Korean J. Math. **23** (2015), No. 3, pp. 457–478
http://dx.doi.org/10.11568/kjm.2015.23.3.457

# A HYBRID METHOD FOR A SYSTEM INVOLVING EQUILIBRIUM PROBLEMS, VARIATIONAL INEQUALITIES AND NONEXPANSIVE SEMIGROUP

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Abstract. In this paper we propose an iteration hybrid method for approximating a point in the intersection of the solution-sets of pseudomonotone equilibrium and variational inequality problems and the fixed points of a semigroup-nonexpensive mappings in Hilbert spaces. The method is a combination of projection, extragradient-Armijo algorithms and Manns method. We obtain a strong convergence for the sequences generated by the proposed method.

## 1. Introduction

Throughout the paper we suppose that $\mathcal{H}$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$, that $C$ is a closed convex subset in $\mathcal{H}$, and that $f : C \times C \to \mathbb{R}$, $A : C \to \mathcal{H}$, $T(h) : C \to C$, $h \geq 0$. Conditions for $f$, $A$ and $T(h)$ will be detailed later. In this paper we consider a system that consists of an equilibrium problem, a variational inequality and the fixed point problem for a semimgroup of nonexpansive mappings $\{T(h) : h \geq 0\}$ from $C$ into itself . Namely, we are interested in a solution method for the system defined as

$$\text{Find } x^* \in C : \ f(x^*, y) \geq 0 \ \ \forall\, y \in C, \tag{1}$$

Received December 24, 2014. Revised August 31, 2015. Accepted September 10, 2015.

2010 Mathematics Subject Classification: 65K10, 65K15, 90C25, 90C33.

Key words and phrases: fixed point, variational inequality, equilibrium problems, nonexpansive mapping, pseudomonotonicity, semigroup.

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$$\langle Ax^*, x - x^* \rangle \geq 0 \ \forall x \in C, \tag{2}$$

$$\mathrm{x}^* = T(h)(x^*) \ \forall h > 0. \tag{3}$$

The equilibrium problem (1), the variational inequality (2) and the fixed point problem for a semigroup of nonexpansive mappings are important topics of Applied Analysis, and they attracted much attention of researchers and users (see e.g. [1], [3] [4] [5], [8], [10] [11],... and the references cited therein).

In recent year, the problem of solving a system involving equilibrium problems, variational inequalities and fixed point of a semigroup nonexpansive mappings in Hilbert spaces has attracted attention of some authors (see e.g. [3], [4], [9], [14], [16], [17], [19], [20], [22],... and the references therein). The common approach in these papers is to use a proximal point algorithm for handling the equilibrium problem. For monotone equilibrium problems the subproblems needed to solve in the proximal point method are strongly monotone, and therefore they have a unique solution that can be approximated by available methods. However, for pseudomonone problems the subproblems, in general, may have nonconvex solution-set due to the fact that the regularized bifunctions do not inherit any pseudomonotoniciy property from the original one.

In this article we propose a method for finding a common element in the solution-sets of a pseudomonotone equilibrium problem, and a monotone variational inequality and the set of fixed points for a nonexpansive semigroup in Hilbert spaces. The main point here is that we use the hybrid idea from [12] combining with an extragradient-Armijo procedure rather than a proximal point algorithm. This algorithm thus can be used for pseudomonotone equilibrium problems.

The paper is organized as follows. in the next section we recall some notions and results that will be used for convergence analysis. Then we describe the method at the end of the section. The convergence analysis for the proposed method is detailed in Section 3. Some applications are given in the last section.

## 2. Preliminaries

In what follows by $x^n \rightharpoonup \bar{x}$ we mean that the sequence $\{x^n\}$ converges to $\bar{x}$ in weak topology. We recall that mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for al } x, y \in C.$$

Let $F(T)$ denote the set of fixed points of $T$. A family $\{T(s) : \ s \in \mathbb{R}_+\}$ of mappings from $C$ into itself is called a *nonexpansive semigroup* on $C$ if it satisfies the following conditions:

(i) for each $s \in \mathbb{R}_+,\ T(s)$ is a nonexpansive mapping on $C$;
(ii) $T(0)x = x$ for all $x \in C$;
(iii) $T(s_1 + s_2) = T(s_1) \circ T(s_2)$ for all $s_1, s_2 \in \mathbb{R}_+$;
(iv) for each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into $C$ is continuous.

Let $\mathcal{F} = \bigcap_{s\geq 0} F(T(s))$ be the set of all common fixed points of $\{T(s) : s \in \mathbb{R}_+\}$. We know that $\mathcal{F}$ is nonempty if $C$ is bounded (see [2]).

A mapping $A : C \to \mathcal{H}$ is called *monotone* on $C$ if

$$\langle Ax - Ay, x - y\rangle \geq 0 \text{ for all } x, y \in C;$$

strictly monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0 \text{ for all } x \neq y;$$

$\beta-$inverse strongly monotone mapping if

$$\langle Ax - Ay, x - y\rangle \geq \beta\|Ax - Ay\|^2 \text{ for all } x, y \in C;$$

and $L-$Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\| \text{ for all } x, y \in C.$$

It is clear that if $A$ is $\beta-$ inverse strongly monotone, then $A$ is monotone and Lipschitz continuous.

The bifunction $f$ is called *monotone* on $C$ if

$$f(x, y) + f(y, x) \leq 0 \text{ for al } x, y \in C;$$

*pseudomonotone* on $C$ if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \text{ for al } x, y \in C.$$

We suppose the following assumptions:

$(A_0)$ $A$ is monotone and Lipschitz on $C$ with constant $L > 0$;
$(A_1)$ $f(x, x) = 0$ for all $x \in C$;
$(A_2)$ $f$ is pseudomonotone on $C$;
$(A_3)$ $f$ is continuous on $C$;
$(A_4)$ for each $x \in C,\ f(x, \cdot)$ is convex and subdifferentiable on C;
$(A_5)$ $\overline{\lim}_{t\to+0} f((1 - t)u + tz, v) \leq f(u, v)$ for all $(u, z, v) \in C \times C \times C$.

For each point $x \in \mathcal{H}$, let $P_C(x)$ denote the projection of $x$ onto $C$. The following well known lemma will be used in the sequel.

LEMMA 2.1. *Let $C$ be a nonempty closed convex subset of $\mathcal{H}$. Given $x \in \mathcal{H}$ and $y \in C$ then*

(i) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$ *for all* $x, y \in \mathcal{H}$;
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, *for all* $t \in [0, 1]$ *and for all* $x, y \in \mathcal{H}$;
(iii) $y = P_C(x)$ *if and only if* $\langle x - y, y - z\rangle \geq 0$ *for all* $z \in C$;
(iv) $P_C$ *is a nonexpansive mapping on* $C$;
(v) $\langle x - y, P_C(x) - P_C(y) \geq \|P_C(x) - P_C(y)\|^2$ *for all* $x, y \in \mathcal{H}$;
(vi) $\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$ *for any* $x \in \mathcal{H}$ *and for all* $y \in C$.

Using Lemma 2.1, it is easy to prove the following lemma.

LEMMA 2.2. *A point $u \in C$ is a solution of the variational inequality* (2) *if and only if $u = P_C(u - \lambda Au)$ for any $\lambda > 0$.*

We recall that a set-valued mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is called monotone if for all $x, y \in \mathcal{H}, u \in Tx$, and $v \in Ty$ imply $\langle x - y, u - v\rangle \geq 0$. A monotone mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is *maximal* if the graph $G(T)$ of $T$ is not property contained in the graph of any other monotone mapping. It is known that a monotone mapping $T$ is maximal if and only if for $(x, u) \in \mathcal{H}\times\mathcal{H}$, $\langle x-y, u-v\rangle \geq 0$ for every $(y, v) \in G(T)$ implies $u \in Tx$.

Let $A$ be a monotone mapping of $C$ to $\mathcal{H}$, let $N_C(v)$ be the normal cone to $C$ at $v \in C$, that is, $N_C(v) = \{w \in \mathcal{H} : \langle v - u, w\rangle \geq 0 \text{ for all } u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C(v), & \text{if} \quad v \in C, \\ \emptyset & \text{if} \quad v \notin C. \end{cases}$$

Then it is well-known [15] that $T$ is maximal monotone and $v \in T^{-1}0$ if and only if $v \in Sol(A, C)$.

Now we collect some lemmas which will be used for proving the convergence results for the method to be described below.

LEMMA 2.3. ([6]). *Let $C$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $h : C \to \mathbb{R}$ be a convex and subdifferentiable function on $C$. Then $x^*$ is a solution to the following convex problem:*

$$\min\{g(x) : \ x \in C\}$$

*if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $N_C(x^*)$ is normal cone at $x^*$ on $C$ and $\partial g(\cdot)$ denotes the subdifferential of $g$.*

LEMMA 2.4. ([7]) *Let $C$ be a closed convex subset of a Hilbert space $\mathcal{H}$ and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. If a sequence $\{x^n\} \subset C$ such that $x^n \rightharpoonup z$ and $x^n - Sx^n \to 0$, then $z = Sz$.*

LEMMA 2.5. ( [18]) *Let $C$ be a nonempty bounded closed convex subset of $\mathcal{H}$ and let $\{T(s) : \ s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on $C$. Then, for any $h \geq 0$*

$$\lim_{s\to\infty} \sup_{y\in C} \Big\| T(h)\Big(\frac{1}{s}\int_0^s T(t)ydt\Big) - \frac{1}{s}\int_0^s T(t)ydt\Big\| = 0$$

It is well known that $\mathcal{H}$ satisfies the following Opial's condition [13]: If a sequence $\{x^k\}$ converges weakly to $x$, as $k \to \infty$, then

$$\lim_{n\to\infty} \sup \|x^n - x\| < \lim_{n\to\infty} \sup \|x^n - y\| \text{ for al } y \in \mathcal{H}, \text{ with } x \neq y.$$

Now we are in a position to describe a hybrid iteration method for finding a element in the set $\mathcal{F} \cap Sol(C, f) \cap Sol(C, A)$ under the Assumption $(A_0) - (A_5)$.

ALGORITHM 2.6. Choose positive sequences $\{\mu_n\} \subset [a, 1]$ for some $a \in (0, 1)$, $\{\lambda_n\} \subset [b, c]$ for some $b, c \in (0, 1/\sqrt{2}L)$ and positives number $\beta > 0$, $\sigma \in (0, \frac{\beta}{2})$, $\gamma \in (0, 1)$.

Seek a starting point $x^0 \in C$ and set $n := 0$,

**Step 1.** Solve the strongly convex program

$$y^n = \text{argmin}\{f(x^n, y) + \frac{\beta}{2}||y - x^n||^2 : \ y \in C\}$$

and set $d(x^n) = x^n - y^n$.
If $\|d(x^n)\| \neq 0$ then go to Step 2.
Otherwise, set $u^n = x^n$ and go to Step 3.

**Step 2.** (linesearch) Find the smallest positive integer number $m_n$ such that

$$f\big(x^n - \gamma^{m_n} d(x^n), y^n\big) \leq -\sigma\|d(x^n)\|^2. \tag{4}$$

Compute

$$u^n = P_{C\cap V_n}(x^n),$$

where $\bar{z}^n = x^n - \gamma^{m_n} d(x^n), w^n \in \partial_2 f(\bar{z}^k, \bar{z}^k)$ and

$$V_n = \{x \in \mathcal{H} : \ \langle w^n, x - \bar{z}^n\rangle \leq 0\},$$

then go to Step 3.

**Step 3.** Compute

$$\begin{aligned} v^n &= P_C(u^n - \lambda_n A u^n) \\ z^n &= (1-\mu_n)P_C(x^n) + \mu_n T_n P_C(u^n - \lambda_n A v^n) \\ x^{n+1} &= P_{H_n \cap W_n}(x^0) \\ \textit{where} & \\ H_n &= \{z \in \mathcal{H} : \ \|z^n - z\| \le \|x^n - z\|\}, \\ W_n &= \{z \in \mathcal{H} : \ \langle x^n - z, x^0 - x^n \rangle \ge 0\}. \end{aligned}$$

and $T_n$ is defined as

$$T_n x := \frac{1}{s_n}\int_0^{s_n} T(s)x ds, \ \forall x \in C \text{ with } \lim_{n\to+\infty} s_n = +\infty.$$

Increase $n$ by 1 and go back to Step 1.

We now turn to the convergence of the proposed algorithm.

## 3. Convergence Results

In this section, we show that the sequences $\{x^n\}$, $\{v^n\}$, $\{z^n\}$ and $\{u^n\}$ defined by Algorithm 2.6 strongly convergent to a point in the set $\Omega := \mathcal{F} \cap Sol(C, A) \cap Sol(C, f)$

THEOREM 3.1. *Let $C$ be a nonempty closed convex subset in a real Hilbert space $\mathcal{H}$, $\{T(s) : \ s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on $C$, $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(A_1) - (A_5)$, and $A : C \to \mathcal{H}$ be a monotone $L-$Lipschitz continuous mapping such that $\Omega = \mathcal{F} \cap Sol(f, C) \cap Sol(A, C) \neq \emptyset$. Let $\{x^n\}$, $\{z^n\}, \{v^n\}$ and $\{u^n\}$ be sequences generated by the algorithm, where $\{\mu_n\} \subset [a, 1]$ for some $a \in (0, 1)$, $\{\lambda_n\} \subset [b, c]$ for some $b, c \in (0, 1/\sqrt{2}L)$. Then, $\{x^n\}$, $\{v^n\}, \{z^n\}$ and $\{u^n\}$ converge strongly to an element $p = P_\Omega(x^0)$.*

*Proof.* First, we consider the case when there exists $n_0$ such that $d(x^n) = 0$, i.e. $x^n = y^n$ for all $n \ge n_0$. Then from Step 1, we have $u^n = x^n$ which implies that $x^n$ is a solution of the equilibrium problem $EP(C, f)$ for all $n \ge n_0$. Thus, the algorithm becomes the one in [3] which has been shown there that the sequence $\{x^n\}$ strongly converges to a point in $\mathcal{F} \cap Sol(C, A)$.

Now we may assume that $d(x^n) \neq 0$ for all $n$. For this case we divide the proof into several steps. Some ideas in this proof are taken from the references [3], [12].

**Step 1.** We prove that the linesearch is finite for every $n$, that means that there exists the smallest nonnegative integer $m_n$ satisfying

$$f(x^n - \gamma^{m_n} d(x^n), y^n) \leq -\sigma \|d(x^n)\|^2 \text{ for al } n.$$

Indeed, suppose for contradiction that for every nonnegative integer $m$, one has

$$f\big(x^n - \gamma^m d(x^n), y^n\big) + \sigma \|d(x^n)\|^2 > 0.$$

Taking the limit above inequality as $m \to \infty$, by continuity of $f$, we obtain

$$f\big(x^n, y^n\big) + \sigma \|d(x^n)\|^2 \geq 0. \tag{5}$$

On the other hand, since $y^n$ is the unique solution of the strongly convex problem

$$\min\{f(x^n, y) + \frac{\beta}{2}\|y - x^n\|^2 : \; y \in C\},$$

we have

$$f(x^n, y) + \frac{\beta}{2}\|y - x^n\|^2 \geq f(x^n, y^n) + \frac{\beta}{2}\|y^n - x^n\|^2 \text{ for al } y \in C.$$

With $y = x^n$, the last inequality becomes

$$f(x^n, y^n) + \frac{\beta}{2}||d(x^n)||^2 \leq 0. \tag{6}$$

Combining (5) with (6) yields

$$\sigma \|d(x^n)\|^2 \geq \frac{\beta}{2}\|d(x^n)\|^2.$$

Hence it must be either $\|d(x^n)\| = 0$ or $\sigma \geq \frac{\beta}{2}$. The first case contradicts to $\|d(x^n)\| \neq 0$, while the second one contradicts to the choice $\sigma < \frac{\beta}{2}$.

**Step 2.** We show that $x^n \notin V_n$. In fact, from $\bar{z}^n = x^n - \gamma^{m_n} d(x^n)$, it follows that

$$y^n - \bar{z}^n = \frac{1 - \gamma^{m_n}}{\gamma^{m_n}}(\bar{z}^n - x^n).$$

Then using (4) and the assumption $f(x,x)=0$ for all $x\in C$, we have

$$\begin{aligned} 0 &> -\sigma\|d(x^n)\|^2 \\ &\geq f(\bar{z}^n, y^n) \\ &= f(\bar{z}^n, y^n) - f(\bar{z}^n, \bar{z}^n) \\ &\geq \langle w^n, y^n - \bar{z}^n\rangle \\ &= \frac{1-\gamma^{m_n}}{\gamma^{m_n}}\langle \bar{z}^n - x^n, w^n\rangle. \end{aligned}$$

Hence

$$\langle x^n - \bar{z}^n, w^n\rangle > 0,$$

which implies $x^n \notin V_n$.

**Step 3.** We claim that $u^n = P_{C\cap V_n}(\bar{y}^n)$, where $\bar{y}^n = P_{V_n}(x^n)$.

Indeed, let $K := \{x \in \mathcal{H} : \ \langle t, x - x^0\rangle \leq 0\}$ with $\|t\| \neq 0$. It is easy to check that

$$P_K(y) = y - \frac{\langle t, y - x^0\rangle}{\|t\|^2}t,$$

Hence,

$$\begin{aligned} \bar{y}^n &= P_{V_n}(x^n) \\ &= x^n - \frac{\langle w^n, x^n - \bar{z}^n\rangle}{\|w^n\|^2}w^n \\ &= x^n - \frac{\gamma^{m_n}\langle w^n, d(x^n)\rangle}{\|w^n\|^2}w^n. \end{aligned}$$

Note that, for every $y \in C \cap V_n$, there exists $\lambda \in (0,1)$ such that

$$\hat{x} = \lambda x^n + (1-\lambda)y \in C \cap \partial V_n,$$

where

$$\partial V_n = \{x \in \mathcal{H} : \ \langle w^n, x - \bar{z}^n\rangle = 0\}.$$

Since $x^n \in C$, $\hat{x} \in \partial V_n$ and $\bar{y}^n = P_{V_n}(x^n)$, we have

$$\begin{aligned} \|y - \bar{y}^n\|^2 &\geq (1-\lambda)^2\|y - \bar{y}^n\|^2 \\ &= \|\hat{x} - \lambda x^n - (1-\lambda)\bar{y}^n\|^2 \\ &= \|(\hat{x} - \bar{y}^n) - \lambda(x^n - \bar{y}^n)\|^2 \\ &= \|\hat{x} - \bar{y}^n\|^2 + \lambda^2\|x^n - \bar{y}^n\|^2 - 2\lambda\langle \hat{x} - \bar{y}^n, x^n - \bar{y}^n\rangle \\ &= \|\hat{x} - \bar{y}^n\|^2 + \lambda^2\|x^n - \bar{y}^n\|^2 \\ &\geq \|\hat{x} - \bar{y}^n\|^2. \qquad (7) \end{aligned}$$

At the same time

$$\begin{aligned}\|\hat{x} - x^n\|^2 &= \|\hat{x} - \bar{y}^n + \bar{y}^n - x^n\|^2 \\ &= \|\hat{x} - \bar{y}^n\|^2 - 2\langle \hat{x} - \bar{y}^n, x^n - \bar{y}^n\rangle + \|\bar{y}^n - x^n\|^2 \\ &= \|\hat{x} - \bar{y}^n\|^2 + \|\bar{y}^n - x^n\|^2.\end{aligned}$$

Using $u^n = P_{C\cap V_n}(x^n)$ and the Pythagorean theorem, we can write

$$\begin{aligned}\|\hat{x} - \bar{y}^n\|^2 &= \|\hat{x} - x^n\|^2 - \|\bar{y}^n - x^n\|^2 \\ &\geq \|u^n - x^n\|^2 - \|\bar{y}^n - x^n\|^2 \\ &= \|u^n - \bar{y}^n\|^2. \qquad (8)\end{aligned}$$

From (7) and (8), it follows that

$$\|u^n - \bar{y}^n\| \leq \|y - \bar{y}^n\| \text{ for al } y \in C \cap V_n.$$

Hence

$$u^n = P_{C\cap V_n}(\bar{y}^n).$$

**Step 4.** We show that if $\|d(x^n)\| \neq 0$ then $\Omega \subseteq C \cap V_n$.

Indeed, let $x^* \in \Omega$. Since $f(x^*, x) \geq 0$ for all $x \in C$, by pseudomonotonicity of $f$, we have

$$f(\bar{z}^n, x^*) \leq 0. \qquad (9)$$

It follows from $w^n \in \partial_2 f(\bar{z}^n, \bar{z}^n)$ that

$$\begin{aligned}f(\bar{z}^n, x^*) &= f(\bar{z}^n, x^*) - f(\bar{z}^n, \bar{z}^n) \\ &\geq \langle w^n, x^* - \bar{z}^n\rangle. \qquad (10)\end{aligned}$$

Combining (9) and (10), we get

$$\langle w^n, x^* - \bar{z}^n\rangle \leq 0.$$

On the other hand, by definition of $V_n$, we have $x^* \in V_n$. Thus $\Omega \subseteq C \cap V_n$.

**Step 5.** It holds that $\Omega \subset H_n \cap W_n$ for every $n \geq 0$. In fact, for each $x^* \in \Omega$, one has

$$\|u^n - x^*\| = \|P_{C\cap V_n}(x^n) - P_{C\cap V_n}(x^*)\| \leq \|x^n - x^*\|. \qquad (11)$$

Let $t^n = P_C(u^n - \lambda_n Av^n)$. Applying $(vi)$ in Lemma 2.1, with $x = u^n - \lambda_n Av^n$ and $y = u^n$, monotonicity of $A$, we obtain

$$\begin{aligned}
\|t^n - x^*\|^2 &\le \|u^n - \lambda_n Av^n - x^*\|^2 - \|u^n - \lambda_n Av^n - t^n\|^2 \\
&= \|u^n - x^*\|^2 + \lambda_n^2\|Av^n\|^2 - 2\lambda_n\langle u^n - x^*, Av^n\rangle \\
&\quad -\Big(\|u^n - t^n\|^2 + \lambda_n^2\|Av^n\|^2 - 2\lambda_n\langle u^n - t^n, Av^n\rangle\Big) \\
&= \|u^n - x^*\|^2 - \|u^n - t^n\|^2 \\
&\quad +2\lambda_n\Big(\langle Av^n - Ax^*, x^* - v^n\rangle + \langle Ax^*, x^* - v^n\rangle + \langle Av^n, v^n - t^n\rangle\Big) \\
&\le \|u^n - x^*\|^2 - \|u^n - t^n\|^2 + 2\lambda_n\langle Av^n, v^n - t^n\rangle \\
&= \|u^n - x^*\|^2 - \|u^n - v^n\|^2 - \|v^n - t^n\|^2 \\
&\qquad -2\langle u^n - v^n, v^n - t^n\rangle + 2\lambda_n\langle Av^n, v^n - t^n\rangle \\
&\le \|u^n - x^*\|^2 - \|u^n - v^n\|^2 - \|v^n - t^n\|^2 \\
&\qquad +2\langle \lambda_n Av^n + v^n - u^n, v^n - t^n\rangle. \qquad (12)
\end{aligned}$$

Since $v^n = P_C(u^n - \lambda_n Au^n)$, and $A$ is $L-$Lipschitz continuous, by Lemma 2.1 $(iii)$, we have

$$\begin{aligned}
2\langle \lambda_n Av^n + v^n - u^n, v^n - t^n\rangle &= 2\lambda_n\langle Av^n - Au^n, v^n - t^n\rangle \\
&\quad +2\langle v^n - (u^n - \lambda_n Au^n), v^n - t^n\rangle \\
&\le 2\lambda_n\langle Av^n - Au^n, v^n - t^n\rangle \\
&\le 2\lambda_n L\|v^n - u^n\|\|v^n - t^n\|. \qquad (13)
\end{aligned}$$

Using monotonicity of $A$, $\{\lambda_n\} \subset (0, 1/\sqrt{2}L)$ and nonexpansiveness of $P_C$ we obtain from (12) and (13) that

$$\begin{aligned}
\|t^n - x^*\|^2 &\le \|u^n - x^*\|^2 - \|u^n - v^n\|^2 - \|v^n - t^n\|^2 \\
&\quad +2\lambda_n L\|v^n - u^n\|\|v^n - t^n\| \\
&\le \|u^n - x^*\|^2 - \|u^n - v^n\|^2 \\
&\quad +2\lambda_n L\|v^n - u^n\|\|P_C(u^n - \lambda_n Au^n) - P_C(u^n - \lambda_n Av^n)\| \\
&\le \|u^n - x^*\|^2 - \|u^n - v^n\|^2 + 2\lambda_n^2 L^2\|u^n - v^n\|^2 \\
&= \|u^n - x^*\|^2 + (2\lambda_n^2 L^2 - 1)\|u^n - v^n\|^2 \\
&\le \|u^n - x^*\|^2. \qquad (14)
\end{aligned}$$

By convexity of $\|\cdot\|^2$ and nonexpansiveness of $P_C$, it follows from the definition of $T_n$ and (11), (14) that

$$\begin{aligned}
\|z^n - x^*\|^2 &= \|(1-\mu_n)[P_C(x^n) - P_C(x^*)] + \mu_n(T_n t^n - x^*)\|^2 \\
&\leq (1-\mu_n)\|P_C(x^n) - P_C(x^*)\|^2 + \mu_n\|T_n t^n - T_n x^*\|^2 \\
&\leq (1-\mu_n)\|x^n - x^*\|^2 + \mu_n\|t^n - x^*\|^2 \\
&\leq (1-\mu_n)\|x^n - x^*\|^2 + \mu_n\|u^n - x^*\|^2 \\
&\leq (1-\mu_n)\|x^n - x^*\|^2 + \mu_n\|x^n - x^*\|^2 \\
&= \|x^n - x^*\|^2 \text{ for al } n \geq 0. \qquad (15)
\end{aligned}$$

Then from (15) we have $\|z^n - u^n\| \leq \|x^n - x^*\|$, which implies $x^* \in H_n$. Hence $\Omega \subset H_n$ for all $n \geq 0$.

Next we show $\Omega \subset W_n$ for all $k \geq 0$. Indeed, when $k = 0$, we have $x^0 \in C$ and $W_0 = H$. Consequently, $\Omega \subset H_0 \cap W_0$. By induction, suppose $\Omega \subset H_i \cap W_i$ for some $i \geq 0$. We have to prove that $\Omega \subset H_{i+1} \cap W_{i+1}$. Since $\Omega$ is nonempty closed convex subset of $\mathcal{H}$, there exists a unique element $x^{i+1} \in \Omega$ such that $x^{i+1} = P_\Omega(x^0)$. By Lemma 2.1, for every $z \in \Omega$, it holds that

$$\langle x^{i+1} - z, x^0 - x^{i+1}\rangle \geq 0,$$

which means that $z \in W_{i+1}$. Note that $z \in W_{i+1}$, we can conclude that $\Omega \subset H_n \cap W_n$ for all $n \geq 0$.

**Step 6.** We claim that sequence $\{x^n\}$ and $\{y^n\}$ are bounded.

Since $\Omega$ is a nonempty closed convex subset of $C$, there exists a unique element $z^0 \in \Omega$ such that $z^0 = P_\Omega(x^0)$. Now, from $x^{n+1} = P_{H_n \cap W_n}(x^0)$ we obtain

$$\|x^{n+1} - x^0\| \leq \|z - x^0\| \quad \text{for all } z \in H_n \cap\ W_n. \qquad (16)$$

As $z^0 \in \Omega \subset H_n \cap W_n$, we have

$$\|x^{n+1} - x^0\| \leq \|z^0 - x^0\|, \quad \text{for each } n \geq 0.$$

Hence, the sequence $\{x^n\}$ is bounded.

Since $y^n$ is the unique solution of the mathematical program

$$\min\{f(x^n, y) + \frac{\beta}{2}\|y - x^n\|^2 :\ y \in C\},$$

we have

$$f(x^n, y) + \frac{\beta}{2}\|y - x^n\|^2 \geq f(x^n, y^n) + \frac{\beta}{2}\|y^n - x^n\|^2 \text{ for al } y \in C.$$

With $y = x^n \in C$ and $f(x^n, x^n) = 0$, we can write

$$0 \geq f(x^n, y^n) + \frac{\beta}{2}\|y^n - x^n\|^2. \tag{17}$$

Since $f(x^n, \cdot)$ is convex and subdifferentiable on $C$,

$$f(x^n, y) - f(x^n, x^n) \geq \langle w^n, y - x^n \rangle \text{ for al } y \in C,$$

for any $w^n \in \partial_2 f(x^n, x^n)$. For $y = y^n$, we have

$$f(x^n, y^n) \geq \langle w^n, y^n - x^n \rangle.$$

Combining this inequality and (17), we obtain

$$\langle w^n, y^n - x^n \rangle + \frac{\beta}{2}\|x^n - y^n\|^2 \leq 0,$$

which implies

$$-\|w^n\|\|y^n - x^n\| + \frac{\beta}{2}\|x^n - y^n\|^2 \leq 0.$$

Hence

$$\|x^n - y^n\| \leq \frac{2}{\beta}\|w^n\|. \tag{18}$$

Since $\{x^n\}$ are bounded, by [21], $\{w^n\}$ are bounded, then $\{y^n\}$ is bounded too.

**Step 7.** We claim that $\{x^n\}$, $\{z^n\}$, $\{v^n\}$ and $\{u^n\}$ converge strongly to an element $p \in \Omega$.

In fact, from $x^n = P_{H_{n-1} \cap W_{n-1}}(x^0)$ and $x^{n+1} \in W_n$, it follows that,

$$\|x^n - x^0\| \leq \|x^{n+1} - x^0\| \quad \text{for all } n \geq 0.$$

Thus, there exists a number $c < \infty$ such that $\lim_{n\to\infty} \|x^n - x^0\| = c$. Since $x^n = P_{H_{n-1} \cap W_{n-1}}(x^0)$ and $x^{n+1} \in W_n$, by $(ii)$ in Lemma 2.1, we have

$$\begin{aligned}
\|x^n - x^0\|^2 &\leq \left\|\frac{x^n + x^{n+1}}{2} - x^0\right\|^2 \\
&\leq \left\|\frac{x^n - x^0}{2} + \frac{x^{n+1} - x^0}{2}\right\|^2 \\
&= \frac{\|x^n - x^0\|^2}{2} + \frac{\|x^{n+1} - x^0\|^2}{2} - \frac{\|x^n - x^{n+1}\|^2}{4}.
\end{aligned}$$

So, we get

$$\|x^n - x^{n+1}\|^2 \leq 2(\|x^{n+1} - x^0\|^2 - \|x^n - x^0\|^2)$$

Since $\lim_{n\to\infty} \|x^n - x^0\| = c$, we have

$$\lim_{n\to\infty} \|x^n - x^{n+1}\| = 0. \tag{19}$$

Note that $x^{n+1} \in H_n$, we can write

$$\|z^n - x^n\| \le \|x^n - x^{n+1}\| + \|x^{n+1} - z^n\| \le 2\|x^n - x^{n+1}\|. \tag{20}$$

It follows from (19) and (20) that

$$\lim_{n\to\infty} \|z^n - x^n\| = 0. \tag{21}$$

Now from the second inequality in (15), we can write

$$\|z^n - x^*\|^2 - \|x^n - x^*\|^2 \le \mu_n\Big(\|T_n t^n - x^*\|^2 - \|x^n - x^*\|^2\Big) \le 0 \tag{22}$$

On the other hand, by Lemma 2.1, we have

$$\|z^n - x^*\|^2 - \|x^n - x^*\|^2 = \|z^n - x^n\|^2 + 2\langle z^n - x^n, x^n - x^*\rangle. \tag{23}$$

It follows from (21)-(23) that

$$\lim_{n\to\infty} \mu_n\Big(\|T_n t^n - x^*\|^2 - \|x^n - x^*\|^2\Big) = 0.$$

Since $\{\mu_n\} \subset [a, 1]$ for some $a \in (0, 1)$, we have

$$\lim_{n\to\infty} \Big(\|T_n t^n - x^*\|^2 - \|x^n - x^*\|^2\Big) = 0. \tag{24}$$

Combining (11), (14), (24) and using the nonexpansive property of $T_n$, we obtain

$$0 = \lim_{n\to\infty} \Big(\|T_n t^n - x^*\|^2 - \|x^n - x^*\|^2\Big) \le \lim_{n\to\infty} \Big(\|t^n - x^*\|^2 - \|x^n - x^*\|^2\Big) \le 0.$$

Thus,

$$\lim_{n\to\infty} \Big(\|t^n - x^*\|^2 - \|x^n - x^*\|^2\Big) = 0. \tag{25}$$

On the other hand, from $u^n = P_{C\cap V_n}(x^n)$, by Lemma 2.1 $(vi)$, we have

$$\begin{aligned} \|u^n - x^n\|^2 &\le \|x^n - x^*\|^2 - \|u^n - x^*\|^2 \\ &\le \|x^n - x^*\|^2 - \|t^n - x^*\|^2, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|u^n - x^n\| = 0. \tag{26}$$

Since $\{x^n\}$ is bounded, there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ converging weakly to some element $p$. From (26), (18) and (21), we obtain also that $\{u^n\}$, $\{y^n\}$, $\{z^n\}$ converges weakly to $p$. Since $\{u^{n_j}\} \subset C$ and $C$ is a closed convex subset in $\mathcal{H}$, we have $p \in C$.

Now, we prove that $p \in \Omega$. To this end, first we show that $p \in Sol(C, f)$. Indeed, since $y^n$ is the unique solution of the convex minimization problem

$$y^n = \operatorname{argmin}\{f(x^n, y) + \frac{\beta}{2}||y - x^n||^2 : \ y \in C\},$$

by optimality condition, we have

$$0 \in \partial_2\Big(f(x^n, y^n) + \frac{\beta}{2}||y^n - x^n||^2\Big)(y^n) + N_C(y^n).$$

Thus

$$0 = z + \beta(y^n - x^n) + z_n,$$

for some $z \in \partial_2 f(x^n, y^n)$ and $z^n \in N_C(y^n)$. By the definition of the normal cone $N_C(y^n)$, we get

$$\beta\langle y^n - x^n, y - y^n\rangle \geq \langle z, y^n - y\rangle \text{ for all } y \in C. \tag{27}$$

On the other hand, since $z \in \partial_2 f(x^n, y^n)$, we have

$$f(x^n, y) - f(x^n, y^n) \geq \langle z, y - y^n\rangle \text{ for all } y \in C. \tag{28}$$

Combining (27) with (28), we have

$$f(x^n, y) - f(x^n, y^n) \geq \beta\langle y^n - x^n, y^n - y\rangle \text{ for all } y \in C.$$

Hence

$$(f(x^{n_j}, y) - f(x^{n_j}, y^{n_j})) \geq \beta\langle y^{n_j} - x^{n_j}, y^{n_j} - y\rangle \text{ for all } y \in C.$$

Letting $j \to \infty$ we obtain in the limit that $f(p, y) \geq 0$ for all $y \in C$. Hence $p \in Sol(C, f)$.

Next we show that $p \in Sol(C, A)$. Define

$$Bv := \begin{cases} Av + N_C(v), & v \in C \\ \emptyset & v \notin C, \end{cases}$$

where $N_C(v)$ is normal cone to $C$ at $v$. Then $B$ is a maximal monotone operator. Let $(v, u) \in G(B)$. Since $u - Av \in N_C(v)$, one has

$$\langle v - y, u - Av\rangle \geq 0 \text{ for all } y \in C. \tag{29}$$

On the other hand, by Lemma 2.1 $(iii)$, from $t^n = P_C(u^n - \lambda_n Av^n)$, we have

$$\langle t^n - y, u^n - \lambda_n Av^n - t^n\rangle \geq 0 \text{ for all } y \in C,$$

$$\left\langle y - t^n, \frac{t^n - u^n}{\lambda_n} + Av^n \right\rangle \geq 0 \text{ for all } y \in C.$$

It follows from (29) with $y = t^{n_j}$ and monotonicity of $A$ that

$$\begin{aligned}
\langle v - t^{n_j}, u \rangle &\geq \langle v - t^{n_j}, Av \rangle - \left\langle v - t^{n_j}, \frac{t^{n_j} - u^{n_j}}{\lambda_{n_j}} + Av^{n_j} \right\rangle \\
&\geq \langle v - t^{n_j}, Av - At^{n_j} \rangle + \langle v - t^{n_j}, At^{n_j} - Av^{n_j} \rangle \\
&\quad - \left\langle v - t^{n_j}, \frac{t^{n_j} - u^{n_j}}{\lambda_{n_j}} \right\rangle \\
&\geq \langle v - t^{n_j}, At^{n_j} - Av^{n_j} \rangle - \left\langle v - t^{n_j}, \frac{t^{n_j} - u^{n_j}}{\lambda_{n_j}}. \right\rangle
\end{aligned}$$

Combining (11) and (14) we obtain

$$(30) \qquad (1 - 2\lambda_n^2 L^2)\|u^n - v^n\|^2 \leq \|x^n - x^*\|^2 - \|t^n - x^*\|^2$$

It follows from (25), (30) and the condition $\{\lambda_n\} \subset (0, 1/\sqrt{2}L)$ that

$$(31) \qquad \lim_{n\to\infty} \|u^n - v^n\| = 0.$$

Since $v^n = P_C(u^n - \lambda_n Au^n),\ t^n = P_C(u^n - \lambda_n Av^n)$, from (31) and $A$ is monotone,

$$(32) \qquad \lim_{n\to\infty} \|v^n - t^n\| = 0$$

and

$$\lim_{n\to\infty} \|Av^n - At^n\| = \lim_{n\to\infty} \|v^n - t^n\| = 0,$$

which implies that $\langle v - p, u \rangle \geq 0$ for every $v \in C$. Since $B$ is maximal monotone, we have $p \in B^{-1}0$, and hence $p \in Sol(C, A)$.

Now, we prove that $p = T(h)p$ for all $h > 0$. First, we obtain from Step 3 of the algorithm that

$$
\begin{aligned}
a\|u^n - T_n u^n\| &\leq \mu_n \|u^n - T_n u^n\| \\
&\leq \mu_n\Big(\|u^n - T_n t^n\| + \|T_n t^n - T_n u^n\|\Big) \\
&= \|\mu_n u^n - \mu_n T_n t^n\| + \mu_n \|T_n t^n - T_n u^n\| \\
&= \|\mu_n u^n + (1-\mu_n)P_C(x^n) - z^n\| + \mu_n\|T_n t^n - T_n u^n\| \\
&= \|(1-\mu_n)P_C(x^n) - (1-\mu_n)P_C(u^n) + u^n - z^n\| \\
&\qquad\qquad + \mu_n \|T_n t^n - T_n u^n\| \\
&\leq (1-\mu_n)\|x^n - u^n\| + \|u^n - z^n\| + \mu_n\|t^n - u^n\| \\
&\leq \|x^n - u^n\| + \|u^n - x^n\| + \|x^n - z^n\| + \mu_n \|t^n - u^n\| \\
&\leq 2\|x^n - u^n\| + \|x^n - z^n\| + \mu_n\|t^n - u^n\|.
\end{aligned}
$$

Thus, from (21), (26), (31) and (32) it follows that

$$
\lim_{n\to\infty} \|u^n - T_n u^n\| = 0. \tag{33}
$$

Note that

$$
\begin{aligned}
\|T(h)u^n - u^n\| &\leq \Big\|T(h)u^n - T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)u^n ds\Big)\Big\| \\
&\quad + \Big\|T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)u^n ds\Big) - \frac{1}{s_n}\int_0^{s_n} T(s)u^n ds\Big\| \\
&\quad + \Big\|\frac{1}{s_n}\int_0^{s_n} T(s)u^n ds - u^n\Big\| \\
&\leq 2\Big\|\frac{1}{s_n}\int_0^{s_n} T(s)u^n ds - u^n\Big\| \\
&\quad + \Big\|T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)u^n ds\Big) - \frac{1}{s_n}\int_0^{s_n} T(s)u^n ds\Big\| \qquad (34)
\end{aligned}
$$

We apply Lemma 2.5 to get

$$
\lim_{n\to\infty} \Big\|T(h)\Big(\frac{1}{s_n}\int_0^{s_n} T(s)u^n ds\Big) - \frac{1}{s_n}\int_0^{s_n} T(s)u^n ds\Big\| = 0, \tag{35}
$$

for every $h \in (0,\infty)$ and therefore, by (33), (34) and (35), we obtain

$$
\lim_{n\to\infty} \|T(h)u^n - u^n\| = 0
$$

for each $h > 0$, which, by Lemma 2.4, $p \in F(T(h))$ for all $h > 0$. Hence $p \in \mathcal{F}$.

Finally, we show the strong convergence of the sequences of iterates. Let $z^0 = P_\Omega(x^0)$. Since the norm is weakly lower semicontinuity,

$$\|x^0 - z^0\| \le \|x^0 - p\| \le \lim_{n\to\infty} \inf \|x^0 - x^{n_j}\| \le \lim_{n\to\infty} \sup \|x^0 - x^{n_j}\| \le \|x^0 - z^0\|$$

where the last inequlity comes from (16). Hence, we obtain

$$\lim_{j\to\infty} \|x^{n_j} - x^0\| = \|z^0 - x^0\|,$$

from which, by Opial's property, it follows that $p = z^0$. Since $p$ is any weak limit of sequence $\{x^n\}$, the whole sequence must converge strongly to $p$ as $n \to \infty$. Then the strong convergence of the sequences $\{z^n\}$ and $\{u^n\}$ to $z^0$ is followed from (21) and (26), respectively. Then, by (31), the sequence $\{v^n\}$ strongly converges to $z^0$ as well. The proof of the convergence theorem is complete. □

## 4. Special Cases and a Illustrative Example

If $A \equiv 0$, then Algorithm 2.6 reduces to the following one for finding a common element in the solution-set of a pseudomonotone equilibrium problem and the set of the fixed points of a nonexpansive semigroup in Hilbert spaces.

Corollary 4.1. *Let $C$ be a nonempty closed convex subset in a real Hilbert space $\mathcal{H}, \{T(s) : s \in \mathbb{R}_+\}$ be a nonexpansive semigroup on $C$ and $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(A_1) - (A_5)$ such that $\Omega = \mathcal{F} \cap Sol(f, C) \neq \emptyset$. Let $\{x^n\}$, $\{z^n\}$ and $\{u^n\}$ be sequences generated by*

$$\begin{aligned}
x^0 &\in C \text{ chosen arbitrarily}, \\
y^n &= \text{argmin}\{f(x^n, y) + \frac{\beta}{2}||y - x^n||^2 : \ y \in C\}; \ d(x^n) = x^n - y^n, \\
m_n &= \underset{k}{\text{argmin}}\{m_k : \ f\big(x^n - \gamma^{m_k} d(x^n) y^n\big) \le -\sigma \|d(x^n)\|^2\}, \\
\bar{z}^n &= x^n - \gamma^{m_n} d(x^n), w^n \in \partial_2 f(\bar{z}^k, \bar{z}^k) \\
u^n &= P_{C\cap V_n}(x^n); \ V_n = \{x \in \mathcal{H} : \ \langle w^n, x - \bar{z}^n\rangle \le 0\} \\
z^n &= (1 - \mu_n) x^n + \mu_n T_n u^n, \\
H_n &= \{z \in \mathcal{H} : \ \|z^n - z\| \le \|x^n - z\|\}, \\
W_n &= \{z \in \mathcal{H} : \ \langle x^n - z, x^0 - x^n\rangle \ge 0\}, \\
x^{n+1} &= P_{H_n \cap W_n}(x^0),
\end{aligned}$$

*where* $\{\mu_n\} \subset [a,1]$ *for some* $a \in (0,1)$. *Then,* $\{x^n\}$, $\{z^n\}$ *and* $\{u^n\}$ *converge strongly to an element* $p \in \Omega$. *Further,* $p$ *is the solution of the equilibrium problem*

$$p \in \mathcal{F}: \ f(p,x) \geq 0, \ \forall x \in \mathcal{F}.$$

*Proof.* Taking $A \equiv 0$ in Theorem 3.1, we get the desired conclusion easily. □

If $f(x,y) = 0$ for all $x, y \in C$, Algorithm 2.6 reduces to the following one for finding a common element in the solution-set of a monotone variational inequality problem and the set of the fixed points of a nonexpansive semigroup in Hilbert spaces.

Corollary 4.2. *Let* $C$ *be a nonempty closed convex subset in a real Hilbert space* $\mathcal{H}, \{T(s) : s \in \mathbb{R}_+\}$ *be a nonexpansive semigroup on* $C$ *and* $A : C \to \mathcal{H}$ *be a monotone* $L-$*Lipschitz continuous mapping such that* $\Omega = \mathcal{F} \cap Sol(A,C) \neq \emptyset$. *Let* $\{x^n\}, \{z^n\}$, *and* $\{v^n\}$ *be sequences generated by*

$$\begin{array}{rcl} x^0 & \in & C \text{ chosen arbitrarily}, \\ v^n & = & P_C(x^n - \lambda_n A x^n), \\ z^n & = & (1-\mu_n)x^n + \mu_n T_n P_C(x^n - \lambda_n A v^n), \\ H_n & = & \{z \in \mathcal{H}: \ \|z^n - z\| \leq \|x^n - z\|\}, \\ W_n & = & \{z \in \mathcal{H}: \ \langle x^n - z, x^0 - x^n \rangle \geq 0\}, \\ x^{n+1} & = & P_{H_n \cap W_n}(x^0), \end{array}$$

*where* $\{\mu_n\} \subset [a,1]$ *for some* $a \in (0,1)$, $\{\lambda_n\} \subset [b,c]$ *for some* $b, c \in (0, 1/\sqrt{2}L)$. *Then,* $\{x^n\}$, $\{z^n\}$ *and* $\{v^n\}$ *converge strongly to an element* $p \in \Omega$. *Further,* $p$ *is the solution of the variational inequality*

$$p \in \mathcal{F}: \ \langle Ap, x - p \rangle \geq 0, \ \forall x \in \mathcal{F}.$$

Now putting $f(x,y) = 0$ for all $x, y \in C$ and $A \equiv 0$, we obtain the following result for finding a common fixed point of a nonexpansive semigroup $\{T(s) : s \in \mathbb{R}_+\}$ on $C$.

Corollary 4.3. *Let* $C$ *be a nonempty closed convex subset in a real Hilbert space* $\mathcal{H}, \{T(s) : s \in \mathbb{R}_+\}$ *be a nonexpansive semigroup on* $C$

*such that $\mathcal{F} \neq \emptyset$. Let $\{x^n\}$ and $\{z^n\}$ be sequences generated by*

$$\begin{aligned} x^0 &\in C \text{ chosen arbitrarily}, \\ z^n &= (1-\mu_n)x^n + \mu_n T_n x^n \\ H_n &= \{z \in \mathcal{H} :\ \|z^n - z\| \leq \|x^n - z\|\}, \\ W_n &= \{z \in \mathcal{H} :\ \langle x^n - z, x^0 - x^n\rangle \geq 0\} \\ x^{n+1} &= P_{H_n \cap W_n}(x^0), \end{aligned}$$

*where $\{\mu_n\} \subset [a, 1]$ for some $a \in (0, 1)$. Then, $\{x^n\}$ and $\{z^n\}$ converge strongly to an element $p \in \mathcal{F}$.*

To illustrate the problem we consider the following example.

Let $\mathcal{H} = \mathbb{R}^n$ with the inner product $\langle x, y\rangle := x_1y_1 + \cdots + x_ny_n$ for all $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathcal{H}$. Let $C := [0, 1]^n$ be a $n$-dimensional box in $\mathcal{H}$. For all $x \in C$, we define the operator $A$ by taking

$$Ax := \begin{pmatrix} x_1^m + x_1^{m-1} + \cdots + x_1 \\ x_2^m + x_2^{m-1} + \cdots + x_2 \\ x_3^m + x_3^{m-1} + \cdots + x_3 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, m \in \mathbb{N}, m \geq 2,$$

and consider the variational inequality

$$\text{Find } x^* \in C :\ \langle Ax^*, x - x^*\rangle \geq 0\ \forall x \in C.$$

It is easy to see that the operator $A$ is monotone and $L-$Lipschitz continuous on $C$.

Consider the bifunctions $f$ defined as

$$f(x, y) := y_1^2 + y_2^2 + y_3^2 + y_4^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2).$$

An elementary computation shows that conditions $A_1 - A_5$ are satisfied for $f$.

To define a nonexpansive semigroup let us consider the matrix

$$T(s) = \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},\ s \in \mathbb{R},$$

and let

$$T(s)x = \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} x$$

$$= \begin{pmatrix} e^{-s} & 0 & 0 & \cdots & 0 \\ 0 & e^{-s} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

It is easy to verify that $\{T(s) : \; s \geq 0\}$ is a nonexpansive semigroup on $C$ and that the common solution-set is $\Omega = \mathcal{F} \cap Sol(C, A) \cap Sol(C, f) = \{(0, 0, 0, 0, x_5, \ldots, x_n) : \; x_i \in [0, 1], i = 5, 6, \ldots, n\}$.

**Conclusion** We have proposed a hybrid method for solving a system involving pseudomonotone equilibrium problem, variational inequality and fixed point of a semigroup nonexpansive mappings. For handling pseudomonotone problem we have used the extragrandient with an Armijo lineseach. The strong convergence of the proposed method have established by using cutting planes.

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