^2 = \overline{f_{\bar{z}}}/f_z = g'/h'$. In terms of $f = h + \bar{g}$, the conformal factor λ becomes simply $\lambda = |h'| + |g'|$. Therefore the Gaussian curvature K of S is

$$\begin{aligned} K &= -\frac{\Delta log\lambda}{\lambda^2} = -\left[\frac{4|G'|}{|F|(1+|G|^2)^2}\right]^2 \\ &= \frac{-4|b'|^2}{|h'|^2(|b|^2+1)^4} = \frac{-4|b'|^2}{(|h'|+|g'|)^2(|b|^2+1)^2}. \end{aligned}$$

Let Δ be a domain whose closure is in $\mathbb{D}$, then the total curvature T of the surface restricted to Δ is

$$T = \iint_{\Delta} K\lambda^2 dxdy = -\iint_{\Delta} \left[\frac{2|G'|}{1+|G|^2}\right]^2 dxdy$$
$$= -\iint_{\Delta} \left[\frac{2|b'|}{1+|b|^2}\right]^2 dxdy,$$

where $z = x + iy$.

For more results concerning harmonic mappings related to minimal surfaces, we refer the reader to [2, 3, 5].

THEOREM 4. *Let S be the regular minimal surfaces induced by $F(z)$ and $G(z)$. Then*

$$|K| \leq \frac{16}{(1-a^2)^2}\left(\frac{1+r}{1-r}\right)^4, \quad |z| = r < 1.$$

Proof. Since $b(z)$ is analytic in $\mathbb{D}$ and satisfies the condition $|b(z)| < 1$, the invariant form of Schwarz's lemma implies

$$|b'| \leq \frac{1-|b|^2}{1-|z|^2}.$$

From this inequality and the fact that $1 - |b|^2 \leq 1 + |b|^2$, we have

$$|K| \leq \frac{4}{(1-|z|^2)^2(|h'|+|g'|)^2}. \tag{1}$$

The analytic function $2[h(z) - g(z)]/(1 - a^2) = k(z)$ is univalent and satisfies $k(0) = 0$ and $k'(0) = 1$. Thus we have

$$\frac{(1-a^2)(1-r)}{2(1+r)^3} \leq |h' - g'| \leq \frac{(1-a^2)(1+r)}{2(1-r)^3}$$

by the distortion theorem. This leads to

$$\frac{1}{(|h'|+|g'|)^2} \leq \frac{4(1+r)^6}{(1-a^2)^2(1-r)^2}.$$

By applying this inequality to (1), we obtain

$$|K| \leq \frac{16}{(1-a^2)^2}\left(\frac{1+r}{1-r}\right)^4$$

as desired. □

THEOREM 5. *The Gaussian curvature K at the point $(0,0,0)$ in the minimal surface S is given by*

$$K = -\frac{16(1-a^2)^2}{(1+a^2)^4}.$$

Proof. At the point $(u(0), v(0), H(0)) = (0,0,0)$ on the minimal surface S, the Gaussian curvature is

$$K = \frac{-4|b'(0)|^2}{(|h'(0)| + |g'(0)|)^2(|b(0)|^2+1)^2} = -\frac{16(1-a^2)^2}{(1+a^2)^4}.$$

□

In case of $a = 0$, the minimal surface S induced by $G(z) = i/z$ and $F(z) = z^2/(1-z)^4$ has the Gaussian curvature $K = -16$ at the point $(0,0,0)$. Therefore the estimate in Theorem 4 is sharp in case of $a = 0$.

THEOREM 6. *The total curvature of the minimal surface S induced by the meromorphic function $G(z) = i/z$ and analytic function $F(z) = z^2/(1-z)^4$ is -2π.*

Proof. This is the case of $a = 0$. Let $D_r = \{z : |z| < r\}, r < 1$. Then the total curvature of the minimal surface restricted to D_r is

$$\begin{aligned} T_r &= -\iint_{D_r} \left[\frac{2|b'|}{1+|b|^2}\right]^2 dxdy = -4\iint_{D_r} \frac{1}{(|z|^2+1)^2} dxdy \\ &= -4\int_0^{2\pi}\int_0^r \frac{r}{(r^2+1)^2} drd\theta = \frac{-4\pi r^2}{r^2+1}. \end{aligned}$$

Thus $T_r \to -2\pi$ as $r \to 1$. Therefore the total curvature of the minimal surface S is -2π. □

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Sook Heui Jun
Department of Mathematics
Seoul Women's University
Seoul 01797, Korea
E-mail: `shjun@swu.ac.kr`