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FUNDAMENTAL THEOREM OF UPPER AND LOWER SOLUTIONS FOR A CLASS OF SINGULAR (p_1, p_2)-LAPLACIAN SYSTEMS

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Abstract. We introduce the fundamental theorem of upper and lower solutions for a class of singular (p_1, p_2)-Laplacian systems and give the proof by using the Schauder fixed point theorem. It will play an important role to study the existence of solutions.

1. Introduction

In this paper, we introduce fundamental theorem of the upper and lower solutions for singular general boundary value problem of the form

$$\begin{cases} \varphi_{p_1}(u')' + F_1(t, u, v) = 0, \\ \varphi_{p_2}(v')' + F_2(t, u, v) = 0, \quad t \in (0, 1), \\ u(0) = A_1, u(1) = B_1, v(0) = A_2, v(1) = B_2, \end{cases} \tag{S}$$

where $\varphi_{p_i}(x) = |x|^{p_i-2}x$, $x \in \mathbb{R}$, $p_i > 1$, for $i = 1, 2$, each $A_i, B_i \in \mathbb{R}$ and each $F_i : (0, 1) \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following assumptions

(1) For almost every $t \in (0, 1)$, the function $F_i(t, \cdot, \cdot)$ is continuous.
(2) For each $(u, v) \in \mathbb{R}^2$, the function $F_i(\cdot, u, v)$ is measurable on $(0, 1)$.

Throughout the paper, we denote $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = (0, \infty)$, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $|\cdot|$ the absolute value in $\mathbb{R}$ and $\|(u, v)\| = |u| + |v|$ for $(u, v) \in \mathbb{R}^2$.

In the past years, there have been a lot of studies about the existence of solutions for various separated two-point boundary value problems. Proofs of all the existence results mainly make use of many kinds of nonlinear analytic methods such as the fixed point theorem on cones [1, 3, 17], upper and lower solution method [2, 4] [7]-[11] [14, 15], global bifurcation theorem [6, 12] or global continuation theorem [13, 16]. Especially, the upper and lower solution method is one of well-used methods. When we use the method, we need to construct so

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called the fundamental theorem of upper and lower solutions beside trying to find the upper and lower solutions (see [2, 4] [7]-[11] [14, 15] and the references therein). In this paper, we will introduce the fundamental theorem of upper and lower solutions for a class of singular (p_1, p_2)-Laplacian system like (S) which can cover all the cases emerged in [2, 4] [7]-[11] [14, 15] and will be the newest so far.

Our paper is organized as follows. In Section 2, we introduce the main results including the fundamental theorem of upper and lower solutions. In Section 3, we will give the exact proofs for our main results.

2. Main results

Before introducing the fundamental theorem of the upper and lower solutions for problem (S), we need firstly give some definitions as follows.

Definition 1. We say that (α_u, α_v) is a lower solution of problem (S) if $(\alpha_u, \alpha_v) \in (C[0,1] \times C[0,1]) \cap (C^1(0,1) \times C^1(0,1))$ and satisfies

$$\begin{cases} \varphi_{p_1}(\alpha_u'(t))' + F_1(t, \alpha_u(t), \alpha_v(t)) \geq 0, \\ \varphi_{p_2}(\alpha_v'(t))' + F_2(t, \alpha_u(t), \alpha_v(t)) \geq 0, \quad t \in (0,1), \\ \alpha_u(0) \leq A_1, \alpha_u(1) \leq B_1, \alpha_v(0) \leq A_2, \alpha_v(1) \leq B_2. \end{cases}$$

Similarly, we say that (β_u, β_v) is an upper solution of problem (S) if $(\beta_u, \beta_v) \in (C[0,1] \times C[0,1]) \cap (C^1(0,1) \times C^1(0,1))$ and satisfies the reverse of the above inequalities.

Definition 2. We say that F_1 and F_2 are quasi-monotone nondecreasing with respect to v and u, respectively, if

$$F_1(t, u, v_1) \leq F_1(t, u, v_2), \quad \text{whenever} \quad v_1 \leq v_2,$$

$$F_2(t, u_1, v) \leq F_2(t, u_2, v), \quad \text{whenever} \quad u_1 \leq u_2.$$

Thus, we have the following fundamental theorem of upper and lower solutions for the singular (p_1, p_2)-Laplacian system.

Theorem 2.1. *Let (α_u, α_v) and (β_u, β_v) be a lower and upper solution of (S), respectively, such that*

(a_1) $(\alpha_u(t), \alpha_v(t)) \leq (\beta_u(t), \beta_v(t))$, *for all* $t \in [0,1]$.

Assume also that each $h_i : (0,1) \to \mathbb{R}^+$ *is locally integrable such that*

(a_2) $\int_0^{\frac{1}{2}} \varphi_{p_i}^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau) d\tau\right) ds + \int_{\frac{1}{2}}^1 \varphi_{p_i}^{-1}\left(\int_{\frac{1}{2}}^s h_i(\tau) d\tau\right) ds < \infty$, *for* $i = 1, 2$.

(a_3) $|F_i(t, u, v)| \leq h_i(t)$, *for all* $(t, u, v) \in D_\alpha^\beta$ *and* $i = 1, 2$, *where*

$$D_\alpha^\beta = \{(t, u, v) \in (0,1) \times \mathbb{R}^2 \mid (\alpha_u(t), \alpha_v(t)) \leq (u, v) \leq (\beta_u(t), \beta_v(t))\}.$$

(a_4) F_1 *and* F_2 *are quasi-monotone nondecreasing with respect to* v *and* u, *respectively.*

Then problem (S) has at least one solution (u, v) such that

$$(\alpha_u(t), \alpha_v(t)) \le (u(t), v(t)) \le (\beta_u(t), \beta_v(t)), \quad \text{for all } t \in [0, 1].$$

To prove Theorem 2.1, we need give the following lemma.

Lemma 2.2. *Assume that there exists $h_i : (0, 1) \to \mathbb{R}^+$ is locally integrable satisfying (a_2) such that*

$$|F_i(t, u, v)| \le h_i(t), \quad \text{for all } (t, u, v) \in (0, 1) \times \mathbb{R}^2 \text{ and } i = 1, 2.$$

Then problem (S) has a solution.

Remark 1. From the definition of φ_{p_i}, then for any $x, y \in \mathbb{R}$, we have

$$\varphi_{p_i}^{-1}(|x| + |y|) \le C_{p_i} \left(\varphi_{p_i}^{-1}(|x|) + \varphi_{p_i}^{-1}(|y|)\right),$$

where

$$C_{p_i} = \begin{cases} 1, & p_i > 2, \\ 2^{\frac{2-p_i}{p_i-1}}, & 1 < p_i \le 2. \end{cases}$$

3. Proofs of main results

In this section, we give the exact proofs of main results. For this, we need the following well-known Schauder fixed point theorem.

Theorem 3.1. ([5]) *Let X be a Banach space and let M be a closed, convex and bounded set in X. Assume that $T : M \to M$ is completely continuous. Then T has a fixed point in M.*

To set up the solution operator for (S), let us take $X = C[0, 1] \times C[0, 1]$ as a Banach space with norm $\|(u, v)\|_\infty = \|u\|_\infty + \|v\|_\infty$, for all $(u, v) \in X$. By the assumptions on F_i, we can denote $N_{F_i}(u, v)(t) \triangleq F_i(t, u(t), v(t))$, where $N_{F_i} : X \to \mathbb{R}$ is called the Nemytskii operator corresponding to F_i for $i = 1, 2$. Motivated by the solution operator established in Sim-Lee [16], for $(u, v) \in X$, we define $T^i : X \to C[0, 1]$ by

$$T^i(u, v)(t) = \begin{cases} \int_0^t \varphi_{p_i}^{-1} \left(a^i(N_{F_i}(u, v)) + \int_s^{\frac{1}{2}} F_i(\tau, u(\tau), v(\tau))d\tau\right) ds, t \in [0, \frac{1}{2}], \\ \int_t^1 \varphi_{p_i}^{-1} \left(-a^i(N_{F_i}(u, v)) + \int_{\frac{1}{2}}^s F_i(\tau, u(\tau), v(\tau))d\tau\right) ds, t \in [\frac{1}{2}, 1], \end{cases}$$

where $a^i(N_{F_i}(u, v)) \in \mathbb{R}$ uniquely satisfies

$$\int_0^{\frac{1}{2}} \varphi_{p_i}^{-1} \left(a^i(N_{F_i}(u, v)) + \int_s^{\frac{1}{2}} F_i(\tau, u(\tau), v(\tau))d\tau\right) ds$$
$$= \int_{\frac{1}{2}}^1 \varphi_{p_i}^{-1} \left(-a^i(N_{F_i}(u, v)) + \int_{\frac{1}{2}}^s F_i(\tau, u(\tau), v(\tau))d\tau\right) ds,$$

and

$$T(u, v) = (T^1(u, v), T^2(u, v)).$$

Then we can easily see that problem (S) can be equivalently written as $(u,v) = T(u,v)$ in X. For simplicity, we denote $a^i \triangleq a^i(N_{F_i}(u,v))$.

Remark 2. By the similar arguments in Sim-Lee [16], we can also regard the function a^i as a function of (u,v) and then obtain (1) a^i sends bounded sets in X into bounded sets in $\mathbb{R}$ for $i = 1,2$. (2) $a^i : X \to \mathbb{R}$ is continuous for $i = 1,2$. By these properties of a^i, we can deduce the following lemma.

Lemma 3.2. *$T : X \to X$ is completely continuous.*

Proof. For the continuity of T, let us assume that $(u_n,v_n) \to (u,v)$ in X, then from the continuity of a^i, F_i and Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\|T(u_n,v_n) - T(u,v)\|_\infty \\
&= \sum_{i=1}^{2} \|T^i(u_n,v_n) - T^i(u,v)\|_\infty = \sum_{i=1}^{2} \sup_{t\in[0,1]} |T^i(u_n,v_n) - T^i(u,v)| \\
&\le \sum_{i=1}^{2} \left[\sup_{t\in[0,\frac{1}{2}]} |T^i(u_n,v_n) - T^i(u,v)| + \sup_{t\in[\frac{1}{2},1]} |T^i(u_n,v_n) - T^i(u,v)| \right] \\
&\le \sum_{i=1}^{2} \left[\sup_{t\in[0,\frac{1}{2}]} \left| \int_0^t \varphi_{p_i}^{-1}\left(a_n^i + \int_s^{\frac{1}{2}} F_i(\tau,u_n(\tau),v_n(\tau))d\tau \right) \right.\right. \\
&\left. -\varphi_{p_i}^{-1}\left(a_n^i + \int_s^{\frac{1}{2}} F_i(\tau,u(\tau),v(\tau))d\tau \right) ds \right| \\
&+ \sup_{t\in[\frac{1}{2},1]} \left| \int_t^1 \varphi_{p_i}^{-1}\left(-a_n^i + \int_{\frac{1}{2}}^{s} F_i(\tau,u_n(\tau),v_n(\tau))d\tau \right) \right. \\
&\left.\left. -\varphi_{p_i}^{-1}\left(-a_n^i + \int_{\frac{1}{2}}^{s} F_i(\tau,u(\tau),v(\tau))d\tau \right) ds \right| \right] \to 0.
\end{aligned}$$

Next, let B be a bounded subset of X. Then by Ascoli-Arzela theorem, it is enough to show that $T(B)$ is uniformly bounded and equicontinuous. We first prove that $T(B)$ is uniformly bounded. Indeed, take $K_i = \sup\{|a^i(N_{F_i}(u,v))| \mid (u,v) \in B\}$ and by the assumption on F_i in Lemma 2.2, there is $h_i(t)$ such that $|F_i(t,u,v)| \le h_i(t)$ for *a.e.* $t \in (0,1)$ and all $(u,v) \in B$. Thus, we can compute the bound on the interval $[0,\frac{1}{2}]$ as follows, the bound on the interval $[\frac{1}{2},1]$ can be obtained by the similar way.

$$
\begin{aligned}
\|T(u,v)(t)\| &= \sum_{i=1}^{2} |T^i(u,v)(t)| \\
&= \sum_{i=1}^{2} \left| \int_0^t \varphi_{p_i}^{-1} \left(a^i + \int_s^{\frac{1}{2}} F_i(\tau, u(\tau), v(\tau)) d\tau \right) ds \right| \\
&\le \sum_{i=1}^{2} \left| \int_0^t \varphi_{p_i}^{-1} \left(|a^i| + \int_s^{\frac{1}{2}} |F_i(\tau, u(\tau), v(\tau))| d\tau \right) ds \right| \\
&\le \sum_{i=1}^{2} \int_0^{\frac{1}{2}} \varphi_{p_i}^{-1} \left(K_i + \int_s^{\frac{1}{2}} h_i(\tau) d\tau \right) ds \\
&\le \sum_{i=1}^{2} \left[\frac{1}{2} C_{p_i} \varphi_{p_i}^{-1}(K_i) + C_{p_i} \int_0^{\frac{1}{2}} \varphi_{p_i}^{-1} \left(\int_s^{\frac{1}{2}} h_i(\tau) d\tau \right) ds \right].
\end{aligned}
$$

Finally, we prove the equicontinuity of $T(B)$. Assume that $t_1 < t_2$.
Case 1 ($t_1, t_2 \in [0, \frac{1}{2}]$).

$$
\begin{aligned}
&\|T(u,v)(t_1) - T(u,v)(t_2)\| = \sum_{i=1}^{2} |T^i(u,v)(t_1) - T^i(u,v)(t_2)| \\
&\le \sum_{i=1}^{2} \left| \int_{t_1}^{t_2} \varphi_{p_i}^{-1} \left(|a^i| + \int_s^{\frac{1}{2}} |F_i(\tau, u(\tau), v(\tau))| d\tau \right) ds \right| \\
&\le \sum_{i=1}^{2} \int_{t_1}^{t_2} \varphi_{p_i}^{-1} \left(K_i + \int_s^{\frac{1}{2}} h_i(\tau) d\tau \right) ds \\
&\le \sum_{i=1}^{2} \left[C_{p_i} \varphi_{p_i}^{-1}(K_i)(t_2 - t_1) + C_{p_i} \int_{t_1}^{t_2} \varphi_{p_i}^{-1} \left(\int_s^{\frac{1}{2}} h_i(\tau) d\tau \right) ds \right].
\end{aligned}
$$

The bound of the case above is independent of $(u,v) \in B$ and by the property of h_i^B, we see that the bound converges to 0 as $|t_1 - t_2| \to 0$.
Case 2 ($t_1, t_2 \in [\frac{1}{2}, 1]$).
Proof can be done by the similar argument as Case 1.
Case 3 ($0 < t_1 \le \frac{1}{2} < t_2 < 1$).
Without loss of generality, we assume that $\frac{1}{4} \le t_1 \le \frac{1}{2} < t_2 \le \frac{3}{4}$. Then, by

using the definition of a^i, we have

$$\|T(u,v)(t_1)-T(u,v)(t_2)\| = \sum_{i=1}^{2} |T^i(u,v)(t_1)-T^i(u,v)(t_2)|$$
$$= \sum_{i=1}^{2} \left| \int_0^{t_1} \varphi_{p_i}^{-1}\left(a^i + \int_s^{\frac{1}{2}} F_i(\tau,u(\tau),v(\tau))d\tau\right) ds \right.$$
$$\left. - \int_{t_2}^{1} \varphi_{p_i}^{-1}\left(-a^i + \int_{\frac{1}{2}}^{s} F_i(\tau,u(\tau),v(\tau))d\tau\right) ds \right|$$
$$= \sum_{i=1}^{2} \left| \int_0^{t_1} \varphi_{p_i}^{-1}\left(a^i + \int_s^{\frac{1}{2}} F_i(\tau,u(\tau),v(\tau))d\tau\right) ds \right.$$
$$- \int_0^{\frac{1}{2}} \varphi_{p_i}^{-1}\left(a^i + \int_s^{\frac{1}{2}} F_i(\tau,u(\tau),v(\tau))d\tau\right) ds$$
$$+ \int_{\frac{1}{2}}^{1} \varphi_{p_i}^{-1}\left(-a^i + \int_{\frac{1}{2}}^{s} F_i(\tau,u(\tau),v(\tau))d\tau\right) ds$$
$$\left. - \int_{t_2}^{1} \varphi_{p_i}^{-1}\left(-a^i + \int_{\frac{1}{2}}^{s} F_i(\tau,u(\tau),v(\tau))d\tau\right) ds \right|.$$

Using the properties of a^i and F_i, we obtain

$$\|T(u,v)(t_1)-T(u,v)(t_2)\|$$
$$\le \sum_{i=1}^{2} \left[\int_{t_1}^{\frac{1}{2}} \varphi_{p_i}^{-1}\left(K_i + \int_s^{\frac{1}{2}} h_i(\tau)d\tau\right) ds + \int_{\frac{1}{2}}^{t_2} \varphi_{p_i}^{-1}\left(K_i + \int_{\frac{1}{2}}^{s} h_i(\tau)d\tau\right) ds \right]$$
$$\le \sum_{i=1}^{2} \left[\varphi_{p_i}^{-1}\left(K_i + \|h_i\|_{L^1[\frac{1}{4},\frac{1}{2}]}\right)|t_1-\frac{1}{2}| + \varphi_{p_i}^{-1}\left(K_i + \|h_i\|_{L^1[\frac{1}{2},\frac{3}{4}]}\right)|t_2-\frac{1}{2}| \right]$$
$$\le \sum_{i=1}^{2} \left[2\varphi_{p_i}^{-1}\left(K_i + \|h_i\|_{L^1[\frac{1}{4},\frac{3}{4}]}\right)|t_1-t_2| \right].$$

Conclusion is the same as Case 1 and it completes the proof of equicontinuity. □

Proof of Lemma 2.2. let us take $r=\sum_{i=1}^{2} r_i$ with

$$r_i = \max\left\{ |A_i| + \int_0^{\frac{1}{2}} \varphi_{p_i}^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau)d\tau\right) ds, |B_i| + \int_{\frac{1}{2}}^{1} \varphi_{p_i}^{-1}\left(\int_{\frac{1}{2}}^{s} h_i(\tau)d\tau\right) ds \right\},$$

and $M=\{(u,v)\in X \mid \|(u,v)\|_\infty \le r\}$. For any $(u,v)\in M$, we know $T^i(u,v)\in C[0,1]\cap C^1(0,1)$, $i=1,2$. Thus the maximal point of $|T^i(u,v)(t)|$ must be at the boundary $t=0$, $t=1$ or one extremal point $\sigma_i\in(0,1)$. If the maximal point is $t = 0$ (or $t = 1$), then we get

$$\|T^i(u,v)\|_\infty = |T^i(u,v)(0)| = |A_i| \text{ (or } \|T^i(u,v)\|_\infty = |T^i(u,v)(1)| = |B_i|).$$

If the maximal point is one extremal point $\sigma_i \in (0,1)$, then we also consider two cases $a^i \geq 0$ and $a^i < 0$, applying the same argument in the proof of Theorem 2 for case (1) in Xu and Lee [17], we have

$$\|T^i(u,v)\|_\infty$$
$$\leq \max\left\{|A_i| + \int_0^{\frac{1}{2}} \varphi_{p_i}^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau)d\tau\right) ds, |B_i| + \int_{\frac{1}{2}}^1 \varphi_{p_i}^{-1}\left(\int_{\frac{1}{2}}^s h_i(\tau)d\tau\right) ds\right\} = r_i,$$

and then

$$\|T(u,v)\|_\infty = \sum_{i=1}^2 \|T^i(u,v)\|_\infty \leq r, \text{ for } (u,v) \in M.$$

By Theorem 3.1, we get that T has a fixed point in M. *i.e.*, problem (S) has a solution in M.

Finally, we give the exact proof of Theorem 2.1, mainly using Lemma 2.2.

Proof of Theorem 2.1. Let us consider the modified problem

$$\begin{cases} \varphi_{p_1}(u')' + F_1^*(t,u,v) = 0, \\ \varphi_{p_2}(v')' + F_2^*(t,u,v) = 0, \quad t \in (0,1), \\ u(0) = A_1, u(1) = B_1, v(0) = A_2, v(1) = B_2, \end{cases} \tag{S^*}$$

where $F_i^*(t,u,v) = F_i(t, p_1(t,u,v), p_2(t,u,v))$, for $i = 1,2$, and

$$p_1(t,u,v) = \max\{\alpha_u(t), \min\{u, \beta_u(t)\}\},$$
$$p_2(t,u,v) = \max\{\alpha_v(t), \min\{v, \beta_v(t)\}\}.$$

Then $F_i^* : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the conditions in Lemma 2.2 for $i = 1,2$. By Lemma 2.2, we see that problem (S^*) has a solution (u,v). If we can show that solution (u,v) satisfies

$$(\alpha_u(t), \alpha_v(t)) \leq (u(t), v(t)) \leq (\beta_u(t), \beta_v(t)), \quad \text{for all } t \in [0,1],$$

then problem (S^*) is equivalent to problem (S) and the proof will be done. Suppose that it is not true, then $u(t) \not\leq \beta_u(t)$ or $v(t) \not\leq \beta_v(t)$ for $t \in [0,1]$. Here we assume that $u(t) \not\leq \beta_u(t)$ for $t \in [0,1]$. The other inequality can be proved by the similar argument. By the boundary values of u and β_u, there exist $T_1, T_2 \in (0,1)$ such that

$$u(t) - \beta_u(t) > 0 \text{ on } (T_1,T_2), \; u(T_1) - \beta_u(T_1) = 0 = u(T_2) - \beta_u(T_2).$$

For $t \in (T_1,T_2)$, we have

$$\begin{aligned} &-\varphi_{p_1}(u'(t))' = F_1^*(t,u(t),v(t)) = F_1(t, p_1(t,u(t),v(t)), p_2(t,u(t),v(t))) \\ &= F_1(t, \beta_u(t), p_2(t,u(t),v(t))) = F_1(t, \beta_u(t), \beta_v(t)), \end{aligned}$$

and

$$-\varphi_{p_1}(\beta_u'(t))' \geq F_1(t, \beta_u(t), \beta_v(t)).$$

Thus, we have
$$\varphi_{p_1}(u'(t))' \geq \varphi_{p_1}(\beta_u'(t))'. \tag{1}$$
Since $u - \beta_u \in C_0[T_1, T_2]$, there exists $t_0 \in (T_1, T_2)$ and $0 < \delta < T_2 - t_0$ such that $u(t_0) - \beta_u(t_0) = \max_{t\in[T_1,T_2]}\{u(t) - \beta_u(t)\}$, $u'(t_0) - \beta_u'(t_0) = 0$ and $u'(t) - \beta_u'(t) < 0$, for $t \in (t_0, t_0 + \delta)$. Integrating on both sides of (1) from t_0 to $t \in (t_0, t_0 + \delta)$, then we get
$$\varphi_{p_1}(u'(t)) - \varphi_{p_1}(u'(t_0)) \geq \varphi_{p_1}(\beta_u'(t)) - \varphi_{p_1}(\beta_u'(t_0)).$$
Since φ_{p_1} is increasing, we have
$$u'(t) \geq \beta_u'(t), \text{ for } t \in (t_0, t_0 + \delta),$$
which is a contradiction and it completes the proof.

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