Quasi-Normal Relations - a New Class of Relations

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Abstract. In this paper, concepts of quasi-normal and dually quasi-normal relations are introduced. Characterizations of these relations are obtained. In addition, particularly we show that the anti-order relation \( \not\leq (= \leq^C) \) is a (dually) quasi-normal relation if and only if the partially ordered set \((X, \leq)\) is an anti-chain.

1. Introduction and Preliminaries

The regularity of binary relations was first characterized by Zareckiı ([9],[10]). Further criteria for regularity were given by Hardy and Petrich ([3]), Markowsky ([7]), Schein ([8]) and Xu Xiao-quan and Liu Yingming ([11]) (see also [1] and [2]). The concepts of conjugative relations, dually conjugative relations and dually normal relations were introduced by Guanghao Jiang and Luoshan Xu ([4], [5]), and a characterization of normal relations was introduced and analyzed by Jiang Guanghao, Xu Luoshan, Cai Jin and Han Guiwen ([6]). In this paper, we introduce and analyze two new classes of relations in sets - class of quasi-normal relations and class of dually quasi-normal relations on sets.

Notions and notations which aren’t explicitly exposed but are used in this article, readers can find them from texts [3] and [11], for an example.

For a set \(X\), we call \(\rho\) a binary relation on \(X\), if \(\rho \subseteq X \times X\). Let \(\mathcal{B}(X)\) denote the set of all binary relations on \(X\). For \(\alpha, \beta \in \mathcal{B}(X)\), we define

\[
\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \land (y, z) \in \beta)\}.
\]

The relation \(\beta \circ \alpha\) is called the composition of \(\alpha\) and \(\beta\). It is well known that \((\mathcal{B}(X), \circ)\) is a semigroup. For a binary relation \(\alpha\) on a set \(X\), we define \(\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}\) and \(\alpha^C = (X \times X) \setminus \alpha\).

Let \(A\) and \(B\) be subsets of \(X\). For \(\alpha \in \mathcal{B}(X)\), we set

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\[ Aa = \{ y \in X : (\exists a \in A)(a, y) \in \alpha \} \text{ and } aB = \{ x \in X : (\exists b \in B)(x, b) \in \alpha \}. \]

Specially, we put \( aa \alpha \) instead of \( \{a\} \alpha \) and \( \alpha b \) instead of \( \alpha \{b\} \).

The following classes of elements in the semigroup \( \mathcal{B}(X) \) have been investigated:

- **normal** ([5]) if there exists a relation \( \beta \in \mathcal{B}(X) \) such that
  \[ \alpha = \alpha \circ \beta \circ (\alpha^C)^{-1}. \]

- **dually normal** ([4]) if there exists a relation \( \beta \in \mathcal{B}(X) \) such that
  \[ \alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha. \]

- **conjugative** ([3]) if there exists a relation \( \beta \in \mathcal{B}(X) \) such that
  \[ \alpha = \alpha^{-1} \circ \beta \circ \alpha. \]

- **dually conjugative** ([3]) if there exists a relation \( \beta \in \mathcal{B}(X) \) such that
  \[ \alpha = \alpha \circ \beta \circ \alpha^{-1}. \]

Put \( \alpha^1 = \alpha \). It is easy to see that \( (\alpha^{-1})^C = (\alpha^C)^{-1} \) holds. Previous description gives equality
\[ \alpha = (a^i)^j \circ \beta \circ (a^b)^j \]
for some \( \beta \in \mathcal{B}(X) \) where \( i, j \in \{-1, 1\} \) and \( a, b \in \{1, C\} \). We should investigate all other possibilities since some of possibilities given in the previous equation have been investigated.

Diverse descriptions of regular elements of \( \mathcal{B}(X) \) can be found in [7], [8], and [9]. For any \( \alpha \in \mathcal{B}(X) \), Zaretskiï ([9], Section 3.2) (See, also, paper [3]) introduced the following relation in his study of regular elements of \( \mathcal{B}(X) \):
\[ \alpha^+ = \{ (x, y) \in X \times X : \alpha \circ \{ (x, y) \} \circ \alpha \subseteq \alpha \}. \]
Schein in [8], Theorem 1, proved that \( \alpha^+ = (\alpha^{-1} \circ \alpha^C \circ \alpha^{-1})^C \) is the maximal element in the family of all elements \( \beta \in \mathcal{B}(X) \) such that \( \alpha \circ \beta \circ \alpha \subseteq \alpha \).

**2. Quasi-Normal and Dually Quasi-Normal Relations**

In the following definition we introduce two new classes of elements in \( \mathcal{B}(X) \).

**Definition 2.1.** (a) For relation \( \alpha \in \mathcal{B}(X) \) we say that it is a **quasi-normal** relation on \( X \) if there exists a relation \( \beta \in \mathcal{B}(X) \) such that
\[ \alpha = \alpha^C \circ \beta \circ (\alpha^C)^{-1}. \]
(b) For relation \( \alpha \in \mathcal{B}(X) \) we say that it is a dually quasi-normal relation on \( X \) if there exists a relation \( \beta \in \mathcal{B}(X) \) such that
\[
\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha^C.
\]

**Remark 2.1.** The family of quasi-normal relations is not empty. Let \( \alpha \in \mathcal{B}(X) \) be a relation such that \( \alpha^C \circ (\alpha^C)^{-1} = \text{Id}_X \). Then, we have
\[
\alpha = \text{Id}_X \circ \alpha \circ \text{Id}_X = (\alpha^C \circ (\alpha^C)^{-1}) \circ \alpha \circ (\alpha^C \circ (\alpha^C)^{-1}) = \alpha^C \circ ((\alpha^C)^{-1} \circ \alpha \circ \alpha^C) \circ (\alpha^C)^{-1} = \alpha^C \circ \beta \circ (\alpha^C)^{-1}.
\]
So, such relation \( \alpha \) is a quasi-normal relation on \( X \). Analogously, for relation \( \alpha \in \mathcal{B}(X) \) such that \( (\alpha^C)^{-1} \circ \alpha^C = \text{Id}_X \), we have
\[
\alpha = \text{Id}_X \circ \alpha \circ \text{Id}_X = (\alpha^C)^{-1} \circ \alpha \circ ((\alpha^C)^{-1} \circ \alpha^C) = (\alpha^C)^{-1} \circ (\alpha^C \circ \alpha \circ (\alpha^C)^{-1} \circ \alpha^C = (\alpha^C)^{-1} \circ \beta \circ \alpha^C.
\]
Therefore, this relation \( \alpha \) is a dually quasi-normal relation. So, the family of dually quasi-normal is not empty, either.

Particularly, for the relation \( \nabla = (\text{Id}_X)^C \), since we have
\[
(\nabla^C)^{-1} \circ \nabla^C = \text{Id}_X \circ \text{Id}_X = \text{Id}_X = \nabla^C \circ (\nabla^C)^{-1},
\]
we conclude that it is a quasi-normal relation on \( X \) and it is a dually quasi-normal relation on \( X \) as well.

In following proposition we give a connection between quasi-normal and dually quasi-normal relations.

**Proposition 2.1.** Relation \( \alpha^{-1} \) is a dually quasi-normal relation on \( X \) if and only if \( \alpha \) is a quasi-normal relation on \( X \).

**Proof.** Let \( \alpha \) be a quasi-normal relation on set \( X \). Then there exists a relation \( \beta \) on \( X \) such that \( \alpha = \alpha^C \circ \beta \circ (\alpha^C)^{-1} \). Thus,
\[
\alpha^{-1} = (\alpha^C \circ \beta \circ (\alpha^C)^{-1})^{-1} = ((\alpha^{-1})^C)^{-1} \circ \beta^{-1} \circ (\alpha^{-1})^C.
\]
So, the relation \( \alpha^{-1} \) is a dually quasi-normal if \( \alpha \) is a quasi-normal relation. The second statement we demonstrate by analogy of the previous statement.

Our second proposition is an adaptation of Schein’s concept exposed in [8], Theorem 1 (See, also, [2], Lemma 1.) for our needs.

**Theorem 2.1.** For a binary relation \( \alpha \in \mathcal{B}(X) \), relation
\[
\alpha^* = ((\alpha^C)^{-1} \circ \alpha^C \circ \alpha^C)^C
\]
is the maximal element in the family of all relation \( \beta \in \mathcal{B}(X) \) such that
\[
\alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha.
\]

**Proof.** First, remember ourself that
Theorem 2.2. For a binary relation \( u, v \) we have
\[
\beta \iff (\exists u, v \in X)((u, u) \in \alpha^C \land (u, v) \in \alpha^C \land (v, y) \in (\alpha^C)^{-1}) \iff \\
(\exists u, v \in X)((u, u) \in (\alpha^C)^{-1} \land (u, v) \in \alpha^C \land (y, v) \in (\alpha^C)^{-1}) \iff \\
(\exists u, v \in X)((u, x) \in (\alpha^C)^{-1} \land (x, y) \in \beta \land (y, v) \in \alpha^C \land (u, v) \in \alpha^C) \iff \\
(\exists u, v \in X)((u, v) \in \alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha \land (u, v) \in \alpha^C).
\]

We get a contradiction. So, must be \( \beta \subseteq \alpha^* \).

On the other hand, we should prove that
\[
\alpha^C \circ \alpha^* \circ (\alpha^C)^{-1} \subseteq \alpha.
\]

Let \((x, y) \in \alpha^C \circ \alpha^* \circ (\alpha^C)^{-1}\) be an arbitrary element. Then, there are elements \(u, v \in X\) such that \((x, u) \in (\alpha^C)^{-1}\), \((u, v) \in \alpha^*\) and \((v, y) \in \alpha^C\). So, from
\[
(x, u) \in (\alpha^C)^{-1}, \quad \neg((u, v) \in (\alpha^C)^{-1} \circ \alpha^C \circ \alpha^C), \quad (v, y) \in \alpha^C,
\]
we have \(\neg((x, y) \in \alpha^C)\). Suppose that \((x, y) \in \alpha^C\). Then, we have \((u, v) \in (\alpha^C)^{-1} \circ \alpha^C \circ \alpha^C\), which is impossible. Hence, we have \((x, y) \in \alpha\) and, therefore, \(\alpha^C \circ \alpha^* \circ (\alpha^C)^{-1} \subseteq \alpha\).

Finally, we conclude that \(\alpha^*\) is the maximal element of the family of all relations \(\beta \in \mathcal{B}(X)\) such that \(\alpha^C \circ \beta \circ (\alpha^C)^{-1} \subseteq \alpha\).

\[\square\]

We have the following proposition by dual process to previous theorem:

**Theorem 2.2.** For a binary relation \( \alpha \in \mathcal{B}(X) \), relation
\[
\alpha_* = (\alpha^C \circ \alpha^* \circ (\alpha^C)^{-1})^C
\]
is the maximal element in the family of all relation \( \beta \in \mathcal{B}(X) \) such that
\[
(\alpha^C)^{-1} \circ \beta \circ \alpha^C \subseteq \alpha.
\]

Some properties of relations \(\alpha_*\) and \(\alpha^*\) and connection between them are given in the following proposition.

**Proposition 2.2.** For relation \( \alpha \in \mathcal{B}(X) \) we have:
(a) \(\alpha^* = \{(x, y) \in X \times X : \alpha^C \circ \{(x, y)\} \circ (\alpha^C)^{-1} \subseteq \alpha\}\)
\[= \{(x, y) \in X \times X : x\alpha^C \times y\alpha^C \subseteq \alpha\}.\]
(b) \(\alpha_* = \{(x, y) \in X \times X : (\alpha^C)^{-1} \circ \{(x, y)\} \circ \alpha^C \subseteq \alpha\}\)
\[= \{(x, y) \in X \times X : \alpha^C x \times \alpha^C y \subseteq \alpha\}.\]
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(c) \((\alpha^*)^{-1} = (\alpha^{-1})_*\);
(d) \((\alpha_*)^{-1} = (\alpha^{-1})^*.\)

Proof. (a) By Theorem 2.1, it is straightforward to show that
\[
\alpha^* = \{(x, y) \in X \times X : \alpha^C \circ \{(x, y)\} \circ (\alpha^C)^{-1} \subseteq \alpha\}.
\]
Furthermore, we have
\[
(u, v) \in \alpha^C \circ \{(x, y)\} \circ (\alpha^C)^{-1} \iff (u, x) \in \alpha^C \land (y, v) \in \alpha^C \iff u \in x\alpha^C \land v \in y\alpha^C \iff (u, v) \in x\alpha^C \times y\alpha^C.
\]
(b) Also, without difficulty, we can prove that
\[
\alpha_* = \{(x, y) \in X \times X : (\alpha^C)^{-1} \circ \{(x, y)\} \circ \alpha^C \subseteq \alpha\} = \{(x, y) \in X \times X : \alpha^C x \times \alpha^C y \subseteq \alpha\}
\]
holds.
(c) \((\alpha^*)^{-1} = \{(x, y) \in X \times X : x\alpha^C \times y\alpha^C \subseteq \alpha\}^{-1} = \{(y, x) \in X \times X : x\alpha^C \times y\alpha^C \subseteq \alpha\} = \{(x, y) \in X \times X : y\alpha^C \times x\alpha^C \subseteq \alpha\} = \{(x, y) \in X \times X : (\alpha^C)^{-1} x \times (\alpha^C)^{-1} y \subseteq \alpha^{-1}\} = \{(x, y) \in X \times X : (\alpha^{-1})^C x \times (\alpha^{-1})^C y \subseteq \alpha^{-1}\} = (\alpha^{-1})_*.
(d) The proof of this proposition can be given analogically by previous assertion (c).

In the following proposition we give an essential characterization of dually quasi-normal relations. It is our adaptation of concept exposed in [3], Theorem 7.2.

Theorem 2.3. For a binary relation \(\alpha\) on a set \(X\), the following conditions are equivalent:

(1) \(\alpha\) is a dually quasi-normal relation.
(2) For all \(x, y \in X\), if \((x, y) \in \alpha\), there exists \(u, v \in X\) such that:
   (a) \((x, u) \in \alpha^C \land (y, v) \in \alpha^C\)
   (b) \((\forall s, t \in X)((s, u) \in \alpha^C \land (t, v) \in \alpha^C \implies (s, t) \in \alpha)\).
(3) \(\alpha \subseteq (\alpha^{-1})^{-1} \circ \alpha_* \circ \alpha^C\).

Proof. (1) \(\implies\) (2). Let \(\alpha\) be a dually quasi-normal relation, i.e. let there exist a relation \(\beta\) such that \(\alpha = (\alpha^{-1})^{-1} \circ \beta \circ \alpha^C\). Let \((x, y) \in \alpha\). Then there exist elements \(u, v \in X\) such that
\[
(x, u) \in \alpha^C, (u, v) \in \beta, (v, y) \in (\alpha^C)^{-1}.
\]
From previous, it follows that there exist elements \( u, v \in X \) such that \( (x, u) \in \alpha^C \) and \( (y, v) \in \alpha^C \). This proves condition (a). Now we check the condition (b). Let \( s, t \in X \) be arbitrary elements such that \( (s, u) \in \alpha^C \) and \( (t, v) \in \alpha^C \). Now from \( (s, u) \in \alpha^C \), \( (u, v) \in \beta \) and \( (v, t) \in (\alpha^C)^{-1} \) follows \( (s, t) \in (\alpha^C)^{-1} \circ \beta \circ \alpha^C = \alpha \).

(2) \( \implies \) (1). Let us define a binary relation

\[
\alpha' = \{(u, v) \in X \times X : (\forall s, t \in X)((s, u) \in \alpha^C \wedge (t, v) \in \alpha^C \implies (s, t) \in \alpha)\}
\]

and show that \( \alpha^C \circ \alpha' \circ (\alpha^C)^{-1} = \alpha \) is valid. Let \((x, y) \in \alpha\). Then there exist elements \( u, v \in X \) such that the conditions (a) and (b) hold. We have \((u, v) \in \alpha'\) by definition of relation \( \alpha'\).

Further, from \((x, u) \in \alpha^C\), \((u, v) \in \alpha'\) and \((v, y) \in (\alpha^C)^{-1}\) follows \((x, y) \in (\alpha^C)^{-1} \circ \alpha' \circ \alpha^C\). Hence, we have \( \alpha \subseteq (\alpha^C)^{-1} \circ \alpha' \circ \alpha^C \). Contrary, let \((x, y) \in (\alpha^C)^{-1} \circ \alpha' \circ \alpha^C\). There exist elements \( u, v \in X \) such that \((x, u) \in \alpha^C, (u, v) \in \alpha'\) and \((v, y) \in \alpha^C\). From previous we have \((x, u) \in \alpha^C\) and \((y, v) \in \alpha^C\). Hence, by definition of relation \( \alpha'\), follows \((x, y) \in \alpha\) since \((u, v) \in \alpha'\).

Therefore, \((\alpha^C)^{-1} \circ \alpha' \circ \alpha^C \subseteq \alpha\). So, the relation \( \alpha \) is a dually quasi-normal relation on \( X \) since there exists a relation \( \alpha' \) such that \((\alpha^C)^{-1} \circ \alpha' \circ \alpha^C = \alpha\).

(1) \( \iff \) (3). Let \( \alpha \) be a dually quasi-normal relation. Then there is a relation \( \beta \) such that \( \alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha^C \). Since \( \alpha_* = \bigcup \{ \beta \in \beta(X) : (\alpha^C)^{-1} \circ \beta \circ \alpha^C \subseteq \alpha \} \), we have \( \beta \subseteq \alpha_* \) and \( \alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha^C \subseteq (\alpha^C)^{-1} \circ \alpha_* \circ \alpha^C \). Contrary, let \( \alpha \subseteq (\alpha^C)^{-1} \circ \alpha_* \circ \alpha^C \), for a relation \( \alpha \). Then, we have \( \alpha \subseteq (\alpha^C)^{-1} \circ \alpha_* \circ \alpha^C \subseteq \alpha \).

So, the relation \( \alpha \) is a dually quasi-normal relation on set \( X \).

**Corollary 2.1.** Let \((X, \leq)\) be a poset. Relation \( \leq^C \) is a dually quasi-normal relation on \( X \) if and only if \((X, \leq)\) is an anti-chain.

**Proof.** Let \( \leq^C \) be a dually quasi-normal relation on set \( X \), and let \( x, y \in X \) be elements such that \( x \leq^C y \). Then, by previous theorem, there exist elements \( u, v \in X \) such that:

(a) \( x \leq u \wedge y \leq v \);
(b) \( (\forall s, t \in X)((s \leq u \wedge t \leq v) \implies s \leq^C t) \).

Let \( z \) be an arbitrary element and if we put \( z = s = t \) in formula (b) we have

\[
(z \leq u \wedge z \leq v) \implies z \leq^C z,
\]

which is a contradiction. Hence \( \neg(z \leq u \wedge z \leq v) \), it follows

\[
z \leq^C u \lor z \leq^C v.
\]

However, one can observe that these conditions are satisfied only for the partially ordered sets which are anti-chain. Indeed, let \( X \) have comparable elements, for example, \( y < x \). Then, \( x \leq^C y \). If for elements \( u, v \in X \) the following \( u \geq x \) and \( v \geq y \) hold, then \( y \leq u \wedge y \leq v \). Therefore, there exists \( z \) in \( X \) such that \( z \leq u \wedge z \leq v \) (namely, \( z = y \)). It contradicts to the condition (b). So, there is no different comparable elements in \( X \).
Contrary, let $x, y \in X$ be arbitrary elements such that $x \leq^C y$. There exist elements $u, v \in X$ such that

(a') $x \leq u \land y \leq v$ and

(b') $(\forall z \in X)(z \leq^C u \lor z \leq^C z)$.

Let $s, t \in X$ be arbitrary elements such that $s \leq u$ and $t \leq v$. At the second hand, from proposition (b'), for $z = s$, in this case, $s \leq^C u \lor s \leq^C v$, since the second option $s \leq^C u$ is impossible, the following is $s \leq^C v$. Thus, we have $s \leq^C t \lor t \leq^C v$. Since the second option is also impossible, we have to have $s \leq^C t$. So, elements $u$ and $v$ satisfy condition (b) of Theorem 3.1. So, the relation $\leq^C$ is a dually quasi-normal relation.

\textbf{Remark 2.2.} Corollary 2.1 can be also verified as follows:

Assume first that the relation $\leq^C$ is quasi-normal. It means $\leq^C = \leq^{-1} \circ \beta \circ \leq$ for a relation $\beta$. If $x, y \in X$ with $x < y$ then $(y, x) \in \leq^C$, and so the previous equation implies $x \leq c$ and $d \leq y$ for some $c, d \in X$ such that $(c, d) \in \beta$. Hence $x \leq c \beta d \leq^{-1} x$ follows since $d \leq^{-1} y$ and $x < y$. By the above equation, we deduce $(x, x) \in \leq^C$ which contradicts reflexivity of $\leq$. This shows that there are no elements $x, y \in X$ with $x < y$, and so $(X, \leq)$ is an anti-chain.

Conversely, if $(X, \leq)$ is an anti-chain, $\leq$ is the equality relation and then we have $\leq^{-1} \circ \beta \circ \leq \leq \beta$ for any relation $\beta$. Hence, choosing $\beta$ to be $\leq^C$, we obtain that $\leq^C$ is quasi-normal.

Assertions, analogous to previous theorem and corollary which regards on quasi-normal relations, are presented below.

\textbf{Theorem 2.4.} For a binary relation $\alpha$ on a set $X$, the following conditions are equivalent:

1. $\alpha$ is a quasi-normal relation.
2. For all $x, y \in X$, if $(x, y) \in \alpha$, there exists $u, v \in X$ such that:
   a. $(u, x) \in \alpha^C \land (v, y) \in \alpha^C$
   b. $(\forall s, t \in X)((u, s) \in \alpha^C \land (v, t) \in \alpha^C \Rightarrow (s, t) \in \alpha)$.
3. $\alpha \subseteq \alpha^C \circ \alpha^* \circ (\alpha^C)^{-1}$.

\textbf{Corollary 2.3.} Let $(X, \leq)$ be a poset. Relation $\leq^C$ is a quasi-normal relation on $X$ if and only if $(X, \leq)$ is an anti-chain.

\textbf{References}


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