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SOME INEQUALITIES OF WEIGHTED SHIFTS ASSOCIATED BY DIRECTED TREES WITH ONE BRANCHING POINT

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Abstract. Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space, and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator $T \in B(\mathcal{H})$ has property $B(n)$ if $|T^n| \geq |T|^n$, $n \geq 2$, which generalizes the class A-operator. We characterize the property $B(n)$ of weighted shifts S_λ over (η, κ)-type directed trees which appeared in the study of subnormality of weighted shifts over directed trees recently. In addition, we discuss the property $B(n)$ of weighted shifts S_λ over $(2,1)$-type directed trees with nonzero weights are being distinct with respect to $n \geq 2$. And we give some properties of weighted shifts S_λ over $(2,1)$-type directed trees with property $B(2)$.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space, and let $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be *normal* if $T^*T = TT^*$. And $T \in B(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. An operator $T \in B(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$, $p \in (0, \infty)$. As well, T is said to be ∞-*hyponormal* if it is p-hyponormal for all $p > 0$ ([16]). According to the Lőwner-Heinz inequality([19],[8]), every q-hyponormal operator is p-hyponormal for $p \leq q$. An operator T is said to be *class A-operator* if $\left|T^2\right| \geq |T|^2$, where $|T| = (T^*T)^{1/2}$ (cf. [3]). The structure of class A-operators are developed well recently (cf. [17],[9],[18],[2],[12]). It is well-known that "normal $\Rightarrow$ p-hyponormal $\Rightarrow$ class A-operator". Recall that, for positive integer $n \geq 2$, an operator $T \in B(\mathcal{H})$ has *property* $B(n)$ if $|T^n| \geq |T|^n$ (cf. [15]). The notion of property $B(n)$ will be applied to the gap theory of Hilbert space operators (cf. [14], [13], [1]).

On the other hand, since Mohar's study about connections between operators and graph theory, several operator theoriests have studied such notions. In [6] Fujii-Sasaoka-Watatani defined adjacency operators associated to infinite

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directed graphs and discussed some relations between graphs and bounded adjacency operators (cf. [4],[5],[7]). On the other hand, Jablonski-Jung-Stochel ([10]) introduced and investigated classes of weighted shifts on directed trees which generalize the classical weighted shifts. These operators provided some interesting examples and counter examples concerning moment sequences and subnormal operators (cf. [11]). The structure of such operators has been developed well for recent several years. Basically, the goal of this paper is to give a connection between directed trees and bounded Hilbert space operators.

The organization of this paper is as follows. In Section 2, we recall some terminology and notation concerning directed graphs and weighted shifts on directed trees. In particular, we recall (η, κ)-type directed tree which will be used mainly throughout this paper. In Section 3, we characterize weighted shifts S_λ over (η, κ)-type directed trees with property $B(n)$. And also, we discuss the classes of weighted shifts S_λ over $(2, 1)$-type directed trees with property $B(n)$ are being distinct with respect to $n \geq 2$. Finally, in section 4, we introduce properties of weighted shifts S_λ over $(2, 1)$-type directed trees with property $B(2)$.

We denote $\mathbb{N}[\mathbb{R}, \mathbb{R}_0, \mathbb{R}_+$, or $\mathbb{Z}_+$, resp.] by the set of positive integers[real numbers, nonnegative real numbers, positive real numbers, or nonnegative integers, resp.] throughout this paper.

2. Preliminaries

In this section, we recall some definitions on graph theory which will be used in this paper. First of all, we look at some basic notions of the graph theory. A pair $\mathcal{G} = (V, E)$ is a *directed graph* if V is a nonempty set and E is a subset of $V \times V \setminus \{(v, v) \mid v \in V\}$. We denote by

$$\widetilde{E} = \{\{u, v\} \subseteq V \mid (u, v) \in E \text{ or } (v, u) \in E\}.$$

An element of V is called a *vertex* of $\mathcal{G}$, a member of E is called an *edge* of $\mathcal{G}$, and a member of $\widetilde{E}$ is called an *undirected edge*. A directed graph $\mathcal{G}$ is said to be *connected* if for any two distinct vertices u and v of $\mathcal{G}$, there exists a finite sequence $v_1, \cdots, v_n$ of vertices of $\mathcal{G}(n \geq 2)$ such that $u = v_1$, $\{v_j, v_{j+1}\} \in \widetilde{E}$ for all $j = 1, \cdots, n-1$, and $v_n = v$. Such a sequence will be called an *undirected path* joining u and v. For $u \in V$, put

$$\mathrm{Chi}(u) = \{v \in V \mid (u, v) \in E\}.$$

An element of $\mathrm{Chi}(u)$ is called a *child* of u. If, for a given vertex $u \in V$, there exists a unique vertex $v \in V$ such that $(v, u) \in E$, then we say that u has a parent v and write $\mathrm{par}(u)$ for v. A vertex v of $\mathcal{G}$ is called a *root* of $\mathcal{G}$, or briefly $v \in \mathrm{Root}(\mathcal{G})$, if there is no vertex u of $\mathcal{G}$ such that (u, v) is an edge of $\mathcal{G}$. If $\mathrm{Root}(\mathcal{G})$ is a one-element set, then its unique element is denoted by $\mathrm{root}(\mathcal{G})$, or simply by root if this causes no ambiguity. We write $V^\circ = V \backslash \mathrm{Root}(\mathcal{G})$. A finite sequence $\{u_j\}_{j=1}^n$ $(n \geq 2)$ of distinct vertices is said to be a *circuit* of $\mathcal{G}$

if $(u_j, u_{j+1}) \in E$ for all $j = 1, \cdots, n-1$, and $(u_n, u_1) \in E$. A directed graph $\mathcal{T}$ is a *directed tree* if it satisfies the following conditions

(i) $\mathcal{T}$ is connected,

(ii) $\mathcal{T}$ has no circuits,

(iii) each vertex $v \in V^\circ$ has a parent.

From now on, $\mathcal{T} = (V, E)$ is assumed to be a directed tree. Denote by $\ell^2(V)$ the Hilbert space of all square summable complex functions on V with the standard inner product

$$\langle f, g\rangle = \sum_{u\in V} f(u)\overline{g(u)}, \ \ f, g \in \ell^2(V).$$

For $u \in V$, we define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then the set $\{e_u\}_{u\in V}$ is an orthonormal basis of $\ell^2(V)$. For $\lambda = \{\lambda_v\}_{v\in V^\circ} \subset \mathbb{C}$, we define the operator S_λ on $\ell^2(V)$ by

$$\begin{aligned} D(S_\lambda) &= \{f \in \ell^2(V) : \sum_{u\in V}\left(\sum_{v\in \mathrm{Chi}(u)} |\lambda_v|^2\right)|f(u)|^2 < \infty\}, \\ S_\lambda f &= \Lambda_{\mathcal{T}} f, \qquad f \in D(S_\lambda), \end{aligned}$$

where $\Lambda_{\mathcal{T}}$ is the mapping defined on functions $f : V \to \mathbb{C}$ by

$$(\Lambda_{\mathcal{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathrm{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \text{root}. \end{cases}$$

Then the operator S_λ is called a *weighted shift* on the directed tree $\mathcal{T}$ with weights $\{\lambda_v\}_{v\in V^\circ}$. In particular, if $S_\lambda \in B(\ell^2(V))$, then

$$S_\lambda e_u = \sum_{v\in \mathrm{Chi}(u)} \lambda_v e_v \tag{2.1}$$

(cf. [10, Proposition 3.1.3]) and

$$S_\lambda^* e_u = \begin{cases} \overline{\lambda_u} e_{\mathrm{par}(u)} & \text{if } u \in V^\circ, \\ 0 & \text{if } u \text{ is root,} \end{cases} \tag{2.2}$$

which is used frequently in this paper (cf. [10, Proposition 3.4.1]).

We discuss weighted shifts associated the following models as a central role in this paper. This model is closely related to the subnormality of weighted shifts on directed trees (cf. [10]).

Definition 1. ([10]) Given $\eta, \kappa \in \mathbb{Z}_+ \cup \{\infty\}$ with $\eta \geq 2$, we define the directed tree $\mathcal{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ by

$$\begin{aligned} V_{\eta,\kappa} &= \{-k : k \in J_\kappa\} \cup \{0\} \cup \{(i, j) : i \in J_\eta, j \in \mathbb{N}\}, \\ E_{\eta,\kappa} &= E_\kappa \cup \{(0, (i, 1)) : i \in J_\eta\} \cup \{((i, j), (i, j + 1)) : i \in J_\eta, j \in \mathbb{N}\}, \end{aligned}$$

where $E_\kappa = \{(-k, -k+1) : k \in J_\kappa\}$ and $J_\iota = \{k \in \mathbb{N} : k \leq \iota\}$ for $\iota \in \mathbb{Z}_+ \cup \{\infty\}$. Then the directed tree $\mathcal{T}_{\eta,\kappa}$ is called an (η, κ)-*type directed tree.*

If $\kappa < \infty$, then the directed tree $\mathcal{T}_{\eta,\kappa}$ has a root and $\operatorname{root}(\mathcal{T}_{\eta,\kappa}) = -\kappa$. In turn, if $\kappa = \infty$, then the directed tree $\mathcal{T}_{\eta,\infty}$ is rootless. In the case of $\kappa < \infty$, the (η, κ)-type directed tree can be shown in Figure 2.1 below.

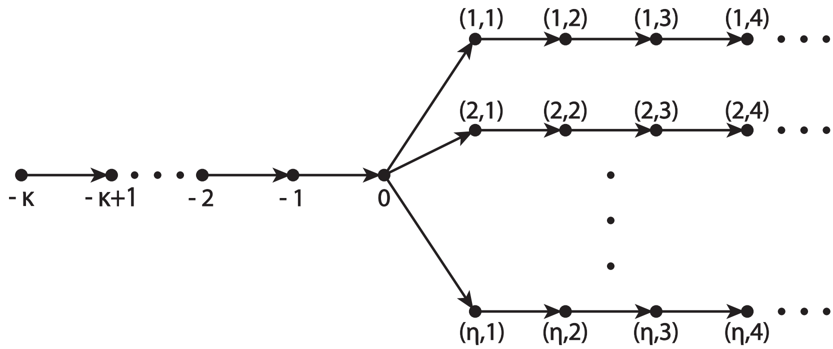

Figure 2.1

For a weight sequence $\lambda = \{\lambda_k\}_{k \in V^\circ_{\eta,\kappa}} \subset \mathbb{C}$, weighted shifts S_λ associated by $\mathcal{T}_{\eta,\kappa}$ are the main model of this paper. And we mention that all operators discussed in the remaining sections of this paper are bounded.

3. A Characterization of property $B(n)$ and related distinction

We firstly characterize weighted shifts with property $B(n)$ over (η, κ)-type directed tree $\mathcal{T}_{\eta,\kappa}$.

Theorem 3.1. *Let $\mathcal{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ be an (η, κ)-type directed tree, and let $\lambda = \{\lambda_k\}_{k \in V^\circ_{\eta,\kappa}} \subset \mathbb{C}$. Suppose $S_\lambda \in B(\ell^2(V_{\eta,\kappa}))$ is an associated weighted shift over $\mathcal{T}_{\eta,\kappa}$. Then the following assertions hold.*

(i) If $n > \kappa$, then S_λ is a weighted shift with property $B(n)$ if and only if the following conditions are satisfied:

$$\prod_{j=0}^{k+1} |\lambda_j|^2 \left(\sum_{i=1}^{\eta} \left(\prod_{j=1}^{n-|k|} |\lambda_{(i,j)}|^2 \right) \right) \geq |\lambda_{k+1}|^{2n}, \quad -\kappa \leq k \leq -1, \tag{3.1}$$

$$\prod_{k=j+2}^{j+n} |\lambda_{(i,k)}| \geq |\lambda_{(i,j+1)}|^{n-1}, \qquad 1 \leq i \leq \eta, \quad j \in \mathbb{N}, \tag{3.2}$$

and

$$\sum_{i=1}^{\eta} (\prod_{j=1}^{n} |\lambda_{(i,j)}|^2) \geq (\sum_{i=1}^{\eta} |\lambda_{(i,1)}|^2)^n. \tag{3.3}$$

(ii) If $n \leq \kappa$, then S_λ is a weighted shift with property $B(n)$ if and only if the following conditions are satisfied:

$$\prod_{i=k+2}^{k+n} |\lambda_i| \geq |\lambda_{k+1}|^{n-1}, \quad -\kappa \leq k \leq -n, \tag{3.4}$$

$$\prod_{j=0}^{k+1} |\lambda_j|^2 \left(\sum_{i=1}^{\eta} \left(\prod_{j=1}^{n-|k|} |\lambda_{(i,j)}|^2 \right) \right) \geq |\lambda_{k+1}|^{2n}, \quad -n+1 \leq k \leq -1, \tag{3.5}$$

$$\prod_{k=j+2}^{j+n} |\lambda_{(i,k)}| \geq |\lambda_{(i,j+1)}|^{n-1}, \quad 1 \leq i \leq \eta, \quad j \in \mathbb{N}, \tag{3.6}$$

and

$$\sum_{i=1}^{\eta} (\prod_{j=1}^{n} |\lambda_{(i,j)}|^2) \geq (\sum_{i=1}^{\eta} |\lambda_{(i,1)}|^2)^n. \tag{3.7}$$

Proof. By simple computation with (2.1), we have

$$S_\lambda^n e_k = \begin{cases} (\prod_{k=j+1}^{j+n} \lambda_{(i,k)}) e_{(i,j+n)} & \text{if } k = (i,j),\ 1 \leq i \leq \eta,\ j \in \mathbb{N}, \\ \sum_{i=1}^{\eta} (\prod_{j=1}^{n} \lambda_{(i,j)}) e_{(i,n)} & \text{if } k = 0, \\ \prod_{j=k+1}^{0} \lambda_j (\sum_{i=1}^{\eta} (\prod_{j=1}^{n-|k|} \lambda_{(i,j)}) e_{(i,n-|k|)}) & \text{if } k = -\kappa, \cdots, -1,\ n > |k|, \\ (\prod_{i=k+1}^{k+n} \lambda_i) e_{k+n} & \text{if } k = -\kappa, \cdots, -1,\ n \leq |k|. \end{cases}$$

By simple computation with (2.2), we may obtain that
$(S_\lambda^*)^n (S_\lambda^n e_k)$

$$= \begin{cases} (\prod_{k=j+1}^{j+n} |\lambda_{(i,k)}|^2) e_{(i,j)} & \text{if } k = (i,j),\ 1 \leq i \leq \eta,\ j \in \mathbb{N}, \\ (\sum_{i=1}^{\eta} (\prod_{j=1}^{n} |\lambda_{(i,j)}|^2)) e_0 & \text{if } k = 0, \\ (\prod_{i=k+1}^{0} |\lambda_i|^2)(\sum_{i=1}^{\eta} (\prod_{j=1}^{n-|k|} |\lambda_{(i,j)}|^2)) e_k & \text{if } k = -\kappa, \cdots, -1,\ n > |k|, \\ (\prod_{i=k+1}^{k+n} |\lambda_i|^2) e_k & \text{if } k = -\kappa, \cdots, -1,\ n \leq |k|. \end{cases} \tag{3.8}$$

Recall that $|S_\lambda^n| e_k = ((S_\lambda^n)^* S_\lambda^n)^{\frac{1}{2}} e_k$ and $|S_\lambda|^n e_k = \|S_\lambda e_k\|^n e_k$ for $k \in V_{\eta,\kappa}$ (see [10, Proposition 3.4.3(iii)]). Since $\|S_\lambda e_k\|^2 = \sum_{u \in \mathrm{Chi}(k)} |\lambda_u|^2$,

$$\|S_\lambda e_k\|^2 = \begin{cases} |\lambda_{(i,j+1)}|^2 & \text{if } k = (i,j),\ 1 \leq i \leq \eta,\ j \in \mathbb{N}, \\ \sum_{i=1}^{\eta} |\lambda_{(i,1)}|^2 & \text{if } k = 0, \\ |\lambda_{k+1}|^2 & \text{if } k = -\kappa, \cdots, -1. \end{cases}$$

Thus

$$|S_\lambda|^n e_k = \begin{cases} |\lambda_{(i,j+1)}|^n e_{(i,j)} & \text{if } k = (i,j),\ 1 \leq i \leq \eta,\ j \in \mathbb{N}, \\ (\sum_{i=1}^{\eta} |\lambda_{(i,1)}|^2)^{\frac{n}{2}} e_0 & \text{if } k = 0, \\ |\lambda_{k+1}|^n e_k & \text{if } k = -\kappa, \cdots, -1. \end{cases} \tag{3.9}$$

Since $(S_\lambda^n)^* S_\lambda^n$ is positive self-adjoint, by (3.8), it follows from [10, Lemma 2.2.1(iii)] that $|S_\lambda^n| e_k = t_k^{\frac{1}{2}} e_k$, where

$$t_k^{\frac{1}{2}} = \begin{cases} \prod_{k=j+1}^{j+n} |\lambda_{(i,k)}| & \text{if } k=(i,j),\ 1 \le i \le \eta,\ j \in \mathbb{N}, \\ (\sum_{i=1}^{\eta} (\prod_{j=1}^{n} |\lambda_{(i,j)}|^2))^{\frac{1}{2}} & \text{if } k = 0, \\ ((\prod_{i=k+1}^{0} |\lambda_i|^2)(\sum_{i=1}^{\eta}(\prod_{j=1}^{n-|k|} |\lambda_{(i,j)}|^2)))^{\frac{1}{2}} & \text{if } k=-\kappa,\cdots,-1,\ n > |k|, \\ \prod_{i=k+1}^{k+n} |\lambda_i| & \text{if } k = -\kappa, \cdots, -1,\ n \le |k|. \end{cases} \tag{3.10}$$

So, by (3.9) and (3.10), we have that S_λ is a weighted shift with property $B(n)$ if and only if (3.1)-(3.7) are satisfied. Hence the proof is complete. □

The following corollay is the special case of Theorem 3.1 with $\eta = 2, \kappa = 1$, which is frequently used in the remaining sections.

Corollary 3.2. *Let $\mathcal{T}_{2,1} = (V_{2,1}, E_{2,1})$ be an $(2,1)$-type directed tree, and let $\lambda = \{\lambda_k\}_{k \in V_{2,1}^\circ} \subset \mathbb{C}$. Suppose $S_\lambda \in B(\ell^2(V_{2,1}))$ is an associated weighted shift over $\mathcal{T}_{2,1}$ (see Figure 3.1).*

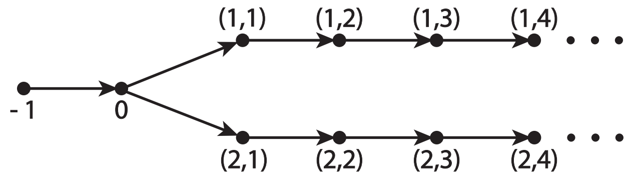

Figure 3.1

Then S_λ is a weighted shift with property $B(n)$ if and only if the following conditions are satisfied:

$$\begin{aligned} &\sum_{i=1}^{2} (\prod_{j=1}^{n-1} |\lambda_{(i,j)}|^2) \ge |\lambda_0|^{2n-2}, \\ &\sum_{i=1}^{2} (\prod_{j=1}^{n} |\lambda_{(i,j)}|^2) \ge (|\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2)^n \\ &\prod_{k=j+2}^{j+n} |\lambda_{(i,k)}| \ge |\lambda_{(i,j+1)}|^{n-1}, \qquad i = 1,2, \quad j \in \mathbb{N}. \end{aligned} \tag{3.11}$$

We now discuss the classes of weighted shifts with property $B(n)$ over directed trees are being distinct with respect to $n \ge 2$. Recall that, for a weighted shift W_α with weight sequence $\alpha = \{\alpha_n\}_{n \in \mathbb{Z}_+} \subset \mathbb{R}_0$, if we denote

$$\mathbb{B}_n = \{W_\alpha : W_\alpha \text{ has property } B(n)\}, \quad n \ge 2,$$

then it follows from [15, Theorem 2.4] that $\mathbb{B}_n$ are distinct one from another. Let $\mathcal{T} = (\mathbb{Z}_+, E)$, where $E = \{(n, n+1) : n \in \mathbb{Z}_+\}$. Then the classical weighted shift W_α is unitarily equivalent to the associated weighted shift S_λ with $\lambda = \{\alpha_n\}_{n\in\mathbb{Z}_+}$ over $\mathcal{T}$ (cf. [10, Theorem 3.2.1]).

Let $\mathcal{T}_{2,1} = (V_{2,1}, E_{2,1})$ be the $(2,1)$-type directed tree. If we consider a weight sequence $\lambda = \{\lambda_k\}_{k\in V_{2,1}^\circ}$ with

$$\lambda_{(1,j)} = 0 \quad \text{or} \quad \lambda_{(2,j)} = 0, \qquad j \in \mathbb{N}, \tag{3.12}$$

then the weighted shift S_λ associated by $\mathcal{T}_{2,1}$ is equivalent to a classical weighted shift W_α. We consider weighted shifts S_λ associated by $\mathcal{T}_{2,1}$ with weight sequence $\lambda = \{\lambda_k\}_{k\in V_{2,1}^\circ}$ without (3.12) to consider more generally in the remaining part of this paper.

Let $\mathbb{B}(n,\eta,\kappa)$ be the set of all bounded weighted shifts with property $B(n)$ over $\mathcal{T}_{\eta,\kappa}$ and $\lambda = \{\lambda_k\}_{k\in V_{2,1}^\circ} \subset \mathbb{C}\setminus\{0\}$. In particular, we denote

$$\mathbb{EB}(n,\eta,\kappa) = \{S_\lambda \in B(\ell^2(V_{\eta,\kappa})) : |S_\lambda^n| = |S_\lambda|^n \text{ and } \lambda = \{\lambda_k\}_{k\in V_{2,1}^\circ} \subset \mathbb{C}\setminus\{0\}\}.$$

Obviously, $\mathbb{EB}(n,\eta,\kappa) \subset \mathbb{B}(n,\eta,\kappa)$. First we consider the classes $\mathbb{B}(n,2,1)$ are being distinct one from another with respect to $n \geq 2$.

Theorem 3.3. *Under the same notation as above, we have that* $\mathbb{B}(m,2,1) \neq \mathbb{B}(n,2,1)$ *if and only if* $m \neq n$.

Proof. Since the sufficiency is obvious, we only consider the necessity. So, suppose $m \neq n$. Without loss of generality, we assume $m < n$. We will claim that there exists S_λ belongs to $\mathbb{EB}(n,2,1)$ but not in $\mathbb{EB}(m,2,1)$. Define $\lambda = \{\lambda_k\}_{k\in V_{2,1}^\circ} \subset \mathbb{C}$ by

$$\lambda_k = \begin{cases} \alpha_i & \text{if } k = (i,1) \text{ for } i = 1,2, \\ \lambda_0 & \text{if } k = 0, \\ \beta_i & \text{if } k = (i,j) \text{ for } i = 1,2 \text{ and } j \geq 2, \end{cases}$$

where $\lambda_0, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}\setminus\{0\}$ with $|\alpha_1|^2 + |\alpha_2|^2 = 1$. Then obviously $S_\lambda \in B(\ell^2(V_{2,1}))$. Observe from [11, Proposition 5.2] that $(S_\lambda^*)^n S_\lambda^n = (S_\lambda^* S_\lambda)^n$, i.e., $|S_\lambda^n|^2 = |S_\lambda|^{2n}$, if and only if

$$\|S_\lambda e_k\|^n = \|S_\lambda^n e_k\|, \qquad k \in V_{2,1}.$$

By a direct computation, we have that $(S_\lambda^*)^n S_\lambda^n = (S_\lambda^* S_\lambda)^n$ if and only if

$$(|\alpha_1\beta_1^{n-2}|^2 + |\alpha_2\beta_2^{n-2}|^2)^{\frac{1}{2}} = |\lambda_0|^{n-1} \text{ and } |\alpha_1\beta_1^{n-1}|^2 + |\alpha_2\beta_2^{n-1}|^2 = 1.$$

Set

$$f(x) = \frac{\log(\frac{\log(2-x)}{-\log x})}{\log(\frac{x}{2-x})}, \qquad x \in (0,2).$$

Then $\lim\limits_{x\to 0^+} f(x) = 0$. Thus there exists $\gamma_n \in (0,1)$ such that $(n-1)f(\gamma_n) \leq 1$. Put

$$g(x) = \gamma_n^{\frac{x}{n-1}} + (2-\gamma_n)^{\frac{x}{n-1}}.$$

Then g is strictly increasing on $[1,\infty)$; see [11, Example 5.5]. Now consider

$$\alpha_1=\alpha_2=(\frac{1}{2})^{\frac{1}{2}},\ \beta_1=\gamma_n^{\frac{1}{2(n-1)}},\ \beta_2=(2-\gamma_n)^{\frac{1}{2(n-1)}},$$

$$\lambda_0=[\frac{1}{2}(\gamma_n^{\frac{n-2}{n-1}}+(2-\gamma_n)^{\frac{n-2}{n-1}})]^{\frac{1}{2(n-1)}}. \tag{3.13}$$

Then

$$(|\alpha_1\beta_1^{n-2}|^2+|\alpha_2\beta_2^{n-2}|^2)^{\frac{1}{2}}=(\frac{1}{2}\gamma_n^{\frac{n-2}{n-1}}+\frac{1}{2}(2-\gamma_n)^{\frac{n-2}{n-1}})^{\frac{1}{2}}=|\lambda_0|^{n-1}$$

and

$$|\alpha_1\beta_1^{n-1}|^2+|\alpha_2\beta_2^{n-1}|^2=\frac{1}{2}\gamma_n+\frac{1}{2}(2-\gamma_n)=1.$$

Thus $|S_\lambda^n|^2=|S_\lambda|^{2n}$. So, $|S_\lambda^n|=|S_\lambda|^n$. Therefore, $S_\lambda\in\mathbb{EB}(n,2,1)$. But

$$\begin{aligned}|\alpha_1\beta_1^{m-1}|^2+|\alpha_2\beta_2^{m-1}|^2 &= \frac{1}{2}(\gamma_n^{\frac{m-1}{n-1}}+(2-\gamma_n)^{\frac{m-1}{n-1}})\\ &= \frac{1}{2}g(m-1)<\frac{1}{2}g(n-1)\\ &= 1.\end{aligned}$$

Thus $S_\lambda\notin\mathbb{EB}(m,2,1)$. So, we have $S_\lambda\in B(\ell^2(V_{2,1}))$ such that $S_\lambda\in\mathbb{EB}(n,2,1)$, but $S_\lambda\notin\mathbb{EB}(m,2,1)$. Since $\mathbb{EB}(n,2,1)\subset\mathbb{B}(n,2,1)$, $S_\lambda\in\mathbb{B}(n,2,1)$. Note that

$$\begin{aligned}\sum_{i=1}^{2}(\prod_{j=1}^{m}|\lambda_{(i,j)}|^2) &= |\alpha_1|^2|\beta_1|^{2(m-1)}+|\alpha_2|^2|\beta_2|^{2(m-1)}\\ &= \frac{1}{2}(\gamma_n^{\frac{m-1}{n-1}}+(2-\gamma_n)^{\frac{m-1}{n-1}})\\ &= \frac{1}{2}g(m-1)\\ &< \frac{1}{2}g(n-1)=1\end{aligned}$$

and also $1=(|\lambda_{(1,1)}|^2+|\lambda_{(2,1)}|^2)^m$. Thus, by (3.11) in Corollary 3.2, $S_\lambda\notin\mathbb{B}(m,2,1)$. Hence $\mathbb{B}(m,2,1)\neq\mathbb{B}(n,2,1)$. □

Remark 1. Let $\mathcal{T}_{\eta,\kappa}$ be an arbitrary (η,κ)-type directed tree. Choose a subtree $\mathcal{T}_{2,1}$ of $\mathcal{T}_{\eta,\kappa}$. We define $\lambda=\{\lambda_k\}_{k\in V_{\eta,\kappa}^\circ}$ such that weights of $\mathcal{T}_{2,1}$ are as in (3.13) and other weights are 0. For any $m,n\geq 2$ such that $m<n$, the weighted shift S_λ associated by $\mathcal{T}_{\eta,\kappa}$ has property $B(n)$ but not property $B(m)$. Of course, if we apply the result of [15, Theorem 2.4] with this idea, we can obtain the same conclusion easily. But, in the case of weight sequence $\lambda=\{\lambda_k\}_{k\in V_{\eta,\kappa}^\circ}\subset\mathbb{C}\setminus\{0\}$, we guess the computations are so complicated. We leave them to the interesting readers.

4. Some Properties

In this section, we discuss some properties about weighted shifts with property $B(n)$ over $(2,1)$-type directed trees.

Theorem 4.1. *Let $\mathcal{T}_{2,1} = (V_{2,1}, E_{2,1})$ be the $(2,1)$-type directed tree as in Figure 4.1, and let $\lambda = \{\lambda_k\}_{k\in V_{2,1}^{\circ}} \subset \mathbb{C}\setminus\{0\}$.*

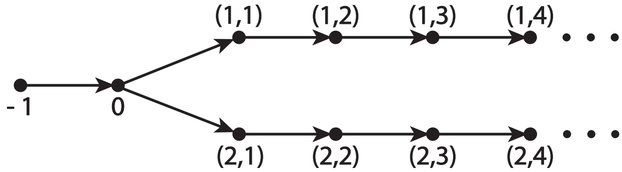

Figure 4.1

If $S_\lambda \in B(\ell^2(V_{2,1}))$ is a weighted shift with property $B(2)$, then the following conditions hold:

(i) the sequence $\{|\lambda_{(i,j)}|\}_{j=2}^{\infty}$ is increasing for $i = 1, 2$,

(ii) $|\lambda_{(1,1)}| \neq |\lambda_{(1,2)}|$ or $|\lambda_{(2,1)}| \neq |\lambda_{(2,2)}|$.

Proof. By Corollary 3.2, S_λ is a weighted shift with property $B(2)$ if and only if

$$|\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2 \geq |\lambda_0|^2, \tag{4.1}$$

$$|\lambda_{(1,1)}|^2|\lambda_{(1,2)}|^2 + |\lambda_{(2,1)}|^2|\lambda_{(2,2)}|^2 \geq (|\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2)^2, \tag{4.2}$$

and

$$|\lambda_{(i,j+1)}| \leq |\lambda_{(i,j+2)}|, \quad i = 1, 2, \quad j \in \mathbb{N}. \tag{4.3}$$

The inequality of (4.3) proves (i) obviously. To show (ii), we suppose that $|\lambda_{(1,1)}| = |\lambda_{(1,2)}|$ and $|\lambda_{(2,1)}| = |\lambda_{(2,2)}|$. Then, by direct computations, we obtain

$$\begin{aligned} |\lambda_{(1,1)}|^2|\lambda_{(1,2)}|^2 + |\lambda_{(2,1)}|^2|\lambda_{(2,2)}|^2 &= |\lambda_{(1,1)}|^4 + |\lambda_{(2,1)}|^4 \\ &< |\lambda_{(1,1)}|^4 + 2|\lambda_{(1,1)}|^2|\lambda_{(2,1)}|^2 + |\lambda_{(2,1)}|^4 \\ &= (|\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2)^2, \end{aligned}$$

which is contradict to (4.2), so the condition (ii) holds. Hence the proof is complete. □

We now discuss some flatness properties of weighted shifts with property $B(2)$ over $(2,1)$-type directed trees.

Theorem 4.2. *Let $S_\lambda \in B(\ell^2(V_{2,1}))$ be an associated weighted shift over $(2,1)$-type directed tree with $\lambda = \{\lambda_k\}_{k\in V_{2,1}^{\circ}} \subset \mathbb{C}\setminus\{0\}$. If S_λ has property $B(2)$, then we have the following assertions.*

(i) If $|\lambda_{(1,2)}| = |\lambda_{(2,2)}| = y$, then $y^2 \geq |\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2 \geq |\lambda_0|^2$.

(ii) If $|\lambda_{(1,1)}| = |\lambda_{(2,1)}| = x$, then $|\lambda_0| \leq \sqrt{2}x$ and $|\lambda_{(1,2)}|^2 + |\lambda_{(2,2)}|^2 \geq 4x^2$.

Proof. (i) Since S_λ has property $B(2)$, applying (4.2) with $|\lambda_{(1,2)}| = |\lambda_{(2,2)}| = y$, we may obtain

$$\begin{aligned} |\lambda_{(1,1)}|^2 y^2 + |\lambda_{(2,1)}|^2 y^2 &= (|\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2) y^2 \\ &\geq (|\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2)^2. \end{aligned}$$

Thus $y^2 \geq |\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2$. By using (4.1), we have

$$y^2 \geq |\lambda_{(1,1)}|^2 + |\lambda_{(2,1)}|^2 \geq |\lambda_0|^2.$$

(ii) Applying (4.1) with $|\lambda_{(1,1)}| = |\lambda_{(2,1)}| = x$, we have

$$(x^2 + x^2)^{\frac{1}{2}} = (2x^2)^{\frac{1}{2}} = \sqrt{2}x \geq |\lambda_0|.$$

By using (4.2), we also obtain

$$x^2|\lambda_{(1,2)}|^2 + x^2|\lambda_{(2,2)}|^2 = x^2(|\lambda_{(1,2)}|^2 + |\lambda_{(2,2)}|^2) \geq 4x^4.$$

Thus $|\lambda_{(1,2)}|^2 + |\lambda_{(2,2)}|^2 \geq 4x^2$. Hence the proof is complete. □

We consider a completion problem of property $B(2)$ on weighted shift over directed tree as Figure 4.2.

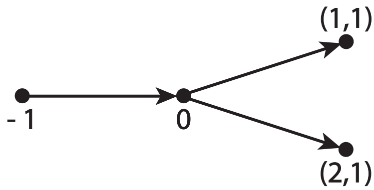

Figure 4.2

Theorem 4.3. *Let $\mathcal{T} = (V, E)$ be a directed tree with $V = \{-1, 0, (1,1), (2,1)\}$ and E is as Figure 4.2. Let the weights of $\mathcal{T}$ be given arbitrarily by*

$$\lambda : \lambda_0 = r,\ \lambda_{(1,1)} = x,\ \lambda_{(2,1)} = y, \qquad r, x, y \in \mathbb{C} \setminus \{0\}.$$

Then the following assertions are equivalent:

(i) *there exist $(2,1)$-type directed tree $\mathcal{T}_{2,1}$ extended from $\mathcal{T}$ and a weight sequence $\widehat{\lambda} = \{\widehat{\lambda}_k\}_{k \in V_{2,1}^\circ}$ extended from λ, i.e., $\widehat{\lambda} \supset \{\lambda_0, \lambda_{(1,1)}, \lambda_{(2,1)}\}$, such that the associated weighted shift $S_{\widehat{\lambda}}$ has property $B(2)$,*

(ii) *the following conditions hold:*
(a) $|x|^2 + |y|^2 \geq |r|^2$,
(b) there exist u and v in $\mathbb{C}$ such that $\sqrt{|x|^2|u|^2 + |y|^2|v|^2} \geq |x|^2 + |y|^2$.

Proof. (i)⇒(ii) Let $\mathcal{T}_{2,1}$ be a $(2,1)$-type directed tree extended from $\mathcal{T}$ and a weight sequence $\widehat{\lambda}$ extended from λ whose associated weighted shift $S_{\widehat{\lambda}}$ has property $B(2)$. Recall that (4.1)-(4.3) are equivalent to the property $B(2)$ of weighted shift S_λ associated by $\mathcal{T}_{2,1}$. The condition (a) comes from (4.1). Seting $u = \widehat{\lambda}_{(1,2)}$ and $v = \widehat{\lambda}_{(2,2)}$, by (4.2), we obtain (b).

(ii)⇒(i) Suppose $|x|^2+|y|^2 \geq |r|^2$ and $\sqrt{|x|^2|u|^2+|y|^2|v|^2} \geq |x|^2+|y|^2$ for some u and v in $\mathbb{C}$. Consider a (2,1)-type directed tree $\mathcal{T}_{2,1}$ as in Figure 4.1, and $\mathcal{T}$ is obviously a subtree of $\mathcal{T}_{2,1}$. Define a weight sequence $\widehat{\lambda} = \{\widehat{\lambda}_k\}_{k \in V_{2,1}^{\circ}}$ of $\mathcal{T}_{2,1}$ by

$$\begin{aligned} \widehat{\lambda}_0 &= \lambda_0 = r,\ \widehat{\lambda}_{(1,1)} = \lambda_{(1,1)} = x,\ \widehat{\lambda}_{(2,1)} = \lambda_{(2,1)} = y, \\ \widehat{\lambda}_{(1,j+1)} &= u, \text{ and } \widehat{\lambda}_{(2,j+1)} = v,\ j \in \mathbb{N}. \end{aligned}$$

Then $\mathcal{T}_{2,1}$ is an extended directed tree from $\mathcal{T}$ and $\widehat{\lambda} = \{\widehat{\lambda}_k\}_{k \in V_{2,1}^{\circ}}$ is extended from λ obviously. Applying the equivalent condition of the property $B(2)$ of weighted shift S_λ associated by $\mathcal{T}_{2,1}$ in (4.1)-(4.3) again, $S_{\widehat{\lambda}}$ has property $B(2)$. Hence the proof is complete. □

References

[1] C. Burnap, I. B. Jung, and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory, **53(2)** (2005), 381-397.

[2] M. Cho and T. Yamazaki, *An operator transform from class A to the class of hyponormal operators and its application*, Integral Equations Operator Theory, **53** (2005), 497-508.

[3] B. Duggal, I. Jeon, and C. Kubrusly, *Contractions satisfying the absolute value property* $|A|^2 \leq |A^2|$, Integral Equations Operator Theory, **49** (2004), 141-148.

[4] J. Fujii, M. Fujii, H. Sasaoka and Y. Watatani, *The spectrum of an infinite directed graph*, Math. Japon., **36** (1991), 607-625.

[5] M. Fujii, T. Sano and H. Sasaoka, *The index of an infinite directed graph*, Math. Japon., **35** (1990), 561-567.

[6] M. Fujii, H. Sasaoka and Y. Watatani, *Adjacency operators of infinite directed graphs*, Math. Japon., **34** (1989), 727-735.

[7] M. Fujii, H. Sasaoka and Y. Watatani, *The numerical radius of an infinite directed graph*, Math. Japon., **35** (1990), 577-582.

[8] T. Furuta, Invitation to Linear Operators, Taylor & Francis Inc., London and New York, 2001.

[9] M. Ito, *On classes of operators generalizing class A and paranormality*, Sci. Math. Japonia., **7** (2002), 353-363.

[10] Z. Jabłoński, I. B. Jung, and J. Stochel, *Weighted shifts on directed trees*, Mem. Amer. Math. Soc. **216** (2012), no. 1017, viii+ 106 pp.

[11] Z. Jabłoński, I. B. Jung, and J. Stochel, *Unbounded quasinormal operatorsrevisited*, Integral Equations Operator Theory **79** (2014), 135-149.

[12] S. Jung, E. Ko and M. Lee, *On class A operators*, Studia Math., **198** (2010), 249-260.

[13] I. B. Jung, M. R. Lee, and P. S. Lim, *Gaps of operators II*, Glasgow Math. J., **47** (2005), 461-469.

[14] I. B. Jung, P. S. Lim, and S. S. Park, *Gaps of operators*, J. Math. Anal. Appl., **304** (2005), 87-95.

[15] E. Y. Lee, M. R. Lee and H. Y. Park, *Inequalities of operator powers*, J. KSIAM, **12** (2008), 1-6.

[16] S. Miyajima and I. Saito, *∞-hyponormal operators and their spectral properties*, Acta Sci. Math. (Szeged) **67** (2001), 357-371.

[17] A. Uchiyama and K. Tanahashi, *On the Riesz idempotent of class A operators*, Math. Inequal. Appl., **5** (2002), 291-298.

[18] A. Uchiyama, K. Tanahashi and J. Lee, *Spectrum of class* $A(s,t)$ *operators*, Acta Sci. Math. (Szeged), **70** (2004), 279-287.

[19] D. Xia, Spectral theory of hyponormal operators, Birkhauser Verlag, Boston, 1983.

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