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WEIGHTED NORM ESTIMATE FOR THE GENERAL HAAR SHIFT OPERATORS VIA ITERATING BELLMAN FUNCTION METHOD

Daewon Chung

Abstract. It is shown that for a general Haar shift operator, and a weight in the A_2 weight class, we establish the weighted norm estimate which linearly depends on A_2-characteristic $[w]_{A_2}$. Although the result is now well known, we introduce the new method, which is called the iterated Bellman function method, to provide the estimate.

1. Introduction

It is well known that a Calderón-Zygmund operator is bounded on the weighted Lebesgue space $L^p(w)$ if the weight w satisfies the famous Muckenhoupt A_p-condition.

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1}\,dx\right) < \infty\,. \tag{1}$$

We call the quantity $[w]_{A_p}$ the A_p-characteristic of the weight w. It has been an long time conjecture finding the best constant in terms of the A_p-characteristic $[w]_{A_p}$. That is, one looks for a function $\phi(x)$, sharp in terms of its growth, such that:

$$\|Tf\|_{L^p(w)} \le C\phi([w]_{A_p})\|f\|_{L^p(w)}\,,$$

where T stands for the general Calderón-Zygmund operator. It was called A_2-*conjecture*, because knowing a bound on $L^2(w)$ is crucial due to the extrapolation theorem. Most recently, it was shown in [4] and [5] based on many beautiful and recently developed techniques, [8], [6], [9], and so on. For more detail arguments and brief history, we refer [4] and [5]. Roughly speaking, the authors in [4] and [5] solved the A_2-conjecture by showing the linear estimate of the general Haar shift operator on $L^2(w)$ with polynomial dependence on

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its complexity via Corona decomposition. In this paper, we present the linear estimate of the general Haar shift operator on $L^2(w)$ by different technique, so called iterated Bellman function method. However, we could not get the polynomial dependence on the complexity. It will be discussed at the end of this paper.

Through out the paper, we denote a constant by C which may change line by line and we indicate its dependence on parameters using a parenthesis. In Section 2 we will introduce notations, some basic facts, and useful lemmas and theorems. We also present the main result of this paper. In Sections 3 and 4 we prove our main result. Finally in Section 5 we will discuss about our result and more.

2. Preliminaries

2.1. Definitions and Main Results

For the simplicity and convenience, we will deal only in the real line, $\mathbb{R}$. Intervals of the form $[kw^{-j},(k+1)2^{-j})$ for some integers j,k are called dyadic intervals. Let us denote $\mathcal{D}$ the collection of all dyadic intervals, and let us denote $\mathcal{D}(J)$ the collection of all dyadic subintervals of J. We say the positive almost everywhere and locally integrable function w, a weight, satisfies the A_2 condition if:

$$[w]_{A_2} := \sup_I \langle w\rangle_I \langle w^{-1}\rangle_I < \infty,$$

where the supremum is taken over all intervals and $\langle w\rangle_I$ stands for the average of w over I. For any interval $I\in\mathcal{D}$, there is a Haar function defined by

$$h_I(x) = \frac{1}{|I|^{1/2}}\left(\chi_{I_+}(x) - \chi_{I_-}(x)\right),$$

where χ_I dentes the characteristic function of the interval I. It is well known fact that the Haar system $\{h_I\}_{I\in\mathcal{D}}$ is an orthonormal system in L^2. Let us introduce a proper orthonormal system for $L^2(w)$ defined by

$$h_I^w(x) = \frac{1}{w(I)^{1/2}}\left[\frac{w(I_-)^{1/2}}{w(I_+)^{1/2}}\chi_{I_+}(x) - \frac{w(I_+)^{1/2}}{w(I_-)^{1/2}}\chi_{I_-}(x)\right],$$

where $w(I)=\int_I w$. We defined the weighted inner product by $\langle f,g\rangle_w = \int fgw$. Then, every function $f\in L^2(w)$ can be written as

$$f = \sum_{I\in\mathcal{D}} \langle f, h_I^w\rangle_w\, h_I^w,$$

where the sum converges a.e. in $L^2(w)$. Moreover,

$$\|f\|^2_{L^2(w)} = \sum_{I\in\mathcal{D}} |\langle f,h_I^w\rangle_w|^2.$$

We now defined the weighted average $\langle f\rangle_{J,w} := w(J)^{-1}\int_J f(x)w(x)dx$. For $I \supsetneq J$, h_I^w is constant on J. We denote this constant by $h_I^w(J)$. Then we can write the weighted averages

$$\langle f\rangle_{J,w} = \sum_{I\in\mathcal{D}:I\supsetneq J} \langle g, h_I^w\rangle_w h_I^w(J)\,.$$

As it was introduced in [5], we say g_I is a generalized Haar function if it is a linear combination of an usual Haar function and χ_I.

Definition 1. For $m,n\in\mathbb{N}$, a general Haar shift operator with complexity $\tau=\max\{m,n\}$ is defined by

$$Sf := \sum_{I\in\mathcal{D}} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} |I|^{-1}\langle f, g_K\rangle g_J\,, \tag{2}$$

where g_K and g_J are generalized Haar functions for the intervals K and J respectively and normalized, that is

$$\|g_L\|_{L^\infty}\cdot\|g_J\|_{L^\infty}\le 1\,.$$

Definition 2. We say an operator given by (2) is an elementary Haar shift operator of

(a) type 1, if g_K and g_J has a mean zero property,
(b) type 2, if one of g_K and g_J have a mean zero property but the other one does not, and they have a property: for all $L\in\mathcal{D}$,

$$\frac{1}{|L|}\sum_{I\in\mathcal{D}(L)} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} |I|\|g_K\|_{L^\infty}^2\|g_J\|_{L^\infty}^2<\infty\,, \tag{3}$$

(c) type 3, if g_K and g_J does not have a mean zero property and they have a property: for all $L\in\mathcal{D}$,

$$\frac{1}{|L|}\sum_{I\in\mathcal{D}(L)} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} |I|\|g_K\|_{L^\infty}\|g_J\|_{L^\infty}<\infty\,. \tag{4}$$

Note that one can easily see that the dyadic paraproduct,

$$\pi_b f := \sum_{I\in\mathcal{D}} \langle b, h_I\rangle\langle f\rangle_I h_I\,,$$

is an elementary Haar shift operator of type 2 and the property (3) is provided by $b\in BMO$. That is, for any $L\in\mathcal{D}$,

$$\frac{1}{|L|}\sum_{I\in\mathcal{D}(L)} \langle b, h_I\rangle^2 \le \|b\|_{BMO}\,.$$

Since

$$\pi_b^*\pi_b f = \sum_{I\in\mathcal{D}} \langle b, h_I\rangle^2 \langle f\rangle_J \frac{\chi_I}{|I|} = \sum_{I\in\mathcal{D}} \langle b, h_I\rangle^2 |I|^{-2}\langle f,\chi_I\rangle\chi_I\,,$$

the composition of an adjoint of dyadic paraproduct and dyadic paraproduct, $\pi_b^*\pi_b$, is an elementary Haar shift operator of type 3. Let us define the constant $B := \max\{B_2, B_3\}$, where

$$B_2 := \sup_{L\in\mathcal{D}} \frac{1}{|L|} \sum_{I\in\mathcal{D}(L)} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} |I| \|g_K\|_{L^\infty}^2 \|g_J\|_{L^\infty}^2$$

$$B_3 := \sup_{L\in\mathcal{D}} \frac{1}{|L|} \sum_{I\in\mathcal{D}(L)} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} |I| \|g_K\|_{L^\infty} \|g_J\|_{L^\infty} .$$

We will indicate the type by using a subscript. For all $i = 1, 2, 3$ and interval $L \in \mathcal{D}$, we define the restricted Haar shift operator S_i^L,

$$S_i^L f := \sum_{I\in\mathcal{D}(L)} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} |I|^{-1}\langle f, g_K\rangle g_J .$$

Then we are going to prove the following Theorem.

Theorem 2.1. *For all $i = 1, 2$, intervals L, and weight $w \in A_2$, there exists a constant $C(m, n)$ such that*

$$\|S_i^L(w^{-1}\chi_L)\|_{L^2(w)} \le C(m, n)[w]_{A_2} w^{-1}(L)^{1/2} . \tag{5}$$

Theorem 2.2. *Let S be a general Haar shift operator with complexity τ. Then, for all $f \in L^p(w)$, there exists a constant only depend on τ and B such that*

$$\|Sf\|_{L^p(w)} \le C(\tau, B)[w]_{A_p}^{\max\{1,\frac{1}{p-1}\}} \|f\|_{L^p(w)} .$$

2.2. Lemmas and Theorems

We now introduce several useful lemmas and theorems. One can find the detail proof in the indicated references, otherwise we will present the proof. We also use the notation $\Delta_I w := (\langle w\rangle_{I_+} - \langle w\rangle_{I_-})/2$.

Lemma 2.3 ([12]). *If $w \in A_2$ then there exist a constant C such that for all dyadic intervals $J \in \mathcal{D}$,*

$$\sum_{I\in\mathcal{D}(J)} \frac{|\Delta_I w|^2 |I|}{\langle w\rangle_I} \le C[w]_{A_2} w(J) .$$

Lemma 2.4 ([10]). *For all dyadic intervals J and all weight w,*

$$\sum_{I\in\mathcal{D}(J)} \frac{|\Delta_I w|^2 |I|}{\langle w\rangle_I^3} \le C w^{-1}(J) .$$

Lemma 2.5 ([1]). *For any weight w, so that w^{-1} is a weight as well, if $\{\lambda_I\}$ is a Carleson sequence of nonnegative numbers, that is, there exists a constant*

$Q > 0$ *such that for all dyadic intervals* $J \in \mathcal{D}$,

$$\sum_{I \in \mathcal{D}(J)} \lambda_I \leq Q|J|\,,$$

then

$$\sum_{I \in \mathcal{D}(J)} \frac{\lambda_I}{\langle w \rangle_I} \leq 4Qw^{-1}(J)\,.$$

Furthermore, if $w \in A_2$ *then we have*

$$\sum_{I \in \mathcal{D}(J)} \lambda_I \langle w \rangle_I \leq 4Q[w]_{A_2} w(J)\,.$$

Lemma 2.6 ([1])**.** *If* $w \in A_2$ *then there exist a constant* $C > 0$ *such that for all dyadic intervals* $J \in \mathcal{D}$,

$$\sum_{I \in \mathcal{D}(J)} \frac{|\Delta_I w|^2 \langle w^{-1} \rangle_I |I|}{\langle w \rangle_I} \leq C[w]_{A_2}|J|\,.$$

The following lemma is the one of our main result in this paper, which will be proved in Section 4.

Lemma 2.7. *There is a positive constant* $C(m)$ *so that for all dyadic intervals* $J \in \mathcal{D}$

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} |I| \langle w \rangle_I^{-7/4} \langle w^{-1} \rangle_I^{1/4} \left(\sum_{j=1}^{m+1} 2^{m+1-j} \sum_{K \in \mathcal{D}(I): |K| = 2^{1-j}|I|} \Delta_K w \right)^2 \leq C(m) \langle w \rangle^{1/4} \langle w^{-1} \rangle_J^{1/4}\,,$$

whenever w *is a weight. Moreover, if* $w \in A_2^d$ *then for all* $J \in \mathcal{D}$

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} |I| \langle w \rangle_I^{-1} \langle w^{-1} \rangle_I \left(\sum_{j=1}^{m+1} 2^{m+1-j} \sum_{K \in \mathcal{D}(I): |K| = 2^{-m}|I|} \Delta_K w \right)^2 \leq C(m)[w]_{A_2^d}\,.$$

Lemma 2.7 is, in fact, a general version of Lemma 2.6, i.e. Lemma 2.6 is the special case ($j = 1$) of Lemma 2.7. The case of $j = 2$ was also appeared in [2]. One of the main tool is a two-weighted bilinear embedding theorem. The original version of such a theorem appeared in [7] and the authors presented the necessary and sufficient conditions. The author in [10] presented the following Theorem, which contains more easy necessary condition but not sufficient, by a Bellman function method. It is not hard to see that this version is a corollary of the original theorem in [7].

Theorem 2.8 (Bilinear Embedding Theorem,[10])**.** *Let* w *and* u *be weights so that* $\langle w \rangle_I \langle u \rangle_I \leq Q$ *for all intervals* I *and let* $\{\alpha_I\}$ *be a non-negative sequence so that the three estimates below hold for all* J

$$\sum_{I \in \mathcal{D}(J)} \frac{\alpha_I}{\langle w \rangle_I} \leq Qu(J), \quad \sum_{I \in \mathcal{D}(J)} \frac{\alpha_I}{\langle u \rangle_I} \leq Qw(J), \quad \text{and} \sum_{I \in \mathcal{D}(J)} \alpha_I \leq Q|J|\,.$$

Then there is C such that for all $f \in L^2(w)$ and $g \in L^2(u)$

$$\sum_{I \in \mathcal{D}} \alpha_I \langle f \rangle_{I,w} \langle g \rangle_{I,v} \le CQ \|f\|_{L^2(w)} \|g\|_{L^2(v)} .$$

3. Proof of main theorem

We will prove Theorem 2.1 for each type separately.

3.1. The case: Haar shift operator of type 1

We are going to show (5) for $m > n$ by duality arguments, that is for any positive function $f \in L^2(w)$ there exist a constant C such that

$$\left| \left\langle S_i^L(w^{-1}\chi_L), f \right\rangle_w \right| \le C[w]_{A_2} w^{-1}(L)^{1/2} \|f\|_{L^2(w)} . \tag{6}$$

To see the case: $m = n$, one needs almost same arguments with the case $m > n$, and the other case: $m < n$ is in fact the dual operator of the case $m > n$. We now fix the index $m > n$ and split (6) by two parts.

$$\begin{aligned}
\left| \left\langle S_2^L(w^{-1}\chi_L), f \right\rangle_w \right| &\le \left| \left\langle S_{2,1}^L(w^{-1}\chi_L), f \right\rangle_w \right| + \left| \left\langle S_{2,2}^L(w^{-1}\chi_L), f \right\rangle_w \right| \\
&= \left| \left\langle \sum_{I \in \mathcal{D}(L)} \sum_{\substack{J,K \in \mathcal{D}(I); J \subset K \\ |J| = 2^{-m}|I|,\, |K| = 2^{-n}|I|}} |I|^{-1} \langle w^{-1}, h_K \rangle h_J, f \right\rangle_w \right| \\
&\quad + \left| \left\langle \sum_{I \in \mathcal{D}(L)} \sum_{\substack{J,K \in \mathcal{D}(I); J \cap K = \emptyset \\ |J| = 2^{-m}|I|,\, |K| = 2^{-n}|I|}} |I|^{-1} \langle w^{-1}, h_K \rangle h_J, f \right\rangle_w \right| \\
&= \mathfrak{L}_1 + \mathfrak{L}_2 .
\end{aligned}$$

Expanding f in the weighted Haar system in $L^2(w)$, we have

$$\mathfrak{L}_1 = \left| \sum_{I \in \mathcal{D}(L)} \sum_{\substack{J,K \in \mathcal{D}(I); J \subset K \\ |J| = 2^{-m}|I|,\, |K| = 2^{-n}|I|}} |I|^{-1} \langle w^{-1}, h_K \rangle \sum_{I' \in \mathcal{D}} \langle f, h_{I'}^w \rangle_w \langle h_J, h_{I'}^w \rangle_w \right| . \tag{7}$$

We will estimate the sum (7) uniformly on J and K. So, we now fix J and K so that $J \subset K \subset I$ and $2^m|J| = 2^n|K| = |I|$. Since $\langle h_J, h_{I'}^w \rangle_w$ can be non-zero only for $J \subseteq I'$, we spilt the sum (7) into $m+2$ sums, $I' = J, J^1, ..., J^{m-n} = K, J^{m-n+1} = K^1, ..., J^m = I$ and $I' \supsetneq I$. (J^k means J's k-the parent).

Let $J \subseteq K$, h_J be a normalized Haar function with mean zero property, and h_K^w be a usual weighted Haar function (not normalized). Then one can easily check that

$$|\langle h_J, h_K^w \rangle_w| \le \|h_J\|_{L^\infty} w(J)^{1/2} .$$

For the sum $I' = \{J^k\}_{k=0,\dots,m}$, we have

$$\sum_{I\in\mathcal{D}(L)}\sum_{k=0}^{m}|I|^{-1}\langle w^{-1},h_K\rangle\langle f,h^w_{J^k}\rangle\langle h_J,h^w_{J^k}\rangle_w$$
$$= \Big(\sum_{I\in\mathcal{D}(L)}\sum_{k=0}^{m}\langle f,h^w_{J^k}\rangle^2\Big)^{1/2}\Big(\sum_{I\in\mathcal{D}(L)}\sum_{k=0}^{m}|I|^{-2}\langle w^{-1},h_K\rangle^2\langle h_J,h^w_{J^k}\rangle_w^2\Big)^{1/2}$$
$$\le (m+1)^{1/2}\|f\|_{L^2(w)}\Big(\sum_{I\in\mathcal{D}(L)}\sum_{k=0}^{m}|I|^{-2}|K|^2\|h_K\|^2_{L^\infty}|\Delta_K w^{-1}|^2\|h_J\|^2_{L^\infty}w(J)\Big)^{1/2}$$
$$\le (m+1)^{1/2}\|f\|_{L^2(w)}\Big(\sum_{I\in\mathcal{D}(L)}\sum_{k=0}^{m}\frac{|K|^2}{|I|^2}|I||\Delta_K w^{-1}|^2\langle w\rangle_I\Big)^{1/2}$$
$$\le C2^{-n}(m+1)[w]^{1/2}_{A_2}\|f\|_{L^2(w)}\Big(\sum_{I\in\mathcal{D}(L)}\frac{|I||\Delta_K w^{-1}|^2}{\langle w^{-1}\rangle_I}\Big)^{1/2}$$
$$\le C(m+1)2^{-n}C(n))^{1/2}[w]_{A_2}w^{-1}(L)^{1/2}\|f\|_{L^2(w)}\,. \tag{8}$$

For the sum $I' \supsetneq I$, we have

$$\Big|\sum_{I\in\mathcal{D}(L)}|I|^{-1}\langle w^{-1},h_K\rangle\sum_{I':I'\supsetneq I}\langle f,h^w_{I'}\rangle_w\langle h_J,h^w_{I'}\rangle_w\Big|$$
$$\le \sum_{I\in\mathcal{D}(L)}|I|^{-1}|\langle w^{-1},h_K\rangle|\,|\langle w,h_J\rangle|\langle f\rangle_{I,w}\langle \chi_L\rangle_{I,w^{-1}}\,. \tag{9}$$

Although the sum is taken over the interval I, for each I, K and J are uniquely determined. By Bilinear Embedding Theorem, our desired estimate for (9) holds, provided the following three embedding conditions hold:

$$\sum_{I\in\mathcal{D}(L)}\frac{|\langle w^{-1},h_K\rangle|\,|\langle w,h_J\rangle|}{|I|\langle w\rangle_I}\le C[w]_{A_2}w^{-1}(L) \tag{10}$$

$$\sum_{I\in\mathcal{D}(L)}\frac{|\langle w^{-1},h_K\rangle|\,|\langle w,h_J\rangle|}{|I|\langle w^{-1}\rangle_I}\le C[w]_{A_2}w(L) \tag{11}$$

$$\sum_{I\in\mathcal{D}(L)}\frac{|\langle w^{-1},h_K\rangle|\,|\langle w,h_J\rangle|}{|I|}\le C[w]_{A_2}|L|\,. \tag{12}$$

Embedding condition (10): After applying Cauchy-Schwarz inequality, using Lemma 2.3 and Lemma 2.4 yields that.

$$\sum_{I\in\mathcal{D}(L)}\frac{|\langle w^{-1},h_K\rangle|\,|\langle w,h_J\rangle|}{|I|\langle w\rangle_I}$$

$$= \sum_{I\in\mathcal{D}(L)} \sqrt{\frac{|K|\,|J|}{|I|\,|I|}} \frac{\|h_K\|_{L^\infty}\sqrt{|K|}|\Delta_K w^{-1}|\|h_J\|_{L^\infty}\sqrt{|J|}|\Delta_J w|}{\langle w\rangle_I}$$
$$\le 2^{\frac{-(m+n)}{2}} \Big(\sum_{I\in\mathcal{D}(L)} |\Delta_K w^{-1}|^2|K|\langle w\rangle_K\Big)^{1/2} \Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_J w|^2|J|}{\langle w\rangle_I^2\langle w\rangle_K}\Big)^{1/2}$$
$$\le 2^{\frac{-(m+n)}{2}} [w]_{A_2}^{1/2} \Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_K w^{-1}|^2|K|}{\langle w^{-1}\rangle_K}\Big)^{1/2} \Big(2^{3m-n}\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_J w|^2|J|}{\langle w\rangle_J^3}\Big)^{1/2}$$
$$\le C2^{\frac{-(m+n)}{2}+\frac{3m-n}{2}}[w]_{A_2}(w^{-1}(J_L)w^{-1}(K_L))^{1/2}$$
$$\le C2^{m-n}[w]_{A_2}w^{-1}(L)\,.$$

Embedding condition (11): One can also uses Cauchy-Schwarz inequality, Lemma 2.4 and Lemma 2.3

$$\sum_{I\in\mathcal{D}(L)} \frac{|\langle w^{-1},h_K\rangle|\,|\langle w,h_J\rangle|}{|I|\langle w^{-1}\rangle_I}$$
$$= \sum_{I\in\mathcal{D}(L)} \sqrt{\frac{|K|\,|J|}{|I|\,|I|}} \frac{\|h_K\|_{L^\infty}\sqrt{|K|}|\Delta_K w^{-1}|\|h_J\|_{L^\infty}\sqrt{|J|}|\Delta_J w|}{\langle w^{-1}\rangle_I}$$
$$\le 2^{\frac{-(m+n)}{2}} \Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_K w^{-1}|^2|K|}{\langle w^{-1}\rangle_I^3}\Big)^{1/2} \Big(\sum_{I\in\mathcal{D}(L)} |\Delta_J w|^2|J|\langle w^{-1}\rangle_I\Big)^{1/2}$$
$$\le 2^{\frac{-(m+n)}{2}} \Big(2^{3n}\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_K w^{-1}|^2|K|}{\langle w^{-1}\rangle_K^3}\Big)^{1/2} \Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_J w|^2|J|\langle w^{-1}\rangle_I\langle w\rangle_J}{\langle w\rangle_J}\Big)^{1/2}$$
$$\le C2^{\frac{-(m+n)}{2}+\frac{3n}{2}+\frac{m}{2}}([w]_{A_2}w(K_L))^{1/2} \Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_J w|^2|J|}{\langle w\rangle_J}\Big)^{1/2}$$
$$\le C2^n[w]_{A_2}(w(K_L)w(J_L))^{1/2}$$
$$\le C2^n[w]_{A_2}w(L)\,.$$

Embedding condition (12):

$$\sum_{I\in\mathcal{D}(L)} \frac{|\langle w^{-1},h_K\rangle|\,|\langle w,h_J\rangle|}{|I|} = \sum_{I\in\mathcal{D}(L)} \frac{|K|\,|J|}{|I|\,|I|}\|h_K\|_{L^\infty}\sqrt{|I|}|\Delta_K w^{-1}|\|h_J\|_{L^\infty}\sqrt{|I|}|\Delta_J w|$$
$$\le 2^{-(m+n)} \Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_K w^{-1}|^2\langle w\rangle_I|I|}{\langle w^{-1}\rangle_I}\Big)^{1/2} \Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_J w|^2\langle w^{-1}\rangle_I|I|}{\langle w\rangle_I}\Big)^{1/2}$$
$$\le 2^{-(m+n)}(C(m)C(n))^{1/2}[w]_{A_2}|L|\,.$$

Here the last inequality uses Lemma 2.7.

We now turn to the proof for the term $\mathfrak{L}_2$. Again, we expand f in the weighted Haar system in $L^2(w)$. Then we have

(13)

$$\mathfrak{L}_2 = \left| \sum_{I\in\mathcal{D}(L)} \sum_{\substack{J,K\in\mathcal{D}(I);J\cap K=\emptyset \\ |J|=2^{-m}|I|,\,|K|=2^{-n}|I|}} |I|^{-1}\langle w^{-1}, h_K\rangle \sum_{I'\in\mathcal{D}} \langle f, h^w_{I'}\rangle_w \langle h_J, h^w_{I'}\rangle_w \right|.$$

Similarly with the term $\mathfrak{L}_1$, we will estimate $\mathfrak{L}_2$ uniformly on J and K after we fix it such that $K \subseteq I$, $J \subset I$, and $J\cap K = \emptyset$. One can easily see that $K^n = I$ and $J^m = I$. We now split the sum (13) into $m+1$ sums, $I' = J, J^1, ..., J^m = I$ and $I' \supsetneq I$. For the sums $I' = J, J^1, ..., J^m$, we have

$$\begin{aligned}
&\sum_{I\in\mathcal{D}(L)} \sum_{k=0}^{m} |I|^{-1}\langle w^{-1}, h_K\rangle\langle f, h^w_{J^k}\rangle\langle h_J, h^w_{J^k}\rangle_w \\
&\quad\le \Big(\sum_{I\in\mathcal{D}(L)} \sum_{k=0}^{m} \langle f, h^w_{J^k}\rangle^2 \Big)^{1/2} \Big(\sum_{I\in\mathcal{D}(L)} \sum_{k=0}^{m} |I|^{-2}\langle w^{-1}, h_K\rangle^2 \langle h_J, h^w_{J^k}\rangle^2_w \Big)^{1/2} \\
&\quad\le (m+1)^{1/2}\|f\|_{L^2(w)} \Big(\sum_{I\in\mathcal{D}(L)} \sum_{k=0}^{m} \|I|^{-2}|K|^2\|h_K\|^2_{L^\infty}|\Delta_L w^{-1}|^2\|h\|^2_{L^\infty} w(J) \Big)^{1/2} \\
&\quad\le (m+1)^{1/2}\|f\|_{L^2(w)} \Big(\sum_{I\in\mathcal{D}(L)} \sum_{k=0}^{m} \frac{|K|^2}{|I|^2}|I||\Delta_K w^{-1}|^2\langle w\rangle_I \Big)^{1/2} \\
&\quad\le C2^{-2}(m+1)[w]^{1/2}_{A_2}\|f\|_{L^2(w)} \Big(\sum_{I\in\mathcal{D}(L)} \frac{|I||\Delta_K w^{-1}|^2}{\langle w^{-1}\rangle_I} \Big)^{1/2}
\end{aligned}$$

(14)

$$\le C2^{-2}C(n)^{1/2}(m+1)[w]_{A_2}\|f\|_{L^2(w)} w^{-1}(L)^{1/2}.$$

For the sum $I' \supsetneq I$, we use the same argument with the estimate of $\mathfrak{L}_1$ then we have the same upper bounds with (8), i.e.

$$\begin{aligned}
&\left| \sum_{I\in\mathcal{D}(L)} \langle w^{-1}, h_K\rangle \sum_{I':I'\supsetneq I} \langle f, h^w_{I'}\rangle_w \langle h_J, h^w_{I'}\rangle_w \right| \\
&\qquad\qquad \le C2^{-n}((m+1)C(n))^{1/2}[w]_{A_2} w^{-1}(L)^{1/2}\|f\|_{L^2(w)}.
\end{aligned}$$

3.2. The case: Haar shift operator of type 2

We now assume that g_J has a mean zero property and g_K does not. Then we can write $g_J = \|g_J\|_{L^\infty}\cdot h_J = a_J h_J$ and $g_K = \|g_K\|_{L^\infty}\cdot \chi_K = a_K\chi_J$. The other case is an adjoint of this. We will prove for each J and K such that $2^m|J| = 2^n|K| = |I|$, and we will denote $a_I = a_J a_K$ for each fixed J and K. Then

$$\begin{aligned}
\left| \left\langle \sum_{I\in\mathcal{D}} |I|^{-1}\langle w^{-1}\chi_I, h_J\rangle h_K, f \right\rangle_w \right| &\le \sum_{I\in\mathcal{D}(L)} |a_I||\Delta_I w^{-1}|\langle w\rangle_I |I|\langle f\rangle_{I,w}\langle \chi_I\rangle_{I,w^{-1}}
\end{aligned}$$

(15)

$$\le C[w]_{A_2} w^{-1}(L)^{1/2}\|f\|_{L^2(w)}.$$

Here the inequality (15) is provided by the following three embedding conditions. To see the each embedding condition, we need to use the property (3) which is, in this setting, $\sum_{I\in\mathcal{D}(L)} a_I^2|I| \le C|L|$.

$$\sum_{I\in\mathcal{D}(L)} \frac{|a_I||\Delta_I w^{-1}|\langle w\rangle_I|I|}{\langle w\rangle_I} \le C[w]_{A_2} w^{-1}(L)\,, \tag{16}$$

$$\sum_{I\in\mathcal{D}(L)} \frac{|a_I||\Delta_I w^{-1}|\langle w\rangle_I|I|}{\langle w^{-1}\rangle_I} \le C[w]_{A_2} w(L)\,, \tag{17}$$

$$\sum_{I\in\mathcal{D}(L)} |a_I||\Delta_I w^{-1}|\langle w\rangle_I|I| \le C[w]_{A_2}|L|\,. \tag{18}$$

Embedding condition (16): We use Lemma (2.3) and Lemma(2.5) with $\lambda_I = |a_I|^2|I|$.

$$\begin{aligned}\sum_{I\in\mathcal{D}(L)} |a_I||\Delta_I w^{-1}||I| &\le \Big(\sum_{I\in\mathcal{D}(L)} |a_I|^2|I|\langle w^{-1}\rangle_I\Big)^{1/2}\Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_I w^{-1}|^2|I|}{\langle w^{-1}\rangle_I}\Big)^{1/2}\\ &\le 4C[w]_{A_2} w^{-1}(L)\,.\end{aligned}$$

Embedding condition (17): We use Lemma (2.5) twice, first with $\lambda_I = |a_I|^2|I|$ and second with $\lambda_I = |a_I|^2|I|/\langle w^{-1}\rangle_I$, and Lemma (2.6).

$$\begin{aligned}\sum_{I\in\mathcal{D}(L)} \frac{|a_I||\Delta_I w^{-1}|\langle w\rangle_I|I|}{\langle w^{-1}\rangle_I} &\le \Big(\sum_{I\in\mathcal{D}(L)} \frac{|a_I|^2|I|\langle w\rangle_I}{\langle w^{-1}\rangle_I}\Big)^{1/2}\Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_I w^{-1}|^2|I|\langle w\rangle_I}{\langle w^{-1}\rangle_I}\Big)^{1/2}\\ &\le 4C[w]_{A_2} w(L)\,.\end{aligned}$$

Embedding condition (18): We use the assumption and Lemma (2.6).

$$\begin{aligned}\sum_{I\in\mathcal{D}(L)} |a_I||\Delta_I w^{-1}|\langle w\rangle_I|I| &\le \Big(\sum_{I\in\mathcal{D}(L)} |a_I|^2|I|\langle w\rangle_I\langle w^{-1}\rangle_I\Big)^{1/2}\Big(\sum_{I\in\mathcal{D}(L)} \frac{|\Delta_I w^{-1}|^2|I|\langle w\rangle_I}{\langle w^{-1}\rangle_I}\Big)^{1/2}\\ &\le C[w]_{A_2}|L|\,.\end{aligned}$$

3.3. The case: Haar shift operator of type 3

We now turn to the Haar shift operator with non-zero mean case.

$$\begin{aligned}Sf &= \sum_{I\in\mathcal{D}} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} |I|^{-1}\langle f, g_K\rangle g_J\\ &= \sum_{I\in\mathcal{D}} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} a_K a_J |I|^{-1}\langle f, \chi_K\rangle \chi_J\,,\end{aligned}$$

where $a_K = \|h_K\|_{L^\infty}$ and $a_J = \|h_J\|_{L^\infty}$. We are going to prove an analog of (6). That is,

$$\left|\left\langle \sum_{I\in\mathcal{D}} \sum_{\substack{J,K\in\mathcal{D}(I)\\ |J|=2^{-m}|I|,|K|=2^{-n}|I|}} a_K a_J |I|^{-1}\langle w^{-1}\chi_L, \chi_K\rangle \chi_J, f\right\rangle_w\right|$$

$$\leq C[w]_{A_2} w^{-1}(L)^{1/2}\|f\|_{L^2(w)}\,. \tag{19}$$

Similarly with mean zero case, we fix m and n and estimate uniformly on K and J. Then we need to prove the follow.

$$\left|\sum_{I\in\mathcal{D}(L)} a_I |I|^{-1} w^{-1}(K)\langle f\rangle_{J,w} w(J)\right| \leq C[w]_{A_2} w^{-1}(L)^{1/2}\|f\|_{L^2(w)}\,. \tag{20}$$

$$\begin{aligned}\left|\sum_{I\in\mathcal{D}(L)} a_I |I|^{-1} w^{-1}(K)\langle f\rangle_{J,w} w(J)\right| &\leq \sum_{I\in\mathcal{D}(L)} |a_I|\,|I|\langle w^{-1}\rangle_I \langle w\rangle_I \langle f\rangle_{I,w}\langle \chi_L\rangle_{I,w^{-1}}\\ &\leq [w]_{A_2} \sum_{I\in\mathcal{D}(L)} |a_I|\,|I|\langle f\rangle_{I,w}\langle \chi_L\rangle_{I,w^{-1}}\\ &\leq C[w]_{A_2} w^{-1}(L)^{1/2}\|f\|_{L^2(w)}\,. \end{aligned}\tag{21}$$

Inequality (21) is provided by Bilinear Embedding Theorem and the following three embedding conditions:

$$\sum_{I\in\mathcal{D}(L)} \frac{|a_I||I|}{\langle w\rangle_I} \leq C w^{-1}(L) \tag{22}$$

$$\sum_{I\in\mathcal{D}(L)} \frac{|a_I||I|}{\langle w^{-1}\rangle_I} \leq C w(L) \tag{23}$$

$$\sum_{I\in\mathcal{D}(L)} |a_I||I| \leq C|L|\,. \tag{24}$$

Embedding condition (24) can be deduced from the Carleson condition of $a_K a_L$. Then the other conditions are easy consequence of Lemma 2.5 with (24).

4. Proof of the iterating 4th root Lemma

Let us consider the function $B(u,v) = \sqrt[4]{uv}$ and the domain $\mathfrak{D}_m$ which is given by

$$\mathfrak{D}_m = \{(u,v)\in\mathbb{R}^2_+ \,|\, uv \geq 1/2^m\}\,.$$

It is known [1] that $B(u,v)$ satisfies that the following differential inequality in $\mathfrak{D}_0$

$$-(du,dv)d^2B(u,v)\begin{pmatrix} du \\ dv\end{pmatrix} \geq \frac{1}{8}\frac{v^{1/4}}{u^{7/4}}|du|^2\,. \tag{25}$$

Furthermore, this implies that the following convexity condition. For all (u,v), $(u_\pm, v_\pm) \in \mathfrak{D}_1$,

$$B(u,v) - \frac{B(u_+,v_+) + B(u_-,v_-)}{2} \geq \frac{1}{18\cdot 2^{7/4}}\frac{v^{1/4}}{u^{7/4}}(\Delta_1^1 u)^2\,, \tag{26}$$

where $u = (u_+ + u_-)/2$, $v = (v_+ + v_-)/2$ and $\Delta_1^1 u = (u_+ - u_-)/2$. One can find the constant in (26) by using (25) and Taylor's theorem. We now consider the function

$$A_1(u,v,\Delta_1^1 u) := (27\cdot 2^{7/4})B(u,v) + B(u+\Delta_1^1, u) + B(u - \Delta_1^1 u, v)\,.$$

We also consider the domain

$$\mathfrak{F}_1 = \{(u,v,\Delta_1^1 u)\,|\,(u,v),\ (u+\Delta_1^1 u, v),\ \text{and } (u-\Delta_1^1 u, v)\in\mathfrak{D}_1\}\,.$$

Then, $A_1(u,v,\Delta_1^1 u)$ satisfies the following size and convexity conditions. For all $(u,v,\Delta_1^1 u)\in\mathfrak{F}_1$,

$$0 \leq A_1(u,v,\Delta_1^1 u) \leq (C_1+2)\sqrt[4]{uv}\,. \tag{27}$$

$$A_1(u,v,\Delta_1^1 u) - \frac{A_1(u_+,v_+,\Delta_2^2 u) + A_1(u_-,v_-,\Delta_2^1 u)}{2} \geq D_1\frac{v^{1/4}}{u^{7/4}}\Big(4(\Delta u_1^1)^2 + 2(\Delta_2^1 u)^2 + 2(\Delta_2^2 u)^2\Big)\,. \tag{28}$$

This appeared in [2]. Here $\Delta_2^2 u = (u_{++} - u_{+-})/2$ and $\Delta_2^1 u = (u_{-+} - u_{--})/2$. The size condition (27) can be seen easily using the definitions of A_1 and B and the inequality $(\sqrt[4]{u} + \sqrt[4]{v})/2 \leq \sqrt[4]{(u+v)/2}$. To see more detail, let us consider the function

$$\begin{aligned} A_2(u,&v,\Delta_1^1 u,\Delta_2^1 u,\Delta_2^2 u) := aA_1(u,v,\Delta_1^1 u, v) + B(u+\Delta_1^1 u + \Delta_2^2 u, v)\\ &+ B(u+\Delta_1^1 u - \Delta_2^2 u, v) + B(u - \Delta_1^1 u + \Delta_2^1 u, v) + B(u-\Delta_1^1 u - \Delta_2^1 u, v)\,. \end{aligned}$$

on the domain $\mathfrak{F}_2$. Here $(u,v,\Delta_1^1 u,\Delta_2^1 u,\Delta_2^2 u)\in\mathfrak{F}_2$ means all pairs (u,v), $(u+\Delta_1^1, v)$, $(u-\Delta_1^1, v)$, $(u+\Delta_1^1 u+\Delta_2^2 u, v)$, $(u+\Delta_1^1 u - \Delta_2^2 u, v)$, $(u-\Delta_1^1 u + \Delta_2^1 u, v)$ and $(u-\Delta_1^1 u - \Delta_2^1 u, v)$ belong to $\mathfrak{D}_2$. Then A_2 has the size property: if $(u,v,\Delta_1^1 u,\Delta_2^1 u,\Delta_2^2 u)\in\mathfrak{F}_2$, then

$$0 \leq A_2(u,v,\Delta_1^1 u,\Delta_2^1 u,\Delta_2^2 u) \leq (C_2 + 2^2)\sqrt[4]{uv}\,, \tag{29}$$

with $C_2 = a(C_1+2)$, and the convexity property

$$A_2(u,v,\Delta_1^1 u,\Delta_2^1,\Delta_2^2 u) - \frac{A_2(u_+,v_+,\Delta_2^2 u,\Delta_3^3 u\Delta_3^4 u) + A_2(u_-,v_-,\Delta_2^1 u,\Delta_3^1 u\Delta_3^2 u)}{2} \geq D_2\frac{v^{1/4}}{u^{7/4}}\Big(8(\Delta_1^1 u)^2 + 4\sum_{i=1}^{2}(\Delta_2^i u)^2 + 2\sum_{i=1}^{4}(\Delta_3^i u)^2\Big)\,, \tag{30}$$

where $\Delta_3^1 u = \frac{u_{--+} - u_{---}}{2}$, $\Delta_3^2 u = \frac{u_{-++} - u_{-+-}}{2}$, $\Delta_3^3 u = \frac{u_{+-+} - u_{+--}}{2}$, and $\Delta_3^4 u = \frac{u_{+++} - u_{++-}}{2}$.

Proof of (30). We rewrite the left hand side of the inequality (30) as follows.

$$\begin{aligned}
& aA_1(u,v,\Delta_1^1 u) + B(u+\Delta_1^1 u+\Delta_2^2 u, v) + B(u+\Delta_1^1 u - \Delta u_2^2, v) \\
&\quad + B(u-\Delta_1^1 u + \Delta_2^1 u, v) + B(u-\Delta_1^1 u - \Delta_1^2 u, v) \\
&\quad - \frac{1}{2}\Big(aA_1(u_+,v_+,\Delta_2^2 u) + B(u_+ + \Delta_2^2 u + \Delta_3^4 u, v_+) \\
&\quad + B(u_+ + \Delta_2^2 u - \Delta u_3^4, v_+) + B(u_+ - \Delta_2^2 u + \Delta_3^3 u, v_+) \\
&\quad + B(u_+ - \Delta_2^2 u - \Delta_3^3 u, v_+) + aA_1(u_-,v_-,\Delta_2^1 u) \\
&\quad + B(u_- + \Delta_2^1 u + \Delta_3^2 u, v_-) + B(u_- + \Delta_2^1 u - \Delta_3^2 u, v_-) \\
&\quad + B(u_- - \Delta_2^1 u + \Delta_3^1 u, v_-) + B(u_- - \Delta_2^1 u - \Delta_3^1 u, v_-)\Big) \\
&= a\Big(A_1(u,v,\Delta_1^1 u) - \frac{1}{2}\big(A_1(u_+,v_+,\Delta_2^2 u) + A_1(u_-,v_-,\Delta_2^1 u)\big)\Big) \qquad (31)\\
&\quad + B(u_{++},v) + B(u_{+-},v) + B(u_{-+},v) + B(u_{--},v) \\
&\quad - \frac{1}{2}\Big(B(u_+ + \Delta_2^2 u + \Delta_3^4 u, v_+) + B(u_+ + \Delta_2^2 u - \Delta_3^4 u, v_+) \\
&\quad + B(u_+ - \Delta_2^2 u + \Delta_3^3 u, v_+) + B(u_+ - \Delta_2^2 u - \Delta_3^3 u, v_+) \\
&\quad + B(u_- + \Delta_2^1 u + \Delta_3^2 u, v_-) + B(u_- + \Delta_2^1 u - \Delta_3^2 u, v_-) \\
&\quad + B(u_- - \Delta_2^1 u + \Delta_3^1 u, v_-) + B(u_- - \Delta_2^1 u - \Delta_3^1 u, v_-)\Big)\,.
\end{aligned}$$

Using Taylor's theorem and the differential convexity condition (25) of $B(u,v)$, we can estimate the following. For $\alpha \le (k-1)u$ and $|\beta| < v$,

$$\begin{aligned}
B(u+\alpha, v+\beta) &= B(u,v) + \nabla B(u,v)(\alpha,\beta)^t \\
&\qquad + \int_0^1 (1-s)(\alpha,\beta) d^2 B(u+s\alpha, v+s\beta)(\alpha,\beta)^t ds \\
&\le B(u,v) + \nabla B(u,v)(\alpha,\beta)^t - \frac{1}{8}\int_0^1 (1-s)\frac{(v+s\beta)^{1/4}}{(u+s\alpha)^{7/4}}\alpha^2\, ds \\
&\le B(u,v) + \nabla B(u,v)(\alpha,\beta)^t - \frac{\alpha^2}{8(ku)^{7/4}}\int_0^1 (1-s)(v+s\beta)^{1/4} ds \\
&\le B(u,v) + \nabla B(u,v)(\alpha,\beta)^t - \frac{\alpha^2 v^{1/4}}{8(ku)^{7/4}}\int_0^1 (1-s)(1+s\frac{\beta}{v})^{1/4}\, ds \\
&\le B(u,v) + \nabla B(u,v)(\alpha,\beta)^t - \frac{4\alpha^2 v^{1/4}}{8\cdot 9(ku)^{7/4}}\,. \qquad (32)
\end{aligned}$$

Here we use the differential convexity condition (25) and the following simple estimate:

$$\text{for } |\gamma| < 1\,, \quad \int_0^1 (1-s)(1+\gamma s)^{1/4}\, ds \ge \int_0^1 (1-s)^{5/4}\, ds = \frac{4}{9}\,.$$

One can easily check that, for $i = 1,2$ and $j = 1,2,3,4$,

$$|\Delta_1^1 u| + |\Delta_2^i u_i| + |\Delta_3^j u| < 7u \text{ and } |\Delta_1^1 v| < v\,.$$

Using inequality (32), we can obtain the following lower bounds for the term (31).

$$(3\not{3})4B(u,v) + \frac{4}{2\cdot 8\cdot 9(2^3)^{7/4}}\frac{v^{1/4}}{u^{7/4}}\Big(8(\Delta_1^1 u)^2 + 4\sum_{i=1}^{4}(\Delta_2^i u)^2 + 2\sum_{i=1}^{4}(\Delta_3^i)^2\Big)$$

On the other hand,

$$B(u+\alpha,v) = B(u,v) + \Delta B(u,v)(\alpha,0)^t + \int_0^1 (1-s)(\alpha,0)d^2B(u+s\alpha,v)(\alpha,0)^t\,ds$$

$$= B(u,v) + \Delta B(u,v)(\alpha,0)^t - \frac{3v^{1/4}\alpha^2}{16}\int_0^1 \frac{1-s}{(u+s\alpha)^{7/4}}\,ds\,. \tag{34}$$

If $u+\alpha \geq 0$ then $0 < u - su \leq u + s\alpha$, for $0 < s < 1$, and

$$\int_0^1 \frac{1-s}{(u+s\alpha)^{7/4}}\,ds \leq \int_0^1 \frac{1-s}{(u-su)^{7/4}}\,ds = \frac{1}{u^{7/4}}\int_0^1 (1-s)^{-3/4}\,ds = \frac{4}{u^{7/4}}\,.$$

Thus, we have the lower bound of (34):

$$B(u+\alpha,v) \geq B(u,v) + \nabla B(u,v)\begin{pmatrix}\alpha\\0\end{pmatrix} - \frac{3v^{1/4}\alpha^2}{4u^{7/4}}\,.$$

Since $u_{++} = u + \Delta_1^1 u + \Delta_2^2 u$, $u_{+-} = u + \Delta_1^1 u - \Delta u_2^2$, $u_{-+} = u - \Delta_1^1 u + \Delta_1^2 u_1$, $u_{--} = u - \Delta_1^1 u - \Delta_2^1 u$, are all positive, we have

$$\begin{aligned}
&B(u_{++},v) + B(u_{+-},v) + B(u_{-+},v) + B(u_{--},v)\\
&\quad\geq 4B(u,v) + \nabla B(u,v)\begin{pmatrix}\Delta_1^1 u + \Delta_2^2 u\\0\end{pmatrix} + \nabla B(u,v)\begin{pmatrix}\Delta_1^1 u - \Delta_2^2 u\\0\end{pmatrix}\\
&\qquad + \nabla B(u,v)\begin{pmatrix}-\Delta_1^1 u + \Delta_2^1 u\\0\end{pmatrix} + \nabla B(u,v)\begin{pmatrix}-\Delta_1^1 u - \Delta_2^1 u\\0\end{pmatrix}\\
&\qquad - \frac{3v^{1/4}}{4u^{7/4}}\Big((\Delta_1^1 u + \Delta_2^2 u)^2 + (\Delta_1^1 u - \Delta_2^2 u)^2 + (-\Delta_1^1 u + \Delta u_2^1)^2\\
&\qquad + (-\Delta_1^1 u - \Delta_2^1 u)^2\Big)
\end{aligned}$$

$$\quad = 4B(u,v) - \frac{3v^{1/4}}{4u^{7/4}}\Big(4(\Delta_1^1 u)^2 + 2(\Delta_2^2 u)^2 + 2(\Delta_2^1 u)^2\Big)\,. \tag{35}$$

Using (28), (33), and (35), we can estimate the left hand side of (30) as follows.

$$\begin{aligned}
&\frac{a}{36\cdot 4^{7/4}}\frac{v^{1/4}}{u^{7/4}}\Big((4\Delta_1^1 u)^2 + 2(\Delta_2^1 u)^2 + 2(\Delta_2^2 u)^2\Big)\\
&\quad + \frac{4}{2\cdot 8\cdot 9(2^3)^{7/4}}\frac{v^{1/4}}{u^{7/4}}\Big(8(\Delta_1^1 u)^2 + 4\sum_{i=1}^{2}(\Delta_2^i u)^2 + 2\sum_{i=1}^{4}(\Delta_3^i)^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{3v^{1/4}}{4u^{7/4}}\Big(4(\Delta_1^1 u)^2 + 2(\Delta_2^2 u)^2 + 2(\Delta_2^1 u)^2\Big) \\
\geq &\Big(\frac{a}{36\cdot 4^{7/4}} - \frac{3}{4}\Big)\frac{v^{1/4}}{u^{7/4}}\Big(4(\Delta_1^1 u)^2 + 2(\Delta_2^1 u)^2 + 2(\Delta_2^1 u)^2\Big) \\
&+ \frac{1}{2}\cdot\frac{1}{8(2^3)^{7/4}}\cdot\frac{4}{9}\frac{v^{1/4}}{u^{7/4}}\Big(8(\Delta_1^1 u)^2 + 4\sum_{i=1}^{4}(\Delta_2^i u)^2 + 2\sum_{i=1}^{4}(\Delta_3^i)^2\Big)
\end{aligned}
$$

Choosing $a = 27\cdot 4^{7/4}$ return the convexity condition (30) with $D_2 = \frac{1}{36\cdot 8^{7/4}}$. $\square$

4.1. General case

Let us define function inductively,

$$A_m(u, v, \{\Delta_j^i u\}_{j=1,\dots,m}^{i=1,\dots,2^{j-1}}) = a_m A_{m-1}(u, v, \{\Delta_j^i u\}_{j=1,\dots,m-1}^{i=1,\dots,2^{j-1}}) + \sum_{\sigma\in\Sigma_m} B(u_\sigma, v)\,,$$

where $\{\Delta_j^i u\}_{j=1,\dots,m}^{i=1,\dots,2^{j-1}}$ means $2^{m+1}-1$ tuples of $\Delta_j^i u$, for instance,

$$\{\Delta_j^i u\}_{j=1,2,3}^{i=1,\dots,2^{j-1}} = \{\Delta_1^1 u, \Delta_2^1 u, \Delta_2^2 u, \Delta_3^1 u, \Delta_3^2 u, \Delta_3^3 u, \Delta_3^4 u\}\,.$$

And Σ_m is a signature set, that is each entry of Σ_m is a m combination of $\pm$, for example,

$$\Sigma_3 = \{---, --+, -+-, -++, +--, +-+, ++-, +++\} = (+\Sigma_2)\cup(-\Sigma_2)\,.$$

In general $\Sigma_m = (+\Sigma_{m-1}) \cup (-\Sigma_{m-1})$ where $\pm\Sigma_k$ means adjoining $\pm$ on the left of the given k-tulpe in Σ_k to create a $(k+1)$-tuple

Lemma 4.1. *For $m \geq 2$, if A_{m-1} satisfies the following size and convexity condition. For all $(u, v, \{\Delta_j^i u\}_{j=1,\dots,m-1}^{i=1,\dots,2^{j-1}}) \in \mathfrak{F}_{m-1}$,*

$$0 \leq A_{m-1} \leq (C_{m-1} + 2^{m-1})\sqrt[4]{uv} \tag{36}$$

and

$$
\begin{aligned}
&A_{m-1}(u, v, \{\Delta_j^i u\}_{j=1,\dots,m-1}^{i=1,\dots,2^{j-1}}) \\
&- \frac{A_{m-1}(u_+, v_+, \{\Delta_j^i u\}_{j=2,\dots,m}^{i=1,\dots,2^{j-2}}) + A_{m-1}(u_-, v_-, \{\Delta_j^i u\}_{j=2,\dots,m}^{i=2^{j-2}+1,\dots,2^{j-1}})}{2} \\
&\geq D_{m-1}\frac{v^{1/4}}{u^{7/4}}\Big(\sum_{j=1}^{m} 2^{m+1-j}\sum_{i=1}^{2^j-1}(\Delta_m^i u)^2\Big)\,.
\end{aligned}
\tag{37}
$$

Then for all $(u, v, \{\Delta_j^i u\}_{j=1,\dots,m}^{i=1,\dots,2^{j-1}}) \in \mathfrak{F}_m$, A_m satisfies the following size and convexity properties.

$$0 \leq A_m(u, v, \{\Delta_j^i u\}_{j=1,\dots,m}^{i=1,\dots,2^{j-1}}) \leq (a_m(C_{m-1} + 2^{m-1}) + 2^m)\sqrt[4]{uv} \tag{38}$$

and

$$A_m(u, v, \{\Delta_j^i u\}_{j=1,\dots,m}^{i=1,\dots,2^{j-1}})$$

$$- \frac{A_m(u_+, v_+, \{\Delta_j^i u\}_{j=2,\dots,m+1}^{i=1,\dots,2^{j-2}}) + A_m(u_-, v_-, \{\Delta_j^i u\}_{j=2,\dots,m+1}^{i=2^{j-2}+1,\dots,2^{j-1}})}{2}$$

$$\geq D_m \frac{v^{1/4}}{u^{7/4}} \sum_{j=1}^{m+1} 2^{m+1-j} \sum_{i=1}^{2^{j+1}-1} (\Delta_j^i u)^2 \,. \tag{39}$$

Proof. For the size condition (38), one can easily verify using

$$\sum_{\sigma \in \Sigma_m} B(u_\sigma, v) \leq 2^m B(u, v) = 2^m \sqrt[4]{uv}\,,$$

which is due to $(\sqrt[4]{u_+} + \sqrt[4]{u_-})/2 \leq \sqrt[4]{(u_+ + u_-)/2} = \sqrt[4]{u}\,$. We now rewrite the left hand side of the inequality (39) as follow.

$$\begin{aligned}
&a_m A_{m-1}(u, v, \{\Delta_j^i u\}_{j=1,\dots,m-1}^{i=1,\dots,2^{j-1}}) + \sum_{\sigma \in \Sigma_m} B(u_\sigma, v) \\
&\quad - \frac{1}{2}\Bigg(a_m A_{m-1}(u_+, v_+, \{\Delta_j^i u\}_{j=2,\dots,m}^{i=1,\dots,2^{j-1}}) + \sum_{\sigma \in \Sigma_{m-1}} B(u_{+\sigma}, v_+) \\
&\quad + a_m A_{m-1}(u_-, v_-, \{\Delta_j^i u\}_{j=2,\dots,m}^{i=1,\dots,2^{j-1}}) + \sum_{\sigma \in \Sigma_{m-1}} B(u_{-\sigma}, v_-)\Bigg)
\end{aligned}$$

$$\begin{aligned}
(40)\quad &= a_m \Bigg(A_{m-1}(u, v, \{\Delta_j^i u\}_{j=1,\dots,m-1}^{i=1,\dots,2^{j-1}}) \\
&\quad - \frac{1}{2}\Bigg(A_{m-1}(u_+, v_+, \{\Delta_j^i u\}_{j=2,\dots,m}^{i=1,\dots,2^{j-1}}) + A_{m-1}(u_-, v_-, \{\Delta_j^i u\}_{j=2,\dots,m}^{i=1,\dots,2^{j-1}})\Bigg)\Bigg) \\
(41)\quad &+ \sum_{\sigma \in \Sigma_m} B(u_\sigma, v) \\
(42)\quad &- \frac{1}{2}\Bigg(\sum_{\sigma \in \Sigma_{m-1}} B(u_{+\sigma}, v_+) + \sum_{\sigma \in \Sigma_{m-1}} B(u_{-\sigma}, v_-)\Bigg).
\end{aligned}$$

By the assumption (37), we already have the lower bounds for the term (40). We now estimate the lower bounds for the term (41) and (42) separately. Since, for all u_σ, $\sigma \in \Sigma_m$, $0 \leq u_\sigma = u + \alpha_m$ and $0 < u - su \leq u + s\alpha_m$, we have

$$\sum_{\sigma \in \Sigma_m} B(u_\sigma, v) \geq 2^m B(u, v) - \frac{3v^{1/4}}{4u^{7/4}} \left(\sum_{j=1}^{m} 2^{m+1-j} \sum_{i=1}^{2^{j-1}} (\Delta_j^i u)^2 \right), \tag{43}$$

by using (34) 2^m times. For the term (42), using (32) and observations $u_{\pm\sigma} < 2^m u$ and $|\Delta v| \leq v$, we can obtain the following lower bounds.

$$- \frac{1}{2}\Bigg(\sum_{\sigma \in \Sigma_m} B(u_{+\sigma}, v_+) + \sum_{\sigma \in \Sigma_m} B(u_{-\sigma}, v_-)\Bigg)$$

$$\geq -2^m B(u,v) + \frac{v^{1/4}}{36\cdot(2^m u)^{7/4}}\left(\sum_{j=1}^{m+1} 2^{m+2-j}\sum_{i=1}^{2^{j-1}}(\Delta_j^i u)^2\right). \tag{44}$$

Combining (37), (43), and (44), we can estimate the lower bound for the left hand side of (39),

$$a_m D_{m-1}\frac{v^{1/4}}{u^{7/4}}\left(\sum_{j=1}^{m} 2^{m+1-j}\sum_{i=1}^{2^j-1}(\Delta_m^i u)^2\right) - \frac{3v^{1/4}}{4u^{7/4}}\left(\sum_{j=1}^{m} 2^{m+1-j}\sum_{i=1}^{2^{j-1}}(\Delta_j^i u)^2\right)$$
$$+\frac{v^{1/4}}{36\cdot(2^m u)^{7/4}}\left(\sum_{j=1}^{m+1} 2^{m+2-j}\sum_{i=1}^{2^{j-1}}(\Delta_j^i u)^2\right)$$

Choosing $a_m = \frac{3}{4D_{m-1}}$ and $D_m = \frac{1}{36\cdot(2^m)^{7/4}}$ prove the Lemma. □

From Lemma 4.1, we can state the follows. Let us define

$$A_m(u,v,\{\Delta_j^i u\}_{j=1,\dots,m}^{i=1,\dots,2^{j-1}}) := \sum_{j=1}^{m} 27^{m+1-j}\, 2^{\frac{7(m+1-j)(m+j)}{8}} B_{j-1} + B_m\,,$$

where $B_m = \sum_{\sigma\in\Sigma_m} B(u_\sigma, v)$, and $B_0 = B(u,v)$, on the domain $\mathfrak{F}_m$. Here $(u,v,\{\Delta_j^i u\}_{j=1,\dots,m}^{i=1,\dots,2^{j-1}}) \in \mathfrak{F}_m$ means all pairs (u,v) and $(u_\sigma, v) \in \mathfrak{D}_m$, for all $\sigma\in\Sigma_j$, $j=1,\dots,m$. Then A_m has the size property,

$$0 \leq A_m \leq \left(\sum_{j=1}^{m} 2^{j-1} 27^{m+1-j}\, 2^{\frac{7(m+1-j)(m+j)}{8}} + 2^m\right)\sqrt[4]{uv}\,, \tag{45}$$

and the convexity property,

$$A_m(u,v,\{\Delta_j^i u\}_{j=1,\dots,m}^{i=1,\dots,2^{j-1}})$$
$$-\frac{A_m(u_+,v_+,\{\Delta_j^i u\}_{j=2,\dots,m+1}^{i=1,\dots,2^{j-2}}) + A_m(u_-,v_-,\{\Delta_j^i u\}_{j=2,\dots,m+1}^{i=2^{j-2}+1,\dots,2^{j-1}})}{2}$$
$$\geq \frac{1}{36\cdot(2^m)^{7/4}}\frac{v^{1/4}}{u^{7/4}}\sum_{j=1}^{m+1} 2^{m+1-j}\sum_{i=1}^{2^{j+1}-1}(\Delta_j^i u)^2\,. \tag{46}$$

Then usual Bellman function arguments will return the Lemma 2.7.

5. Remarks

In the present text we only deal with a general Haar shift operator in the real line. However, one can extend the same result to the multi-variable case, $\mathbb{R}^n$, similarly with [11]. The author in [11] extended her Hilbert transform (one-dimensional) result to the Riesz transform (multi-dimensional) result with careful modifications. Also, one may define a general Haar shift operator with different Haar system. The second author presented, in [3], the way to defined the dyadic operator with different Haar system (Wilson's Haar system) but more convenience and he extend the one-dimensional proof to the multi-dimensional result with the same Bellman function. Furthermore, the argument in [3] allows to obtain dimensionless bounds in the anisotropic case.

In fact, our main result (5) depends on exponentially in the complexity. Thus we can not get the estimate for the general Caldeón-Zygmund operators, but we may be able to get the estimate for the Caldeón-Zygmund operators with sufficiently smooth kernel. To overcome the exponential dependence of the complexity, one may need to revisit Lemma 2.7. In Lemma 2.7 we estimate the the too big quantity, which is bigger than the sum of differences of the weight in the all dyadic interval $K \in \mathcal{D}(I)$ such that $2^{-m}|I| \leq |K| \leq |I|$. Thus careful modification can be return another solution of the A_2-conjecture.

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Daewon Chung
Faculty of Basic Sciences, Mathematics Major, Keimyung University, 1095 Dalgubeol-daero, Daegu, 704-701, Daegu, Korea
E-mail address: dwchcung@kmu.ac.kr