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HYPERMEROMORPHY OF FUNCTIONS ON SPLIT QUATERNIONS IN CLIFFORD ANALYSIS

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Abstract. In this paper, we consider split quaternionic functions defined on an open set of split quaternions and give the split quaternionic functions whose each inverse function is sp-hyperholomorphic almost everywhere on Ω. Also, we describe the definitions and notions of pseudoholomorphic functions for split quaternions.

1. Introduction

Split quaterninons are described and studied with respect to addition, non-commutative multiplication and hyperholomorphy on open sets of split quaternions. Colombo *et al*. [1] introduced the definition of the field of quaternions by using complex numbers and the modified Cauchy-Fueter operator. We consider split quaternionic functions defined on an open set of split quaternions which play the notion of hyperholomorphic functions on an open set of quaternions $\mathbb{H}$. We [4, 5, 8] have investigated the definition and properties hyperholomorphy of functions defined on quaternion variables. Also, we have considered and studied about split quaternions. By using quaternionic calculus and analogous, we [6] obtained the properties of regularity of functions on dual its split quaternions in Clifford analysis. And, we [7] obtained the properties of polar coordinate expressions of hyperholomorphic functions on split quaternions. In 2015, we [9] obtained the properties of the inverse mapping theory on split quaternions in Clifford analysis. From the results of [2, 3], we give properties of sp-hyperholomorphic functions. Referring [10, 11], we characterize the split quaternionic functions whose each inverse function is sp-hyperholomorphic almost everywhere on Ω in $\mathbb{C}^2$. Also, we research sp-hyperholomorphic and sp-hypermeromorphic functions with special properties for multiplication on split quaternions. Furthermore, from the studies of [12], we give the definitions of pseudoholomorphy for split quaternions instead of $\mathbb{H}$.

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2. Preliminaries

We consider the set of split quaternions as follows:

$$\mathbb{S} = \{Z \mid Z = x_0 + x_1 i + x_2 j + x_3 k,\ x_r \in \mathbb{R},\ (r = 0,1,2,3)\},$$

where

$$i^2 = -1\ ,\ j^2 = k^2 = 1,$$
$$ij = k = -ji\ ,\ jk = -i = kj\ ,\ ki = j = -ik.$$

Let $z_1 = x_0 + x_1 i$ and $z_2 = x_2 + x_3 i$. Then for $Z \in \mathbb{S}$, we have $Z = z_1 + z_2 j$ and the conjugate element of Z is $Z^* = \overline{z_1} - z_2 j$. Let $M(Z)$ be the modulus of split quaternions such that

$$M(Z) = ZZ^* = Z^*Z = z_1\overline{z_1} - z_2\overline{z_2} = |z_1|^2 - |z_2|^2.$$

Then the inverse element of Z is:

$$Z^{-1} = \frac{Z^*}{M(Z)} \qquad (M(Z) \neq 0).$$

Consider the differential operators as follows:

$$D := \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \overline{z_2}} j \quad \text{and} \quad D^* = \frac{\partial}{\partial \overline{z_1}} - \frac{\partial}{\partial \overline{z_2}} j.$$

Let Ω be an open set of $\mathbb{S} \cong \mathbb{C}^2$ and $F \in \mathcal{C}^\infty(\Omega, \mathbb{S})$. Then

$$F(Z) = F(z_1 + z_2 j) = f_1(z_1, z_2) + f_2(z_1, z_2) j,$$

where f_1 and f_2 are continuously differential complex functions in Ω.

Definition 1. Let Ω be an open set of $\mathbb{S}$. For $F \in \mathcal{C}^\infty(\Omega, \mathbb{S})$, with $F = f_1 + f_2 j$, a function F is said to be sp-hyperholomorphic if

$$D^*F(Z) = \Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial \overline{f_2}}{\partial \overline{z_2}}\Big) + \Big(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial \overline{f_1}}{\partial \overline{z_2}}\Big) j = 0.$$

From the above equation, we have the split Cauchy-Riemann equations of the sp-hyperholomorphic function F on Ω:

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial \overline{f_2}}{\partial \overline{z_2}} \quad \text{and} \quad \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial \overline{f_1}}{\partial \overline{z_2}}.$$

For the definition of almost everywhere, one may refer to [13].

Proposition 2.1. *Let Ω be an open set of $\mathbb{S}$. Let $F = f_1 + f_2 j$ and $G = g_1 + g_2 j$ be two sp-hyperholomorphic functions defined almost everywhere. Then FG satisfies as follows:*

$$D^*(FG) = (D^*F)G + f_1\frac{\partial G}{\partial \overline{z_1}} - \overline{f_2}\frac{\partial G}{\partial \overline{z_2}} + j\Big(\overline{f_2}\frac{\partial G}{\partial z_1} - f_1\frac{\partial G}{\partial z_2}\Big).$$

Proof. We have

$$FG = f_1 g_1 + f_2\overline{g_2} + (f_1 g_2 + f_2\overline{g_1})j$$

and

$$\begin{aligned} D^*(FG) - (D^*F)G &= f_1\frac{\partial g_1}{\partial \overline{z_1}} + f_2\frac{\partial \overline{g_2}}{\partial \overline{z_1}} - \overline{f_1}\frac{\partial \overline{g_2}}{\partial \overline{z_2}} - \overline{f_2}\frac{\partial g_1}{\partial \overline{z_2}} \\ &\quad + \Big(f_1\frac{\partial g_2}{\partial \overline{z_1}} + f_2\frac{\partial \overline{g_1}}{\partial \overline{z_1}} - \overline{f_1}\frac{\partial \overline{g_1}}{\partial \overline{z_2}} - \overline{f_2}\frac{\partial g_2}{\partial \overline{z_2}}\Big)j \\ &= f_1\frac{\partial G}{\partial \overline{z_1}} - \overline{f_2}\frac{\partial G}{\partial \overline{z_2}} + j\Big(\overline{f_2}\frac{\partial G}{\partial z_1} - f_1\frac{\partial G}{\partial z_2}\Big). \end{aligned}$$

Therefore, the result is obtained. □

Theorem 2.2. *Let Ω be an open set of $\mathbb{S}$. If the function F and its inverse function F^{-1} are sp-hyperholomorphic, when they are defined, then we have the following equations:*

$$\begin{cases} f_1\dfrac{\partial f_2}{\partial \overline{z_1}} - f_2\dfrac{\partial f_1}{\partial \overline{z_1}} = 0, \\[2ex] \overline{f_2}\dfrac{\partial f_2}{\partial \overline{z_2}} - f_2\dfrac{\partial \overline{f_1}}{\partial \overline{z_1}} = 0, \\[2ex] f_2\dfrac{\partial \overline{f_2}}{\partial \overline{z_1}} - \overline{f_2}\dfrac{\partial f_1}{\partial \overline{z_2}} = 0. \end{cases}$$

Proof. Let $N_f = f_1\overline{f_1} - f_2\overline{f_2}$. We have the inverse function $F^{-1} = \frac{\overline{f_1}}{N_f} - \frac{f_2}{N_f}j$ and

$$D^*F^{-1} = \Big(\frac{\partial}{\partial \overline{z_1}}\frac{f_1}{N_f} + \frac{\partial}{\partial \overline{z_2}}\frac{f_2}{N_f}\Big) - \Big(\frac{\partial}{\partial \overline{z_1}}\frac{f_2}{N_f} + \frac{\partial}{\partial \overline{z_2}}\frac{f_1}{N_f}\Big)j.$$

In detail, we have the following equations:

$$\begin{aligned} \frac{\partial}{\partial \overline{z_1}}\frac{f_1}{N_f} &= \frac{\partial \overline{f_1}}{\partial \overline{z_1}}N_f - \overline{f_1}\Big(\frac{\partial f_1}{\partial \overline{z_1}}\overline{f_1} + f_1\frac{\partial \overline{f_1}}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_1}}\overline{f_2} - f_2\frac{\partial \overline{f_2}}{\partial \overline{z_1}}\Big), \\ \frac{\partial}{\partial \overline{z_2}}\frac{f_2}{N_f} &= \frac{\partial \overline{f_2}}{\partial \overline{z_2}}N_f - \overline{f_2}\Big(\frac{\partial f_1}{\partial \overline{z_2}}\overline{f_1} + f_1\frac{\partial \overline{f_1}}{\partial \overline{z_2}} - \frac{\partial f_2}{\partial \overline{z_2}}\overline{f_2} - f_2\frac{\partial \overline{f_2}}{\partial \overline{z_2}}\Big), \\ \frac{\partial}{\partial \overline{z_1}}\frac{f_2}{N_f} &= \frac{\partial f_2}{\partial \overline{z_1}}N_f - f_2\Big(\frac{\partial f_1}{\partial \overline{z_1}}\overline{f_1} + f_1\frac{\partial \overline{f_1}}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_1}}\overline{f_2} - f_2\frac{\partial \overline{f_2}}{\partial \overline{z_1}}\Big), \\ \frac{\partial}{\partial \overline{z_2}}\frac{f_1}{N_f} &= \frac{\partial f_1}{\partial \overline{z_2}}N_f - f_1\Big(\frac{\partial f_1}{\partial \overline{z_2}}\overline{f_1} + f_1\frac{\partial \overline{f_1}}{\partial \overline{z_2}} - \frac{\partial f_2}{\partial \overline{z_2}}\overline{f_2} - f_2\frac{\partial \overline{f_2}}{\partial \overline{z_2}}\Big). \end{aligned}$$

Hence, we have

$$\begin{aligned} D^*F^{-1} &= \Big(-\frac{\partial \overline{f_1}}{\partial \overline{z_1}} f_2\overline{f_2} - \frac{\partial f_1}{\partial \overline{z_1}}\overline{f_1}^2 + \frac{\partial f_2}{\partial \overline{z_1}}\overline{f_1 f_2} + \overline{f_1} f_2 \frac{\partial \overline{f_2}}{\partial \overline{z_1}} \\ &\quad + \frac{\partial \overline{f_2}}{\partial \overline{z_2}} f_1\overline{f_1} - \frac{\partial f_1}{\partial \overline{z_2}}\overline{f_2 f_1} - \overline{f_2} f_1 \frac{\partial \overline{f_1}}{\partial \overline{z_2}} + \frac{\partial f_2}{\partial \overline{z_2}}\overline{f_2}^2\Big) \\ &\quad + \Big(\frac{\partial f_2}{\partial \overline{z_1}} f_1\overline{f_1} - \frac{\partial f_1}{\partial \overline{z_1}} f_2\overline{f_1} - f_2 f_1 \frac{\partial \overline{f_1}}{\partial \overline{z_1}} + f_2^2 \frac{\partial \overline{f_2}}{\partial \overline{z_1}} \\ &\quad - \frac{\partial f_1}{\partial \overline{z_2}} f_2\overline{f_2} - f_1^2 \frac{\partial \overline{f_1}}{\partial \overline{z_2}} + f_1\overline{f_2}\frac{\partial f_2}{\partial \overline{z_2}} + f_1 f_2 \frac{\partial \overline{f_2}}{\partial \overline{z_2}}\Big) j = 0. \end{aligned}$$

By arranging the above terms and applying the split Cauchy-Riemann equations, the equation $D^*F^{-1} = 0$ is equivalent to the following equations:

$$\overline{f_1}\frac{\partial f_2}{\partial \overline{z_1}} - f_1 \frac{\partial \overline{f_1}}{\partial \overline{z_2}} + \overline{f_1}\frac{\partial f_1}{\partial \overline{z_1}} - f_1 \frac{\partial \overline{f_2}}{\partial \overline{z_2}} = 0,$$

$$\overline{f_2}\frac{\partial f_2}{\partial \overline{z_2}} - f_2 \frac{\partial \overline{f_1}}{\partial \overline{z_1}} = 0,\ \overline{f_2}\frac{\partial f_1}{\partial \overline{z_2}} - f_2 \frac{\partial \overline{f_2}}{\partial \overline{z_1}} = 0.$$

Therefore, the result follows. □

For the following definitions, see [11, 12].

Definition 2. Let Ω be an open set of $\mathbb{S}$ and let F be any almost everywhere defined sp-hyperholomorphic function in Ω. The function F is said to be a sp_w-hypermeromorphic function if the inverse function of F is sp-hyperholomorphic almost everywhere.

Definition 3. Let Ω be an open set of $\mathbb{S}$ and let F and G be sp_w- hyperholomorphic functions in Ω. The functions F and G are said to be sp-hypermeromorphic functions if the sum and product of F and G are sp_w-hyperholomorphic in Ω, respectively.

Definition 4. Let Ω be an open set of $\mathbb{S}$. Then a function F is said to be sp-pseudoholomorphic on Ω if F is hypermeromorphic without poles on Ω. Also, F is said to be a smooth hypermeromorphic function (sha function) on Ω if F is hypermeromorphic without zeros and poles on Ω.

Theorem 2.3. *Let Ω be an open set of $\mathbb{S}$. If a function F is sp-pseudoholomorphic on Ω, then we have the following equations:*

$$g_1 \frac{\partial \overline{h_1}}{\partial \overline{z_1}} + \overline{g_1} \frac{\partial \overline{h_2}}{\partial \overline{z_2}} = \overline{g_2} \frac{\partial \overline{h_1}}{\partial \overline{z_2}} + g_2 \frac{\partial \overline{h_2}}{\partial \overline{z_1}}$$

and

$$\overline{g_2} \frac{\partial h_2}{\partial \overline{z_2}} + g_2 \frac{\partial h_1}{\partial \overline{z_1}} = g_1 \frac{\partial h_1}{\partial \overline{z_1}} + \overline{g_1} \frac{\partial h_1}{\partial \overline{z_2}}.$$

Proof. We set a function F to $F = GH^{-1}$, where G is a sp-hyperholomorphic function on Ω and an inverse function H^{-1} of H is sp-hyperholomorphic on Ω. Since a function F is sp-pseudoholomorphic on Ω,

$$D^* G = \Big(\frac{\partial g_1}{\partial \overline{z_1}} - \frac{\partial \overline{g_2}}{\partial \overline{z_2}} \Big) + \Big(\frac{\partial g_2}{\partial \overline{z_1}} - \frac{\partial \overline{g_1}}{\partial \overline{z_2}} \Big) j = 0$$

and we have

$$\begin{aligned} D^* H^{-1} &= \frac{1}{N_h} \Big(\frac{\partial \overline{h_1}}{\partial \overline{z_1}} \ N_h - \frac{\partial N_h}{\partial \overline{z_1}} \ \overline{h_1} + \frac{\partial \overline{h_2}}{\partial \overline{z_2}} \ N_h - \frac{\partial N_h}{\partial \overline{z_2}} \ \overline{h_2} \Big) \\ & \quad - \frac{1}{N_h} \Big(\frac{\partial h_2}{\partial \overline{z_1}} \ N_h - \frac{\partial N_h}{\partial \overline{z_1}} \ h_2 + \frac{\partial h_1}{\partial \overline{z_2}} \ N_h - \frac{\partial N_h}{\partial \overline{z_2}} \ h_1 \Big) j, \end{aligned}$$

where $N_h = h_1 \overline{h_1} - h_2 \overline{h_2}$ and

$$\frac{\partial N_h}{\partial \overline{z_r}} = \frac{\partial h_1}{\partial \overline{z_r}} \ \overline{h_1} + \frac{\partial \overline{h_1}}{\partial \overline{z_r}} \ h_1 - \frac{\partial h_2}{\partial \overline{z_r}} \ \overline{h_2} - \frac{\partial \overline{h_2}}{\partial \overline{z_r}} \ h_2 \quad (r = 0, 1).$$

Hence, by Proposition 2.1, we have

$$\begin{aligned} D^* F &= \frac{1}{N_h} \Big(g_1 \frac{\partial \overline{h_1}}{\partial \overline{z_1}} - \overline{g_2} \frac{\partial \overline{h_1}}{\partial \overline{z_2}} - g_2 \frac{\partial \overline{h_2}}{\partial \overline{z_1}} + \overline{g_1} \frac{\partial \overline{h_2}}{\partial \overline{z_2}} \Big) \\ & \quad + \frac{1}{N_h} \Big(-g_1 \frac{\partial h_1}{\partial \overline{z_1}} + \overline{g_2} \frac{\partial h_2}{\partial \overline{z_2}} + g_2 \frac{\partial h_1}{\partial \overline{z_1}} - \overline{g_1} \frac{\partial h_1}{\partial \overline{z_2}} \Big) j \\ &= 0. \end{aligned}$$

Therefore, we obtain the following equations:

$$g_1 \frac{\partial \overline{h_1}}{\partial \overline{z_1}} + \overline{g_1} \frac{\partial \overline{h_2}}{\partial \overline{z_2}} = \overline{g_2} \frac{\partial \overline{h_1}}{\partial \overline{z_2}} + g_2 \frac{\partial \overline{h_2}}{\partial \overline{z_1}}$$

and

$$\overline{g_2} \frac{\partial h_2}{\partial \overline{z_2}} + g_2 \frac{\partial h_1}{\partial \overline{z_1}} = g_1 \frac{\partial h_1}{\partial \overline{z_1}} + \overline{g_1} \frac{\partial h_1}{\partial \overline{z_2}}.$$

□

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