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NOTE ON MODULAR RELATIONS FOR THE ROGER-RAMANUJAN TYPE IDENTITIES AND REPRESENTATIONS FOR JACOBIAN IDENTITY

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Abstract. Combining and specializing some known results, we establish six identities which depict six modular relations for the Roger-Ramanujan type identities and two equivalent representations for Jacobian identity expressed in terms of combinatorial partition identities and Ramanujan-Selberg continued fraction. Two q-product identities are also considered.

1. Introduction and Preliminaries

In recent years, various families of basic (or q-) series and basic (or q-) polynomials have been investigated rather widely and extensively due mainly to their having been found to be potentially useful in such wide variety of fields as (for example) theory of partitions, number theory, combinatorial analysis, finite vector spaces, Lie theory, particle physics, non-linear electric circuit theory, mechanical engineering, theory of heat conduction, quantum mechanics, cosmology, and statistics (see, for details, [11, pp. 350–351]; see also [10, Chapter 6]). Here, in this paper, we combine and specialize some known results to establish six identities which depict six modular relations for the Roger-Ramanujan type identities and two equivalent representations for Jacobian identity expressed in terms of combinatorial partition identities and Ramanujan-Selberg continued fraction. Many q-product identities are also considered.

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$, and $\mathbb{Z}_0^-$ denote the sets of positive integers, integers, complex numbers, and nonpositive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For the sake of easy reference, we recall the following q-notations.

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The *q-shifted factorial* $(a;q)_n$ is defined by

$$(a;q)_n := \begin{cases} 1 & (n=0) \\ \prod_{k=0}^{n-1} \left(1-a\,q^k\right) & (n \in \mathbb{N}), \end{cases} \tag{1.1}$$

where $a,\, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$.

The *q-shifted factorial* for negative subscript is defined by

$$(a;q)_{-n} := \frac{1}{\left(1-a\,q^{-1}\right)\,\left(1-a\,q^{-2}\right)\cdots\left(1-a\,q^{-n}\right)} \quad (n \in \mathbb{N}_0)\,, \tag{1.2}$$

which yields

$$(a;q)_{-n} = \frac{1}{\left(a\,q^{-n};q\right)_n} = \frac{(-q/a)^n\; q^{\binom{n}{2}}}{(q/a;q)_n} \quad (n \in \mathbb{N}_0)\,. \tag{1.3}$$

We also write

$$\begin{aligned}(a;q)_\infty &:= \prod_{k=0}^{\infty} \left(1-a\,q^k\right) \\ &= \prod_{k=1}^{\infty} \left(1-a\,q^{k-1}\right) \quad (a,\, q \in \mathbb{C};\; |q|<1).\end{aligned} \tag{1.4}$$

It is noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in (1.4) diverges. So, whenever $(a;q)_\infty$ is involved in a given formula, the constraint $|q|<1$ will be tacitly assumed.

It follows from (1.1), (1.2) and (1.4) that

$$(a;q)_n = \frac{(a;q)_\infty}{(a\,q^n;q)_\infty} \quad (n \in \mathbb{Z}), \tag{1.5}$$

which can be extended to $n = \alpha \in \mathbb{C}$ as follows:

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(a\,q^\alpha;q)_\infty} \quad (\alpha \in \mathbb{C};\; |q|<1), \tag{1.6}$$

where the principal value of q^α is taken.

The following notations are also frequently used:

$$(a_1,\, a_2, \cdots, a_m;q)_n := (a_1;q)_n\; (a_2;q)_n \cdots (a_m;q)_n \tag{1.7}$$

and

$$(a_1,\, a_2, \cdots, a_m;q)_\infty := (a_1;q)_\infty\; (a_2;q)_\infty \cdots (a_m;q)_\infty\,. \tag{1.8}$$

Ramanujan introduced to investigate the following general theta function (see [3, p. 34]):

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\, b^{\frac{n(n-1)}{2}} \quad (|ab|<1). \tag{1.9}$$

Jacobi's triple product identity is given as follows (see, *e.g.*, [3, p. 35, Entry 19]):

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.10}$$

Jacobi's triple product identity are specialized as follows:

$$\Phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty; \tag{1.11}$$

$$\Psi(q) := \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}; \tag{1.12}$$

$$f(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty, \tag{1.13}$$

which is known as Euler's pentagonal number theorem. Euler's another well known identity is given as follows:

$$(q;q^2)_\infty^{-1} = (-q;q)_\infty. \tag{1.14}$$

Roger-Ramanujan identities are given as follows:

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}; \tag{1.15}$$

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}. \tag{1.16}$$

Roger-Ramanujan function is given as follows:

$$R(q) := q^{\frac{1}{5}}\frac{H(q)}{G(q)} = q^{\frac{1}{5}}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{1.17}$$

The following identities associated with continued fraction were given (see [4]):

$$\begin{aligned}(q^2;q^2)_\infty(-q;q)_\infty &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\\ &= \left(\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-\cdots}\right) \quad (|q|<1));\end{aligned} \tag{1.18}$$

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \left(\frac{1}{1+}\ \frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^4}{1+}\ \frac{q^5}{1+}\ \frac{q^6}{1+\cdots}\right) \quad (|q|<1); \tag{1.19}$$

$$C(q) := \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = \left(1+\frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^4}{1+}\ \frac{q^5}{1+}\ \frac{q^6}{1+\cdots}\right) \quad (|q|<1). \tag{1.20}$$

Very recently Andrews *et al.* [1] investigated combinatorial partition identities associated with the following general family:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s(n2)+tn} r(l,u,v,w;n), \tag{1.21}$$

where

$$r(l,u,v,w:n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \frac{q^{uv(j\ 2)+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{1.22}$$

The following combinatorial partition identities are recalled (see [1, Theorem 3]):

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty; \tag{1.23}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty; \tag{1.24}$$

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{1.25}$$

Chaudhary [4] obtained several q-product identities. Here, two general q-product identities are presented, which will be required in the next section, asserted by the following lemma.

Lemma 1.1. *Each of the following identities holds true:*

$$\left(q^r;q^r\right)_\infty = \left(q^r, q^{2r}, \ldots, q^{r\ell}; q^{r\ell}\right)_\infty \quad (r,\ \ell \in \mathbb{N}). \tag{1.26}$$

More generally,

$$\left(q^r;q^s\right)_\infty = \left(q^r, q^{r+s}, q^{r+2s}, \ldots, q^{r+\ell s}; q^{r+\ell s}\right)_\infty \quad (r,\ s \in \mathbb{N};\ \ell \in \mathbb{N}_0). \tag{1.27}$$

Proof. We can prove two identities by partitioning the sets of involved integers into the modulus $r\ell$ and $r+\ell s$, respectively. So details of their proofs are omitted. □

2. Some known results

Here, certain known results are recalled for later use. Hahn [7] introduced some interesting analogues of the Rogers-Ramanujan functions as follows:

$$L(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^3;q^7)_\infty(q^4;q^7)_\infty}{(q^2;q^2)_\infty}; \tag{2.1}$$

$$M(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^2;q^7)_\infty(q^5;q^7)_\infty}{(q^2;q^2)_\infty}; \tag{2.2}$$

$$N(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q^7;q^7)_\infty(q;q^7)_\infty(q^6;q^7)_\infty}{(q^2;q^2)_\infty}. \tag{2.3}$$

Baruah and Bora [2] established several modular relations for some analogues of the Rogers-Ramanujan functions as follows:

$$P(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} q^{3n^2}}{(q^3;q^3)_n (q^3;q^3)_{2n}} = \frac{(q^4;q^9)_\infty (q^5;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}; \tag{2.4}$$

$$Q(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} (1-q^{3n+2}) q^{3n(n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q^2;q^9)_\infty (q^7;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}; \tag{2.5}$$

$$R(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1} q^{3n(n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q;q^9)_\infty (q^8;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}. \tag{2.6}$$

Jacobian identity is given as follows (see [6]):

$$\big(q;q^2\big)_\infty^8 + 16q\big(-q^2;q^2\big)_\infty^8 = \big(-q;q^2\big)_\infty^8. \tag{2.7}$$

Ramanujan introduced to investigate the following function (see [8, p. 290]):

$$T(q) := \frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}. \tag{2.8}$$

Ramanujan [8] and Selberg [9] presented, independently, the following interesting continued fraction for $T(q)$:

$$T(q) = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q+q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^2+q^4}{1+\cdots} \quad (|q|<1). \tag{2.9}$$

3. Main Results

Here, we combine and specialize some known results to establish six identities which depict six modular relations for the Roger-Ramanujan type identities and two equivalent representations for Jacobian identity expressed in terms of combinatorial partition identities and Ramanujan-Selberg continued fraction.

Theorem 3.1. *Each of the following identities holds true:*

$$L(-q^{\frac{1}{7}}) = \left\{ \frac{(q^2;q^2)_\infty (-q^{\frac{3}{7}};-q)_\infty (q^{\frac{4}{7}};-q)_\infty}{(q^{\frac{2}{7}};q^{\frac{2}{7}})_\infty} \right\} R(2,1,1,1,2,2); \tag{3.1}$$

$$M(-q^{\frac{1}{7}}) = \left\{ \frac{(q^2;q^2)_\infty (q^{\frac{2}{7}};-q)_\infty (-q^{\frac{5}{7}};-q)_\infty}{(q^{\frac{2}{7}};q^{\frac{2}{7}})_\infty} \right\} R(2,1,1,1,2,2); \tag{3.2}$$

$$N(-q^{\frac{1}{7}}) = \left\{ \frac{(q^2;q^2)_\infty (-q^{\frac{1}{7}};-q)_\infty (q^{\frac{6}{7}};-q)_\infty}{(q^{\frac{2}{7}};q^{\frac{2}{7}})_\infty} \right\} R(2,1,1,1,2,2); \tag{3.3}$$

$$P(-q^{\frac{1}{9}}) = \left\{ \frac{(q^2;q^2)_\infty (q^{\frac{4}{9}};-q)_\infty (-q^{\frac{5}{9}};-q)_\infty}{(-q^{\frac{1}{3}};-q^{\frac{1}{3}})_\infty} \right\} R(2,1,1,1,2,2); \tag{3.4}$$

$$Q(-q^{\frac{1}{9}}) = \left\{ \frac{(q^2;q^2)_\infty (q^{\frac{2}{9}};-q)_\infty (-q^{\frac{7}{9}};-q)_\infty}{(-q^{\frac{1}{3}};-q^{\frac{1}{3}})_\infty} \right\} R(2,1,1,1,2,2); \tag{3.5}$$

$$R(-q^{\frac{1}{9}}) = \left\{ \frac{(q^2;q^2)_\infty (-q^{\frac{1}{9}};-q)_\infty (q^{\frac{8}{9}};-q)_\infty}{(-q^{\frac{1}{3}};-q^{\frac{1}{3}})_\infty} \right\} R(2,1,1,1,2,2). \tag{3.6}$$

Proof. setting $r = 7$ and $\ell = 2$ in (1.26), we have

$$\left(q^7;q^7\right)_\infty = \left(q^7;q^{14}\right)_\infty \left(q^{14};q^{14}\right)_\infty. \tag{3.7}$$

Then the right-hand side of (3.7) is substituted for $\left(q^7;q^7\right)_\infty$ in each of (2.1), (2.2) and (2.3). Further replacing q by $-q^{\frac{1}{7}}$ in the resulting identities, rearranging the terms and using (1.23), we are led to the desired results (3.1), (3.2) and (3.3), respectively.

Next, setting $r = 9$ and $\ell = 2$ in (1.26), we have

$$\left(q^9;q^9\right)_\infty = \left(q^9;q^{18}\right)_\infty \left(q^{18};q^{18}\right)_\infty. \tag{3.8}$$

Then the right-hand side of (3.8) is substituted for $\left(q^9;q^9\right)_\infty$ in each of (2.4), (2.5) and (2.6). Further replacing q by $-q^{\frac{1}{9}}$ in the resulting identities, rearranging the terms and using (1.23), we arrive at the desired results (3.4), (3.5) and (3.6), respectively.

□

Two equivalent representations for Jacobian identity which are expressed in terms of combinatorial partition identities and Ramanujan-Selberg continued fraction are given in the following theorem.

Theorem 3.2. *Each of the following identities holds true:*

$$\left\{(q;q^2)_\infty\right\}^8 = \left\{R(2,1,1,1,2,2)\right\}^8 - 16q\left\{R(2,2,1,1,2,2)\right\}^8 \tag{3.9}$$

and

$$\left\{ \frac{(q;q^2)_\infty}{R(2,1,1,1,2,2)} \right\}^8 + 16q\{T(q)\}^8 = 1. \tag{3.10}$$

Proof. Using (1.23) and (1.25) in (2.8) yields (3.9). Further considering (2.9) in (3.9) is easily seen to prove (3.10). □

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