KYUNGPOOK Math. J. 55(2015), 625-639
http://dx.doi.org/10.5666/KMJ.2015.55.3.625
pISSN 1225-6951 eISSN 0454-8124

Differential Subordinations and Superordinations of Certain Meromorphic Functions associated with an Integral Operator

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Abstract. Differential subordinations and superordinations results are obtained for certain meromorphic functions in the punctured unit disk which are associated with an integral operator. These results are obtained by investigating appropriate classes of a dmissible functions. Sandwich-type results are also obtained.

1. Introduction

Let $H(U)$ denotes the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and Let $H[a, n]$ denotes the subclass of the functions $f \in H(U)$ of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$ $(a \in \mathbb{C})$; with
$H[1, 1] \equiv H$. If $f,\ g \in H(U)$, we say that f is subordinate to g, or g is superordinate to f, if there exists a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$. In such case we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. If $g(z)$ is univalent in U, then the following equivalence relationship holds true.

$$f(z) \prec g(z)(z \in U) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let Σ denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1.1}$$

* Corresponding Author.
Received October 23, 2013; accepted April 22, 2014.
2010 Mathematics Subject Classification: 30C45.
Key words and phrases: Meromorphic functions, Hadamard product, Subordination, Superordination, Sandwich theorems, Integral operator.

which are analytic in the punctured disk $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$ $= U \backslash \{0\}$, with a simple pole at the origin.
Let $f,\ g \in \Sigma$, where f given by (1.1) and g is given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k. \tag{1.2}$$

The Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k := (g * f)(z). \tag{1.3}$$

Motivated essentially by Jung, et al. [7] on the normalized analytic functions, Lashin [10] defined the following integral operators

$$Q_\beta^\alpha : \Sigma \to \Sigma :$$

$$Q_\beta^\alpha = Q_\beta^\alpha f(z) = \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z t^\beta \left(1 - \frac{t}{z}\right)^{\alpha-1} f(t)dt \ \ (\alpha, \beta > 0; z \in U^*). \tag{1.4}$$

where $\Gamma(\alpha)$ is the familiar Gamma function.
Using the integral representation of the Gamma and Beta functions, it can be shown that

Remark 1.1. For $f(z) \in \Sigma$ given by (1.1), we have

$$Q_\beta^\alpha f(z) = \frac{1}{z} + \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta+\alpha+1)} a_k z^k \ \ (\alpha > 0, \beta > 0; z \in U^*). \tag{1.5}$$

By (1.5) we see that

$$J_\beta f(z) = Q_\beta^1 f(z) = \frac{\beta}{z^{\beta+1}} \int_0^z t^\beta f(t)dt \quad (\beta > 0;\ z \in U^*), \tag{1.6}$$

$$z\left(Q_\beta^\alpha f(z)\right)' = (\beta+\alpha-1) Q_\beta^{\alpha-1} f(z) - (\beta+\alpha) Q_\beta^\alpha f(z) (\alpha > 1, \beta > 0). \tag{1.7}$$

To prove our results, we need the following definitions and lemmas.
Let Q be the set of all functions $q(z)$ that are analytic and injective on $\overline{U} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. Further let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$, and $Q(1) \equiv Q_1$.

Definition 1.1.([12, Definition 2.3a, p.27]) Let Ω be a set in $\mathbb{C}, q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of these functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever $r = q(\zeta), s = k\zeta q'(\zeta)$, and

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \geq k\mathrm{Re}\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U,\ \zeta \in \partial U \backslash E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In particular when $q(z) = M\frac{Mz+a}{M+\bar{a}z}$, with $M > 0$ and $|a| < M$, then $q(U) = U_M = \{w : |w| < M\}, q(0) = a,\ E(q) = \varphi$ and $q \in Q(a)$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.2.([13, Definition 3, p.817]) Let Ω be a set in $\mathbb{C},\ q \in H[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of these functions $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \zeta) \in \Omega$ whenever $r = q(z), s = zq'(z)/m$, and

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \leq \frac{1}{m}\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\},$$

where $z \in U,\ \zeta \in \partial U$ and $m \geq n \geq 1$. In particular, We write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$. For the obove two classes of admissible functions, Miller and Mocanu proved the following lemmas.

Lemma 1.1.([12, Theorem 2.3b, p.28]) *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\textit{satisfies}$$

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega$$

then $p(z) \prec q(z)$.

Lemma 1.2.([13, Theorem 1, p.818]) *Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p(z) \in Q(a)$ and*

$$\psi(p(z), zp'(z), z^2 p''(z); z),$$

is univalent in U then

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in U\right\},$$

implies $q(z) \prec p(z)$.

In the present paper, the differential subordination result of Miller and Mocanu [12, Theorem 2.3b, p.28] is extended for functions associated with the integral operator Q^{α}_{β}, and we obtain certain other related results. A similar problem for analytic functions was studied by Aghalary et al. [1], Ali et al. [3], Aouf [4], Aouf et al. [5], Aouf and Seoudy [6], and Kim and Srivastava [9]. Also Ali et al. [2], Liu and Owa [11] and Kamali [8] investigated a subordination problem for meromrphic functions. Additionally, the corresponding superordination problem is investigated, and several differential Sandwich-type results are obtained.

2. Subordination Results Involving the Operator Q_β^α

Unless otherwise mentioned, we assume throughout this paper that $\alpha > 1,\ \beta > 0$

Definition 2.1. Let Ω be a set in $\mathbb{C}$, $q(z) \in Q_1 \cap H$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfies the admissibility condition

$$\varphi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta),\ v = \frac{k\zeta q'(\zeta) + (\beta+\alpha)q(\zeta)}{\beta+\alpha},$$

$$\text{Re}\left\{\frac{w-u}{v-u} - \frac{2\beta+2\alpha-1}{\beta+\alpha-1}\right\} \geq \frac{k}{\beta+\alpha-1}\text{Re}\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U,\ \zeta \in \partial U \backslash E(q)$ and $k \geq 1$.

Theorem 2.1. *Let $\varphi \in \Phi_H[\Omega, q]$. If $f(z) \in \Sigma$ satisfies*

$$\left\{\varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^\alpha f(z), zQ_\beta^{\alpha-1}f(z); z\right) : z \in U\right\} \in \Omega \tag{2.1}$$

then

$$zQ_\beta^{\alpha+1}f(z) \prec q(z).$$

Proof. Define the function $p(z)$in U by

$$p(z) := zQ_\beta^{\alpha+1}f(z), z \in U^*,\ p(0) = 1. \tag{2.2}$$

In view of the relation (1.7), it follows from (2.2) that

$$zQ_\beta^\alpha f(z) = \frac{zp'(z) + (\beta+\alpha)p(z)}{\beta+\alpha}. \tag{2.3}$$

Further compuations show that

$$zQ_\beta^{\alpha-1}f(z) = \frac{z^2p''(z) + 2(\beta+\alpha)zp'(z) + (\beta+\alpha)(\beta+\alpha-1)p(z)}{(\beta+\alpha)(\beta+\alpha-1)}. \tag{2.4}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r,s,t) = r,\ v(r,s,t) = \tfrac{s+(\beta+\alpha)r}{\beta+\alpha},\ w(r,s,t) = \tfrac{t+2(\beta+\alpha)s+(\beta+\alpha)(\beta+\alpha-1)r}{(\beta+\alpha)(\beta+\alpha-1)}. \tag{2.5}$$

Let

$$\psi(r,s,t;z) := \varphi(u,v,w;z) = \varphi\left(r, \tfrac{s+(\beta+\alpha)r}{\beta+\alpha}, \tfrac{t+2(\beta+\alpha)s+(\beta+\alpha)(\beta+\alpha-1)r}{(\beta+\alpha)(\beta+\alpha-1)}; z\right). \tag{2.6}$$

The proof will make use of Lemma 1.1. Using equations (2.2), (2.3) and (2.4), it follows from (2.6) that

$$\psi(p(z), zp'(z), z^2p''(z); z) = \varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right). \tag{2.7}$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\varphi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = (\beta+\alpha-1)\left(\frac{w-u}{v-u} - \frac{2\beta+2\alpha-1}{\beta+\alpha-1}\right),$$

and hence $\psi \in \Psi[\Omega, q]$. By lemma 1.1,

$$p(z) \prec q(z) \quad \text{or} \quad zQ_\beta^{\alpha+1}f(z) \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connceted domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_H[h(U), q]$ is written as $\Phi_H[h, q]$.
The following result is an immediate consequence of Theorem 2.1. □

Theorem 2.2. *Let $\varphi \in \Phi_H[h, q]$ with $q(0) = 1$. If $f(z) \in \Sigma$ satisfies*

$$\varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right) \prec h(z) \ (z \in U), \tag{2.8}$$

then

$$zQ_\beta^{\alpha+1}f(z) \prec q(z).$$

Our next result is an extension of theorem 2.1 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 2.1. *Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in U, $q(0) = 1$. Let $\varphi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If $f(z) \in \Sigma$ satisfies*

$$\varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right) \in \Omega \qquad (z \in U),$$

then

$$zQ_\beta^{\alpha+1}f(z) \prec q(z).$$

Proof. Theorem 2.1 yields $zQ_\beta^{\alpha+1}f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. □

Theorem 2.3. *Let $h(z)$ and $q(z)$ be univalent in U, with $q(0) = 1$. and $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfy one of the following conditions :*
(1) $\varphi \in \Phi_H[h, q_\rho]$, for some $\rho \in (0, 1)$, or
(2) there exists $\rho_0 \in (0, 1)$ such that $\varphi \in \Phi_H[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$. If $f(z) \in \Sigma$ satisfies (2.8), then

$$zQ_\beta^{\alpha+1} f(z) \prec q(z).$$

Proof. The proof is similar to [12, Theorem 2.3d, p.30] and is therefore. omitted. The next theorem yields the best dominant of the differential subordination (2.8)□

Theorem 2.4. *Let $h(z)$ be univalent in U, and $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi\left(p(z), \tfrac{zp'(z)+(\beta+\alpha)p(z)}{\beta+\alpha}, \tfrac{z^2p''(z)+2(\beta+\alpha)zp'(z)+(\beta+\alpha)(\beta+\alpha-1)p(z)}{(\beta+\alpha)(\beta+\alpha-1)}; z\right) = h(z) \tag{2.9}$$

has a solution $q(z)$ with $q(0) = 1$ and one of the following conditions is satisfied:
(1) $q(z) \in Q_1$ and $\varphi \in \Phi_H[h, q]$,
(2) $q(z)$ is univalent in U and $\varphi \in \Phi_H[h, q_\rho]$, for some $\rho \in (0, 1)$, or
(3) $q(z)$ is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\varphi \in \Phi_H[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.
If $f(z) \in \Sigma$ satisfies (2.8), then

$$zQ_\beta^{\alpha+1} f(z) \prec q(z).$$

and $q(z)$ is the best dominant.

Proof. Following the same argument in [12, Theorem 2.3e, p.31], we deduce that $q(z)$ is a dominant from Theorems 2.2 and 2.3. Since $q(z)$ satisfies (2.9) it is also a solution of (2.8) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. □

In the particular case $q(z) = 1 + Mz,\ M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_H[\Omega, q]$ denoted by $\Phi_H[\Omega, M]$ can be expressed in the following form:

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ such that the admissibility condition

$$\varphi\left(1 + Me^{i\theta}, 1 + (\tfrac{k+\beta+\alpha}{\beta+\alpha})Me^{i\theta}, 1 + \tfrac{L+[2k(\beta+\alpha)+(\beta+\alpha)(\beta+\alpha-1)]Me^{i\theta}}{(\beta+\alpha)(\beta+\alpha-1)}; z\right) \notin \Omega \tag{2.10}$$

whenever $z \in U,\ \theta \in R,\ \mathrm{Re}\left(Le^{-i\theta}\right) \ge k(k-1)M$ for all real θ and $k \ge 1$.

Corollary 2.2. *Let $\varphi \in \Phi_H[\Omega, M]$. If $f(z) \in \Sigma$ satisfies*

$$\varphi\left(zQ_\beta^{\alpha+1} f(z), zQ_\beta^{\alpha} f(z), zQ_\beta^{\alpha-1} f(z); z\right) \in \Omega \qquad (z \in U),$$

then

$$\left|zQ_{\beta}^{\alpha+1}f(z)-1\right|<M.$$

In the special case $\Omega=q(U)=\{w:|w-1|<M\}$, the class $\Phi_H[\Omega,M]$ is simply denoted by $\Phi_H[M]$. Corollary 2.2, can be written as:

Corollary 2.3. *Let $\varphi\in\Phi_H[M]$. If $f(z)\in\Sigma$ satisfies*

$$\left|\varphi\left(zQ_{\beta}^{\alpha+1}f(z),zQ_{\beta}^{\alpha}f(z),zQ_{\beta}^{\alpha-1}f(z);z\right)-1\right|<M,$$

then

$$\left|zQ_{\beta}^{\alpha+1}f(z)-1\right|<M.$$

Corollary 2.4. *If $M>0$ and $f(z)\in\Sigma$ satisfies*

$$\left|zQ_{\beta}^{\alpha+1}f(z)-zQ_{\beta}^{\alpha}f(z)\right|<\frac{M}{\beta+\alpha},$$

then

$$\left|zQ_{\beta}^{\alpha+1}f(z)-1\right|<M. \tag{2.11}$$

Proof. The proof follows from Corollary 2.2 by taking $\varphi(u,v,w;z)=v-u$ and $\Omega=h(\mathbb{U})$, where $h(z)=\frac{Mz}{\beta+\alpha}$, $M>0$. To use Corollary 2.2, we need to show that $\varphi\in\Phi_H[\Omega,M]$, that is the admissible condition 2.10 is satisfied. This follows since

$$\begin{aligned}&\left|\varphi\left(1+Me^{i\theta},1+(\tfrac{k+\beta+\alpha}{\beta+\alpha})Me^{i\theta},1+\tfrac{L+[2k(\beta+\alpha)+(\beta+\alpha)(\beta+\alpha-1)]Me^{i\theta}}{(\beta+\alpha)(\beta+\alpha-1)}\right);z)\right|\\ =\;& \frac{kM}{\beta+\alpha}\geq\frac{M}{\beta+\alpha},\end{aligned}$$

where $z\in\mathbb{U}$, $\theta\in\mathbb{R}$, and $k\geq 1$. Hence by Corollary 2.2, we deduce the required result Theorem 2.4 shows that the result is sharp. The differential equation $\frac{zq'(z)}{\beta+\alpha}=\frac{M}{\beta+\alpha}z$ $(\alpha,\beta>0)$ has a univalent solution $q(z)=1+Mz$. It follows from Theorem 2.4 that $q(z)=1+Mz$ is the best dominant. □

Definition 2.3. Let Ω be a set in $\mathbb{C}$ and $q(z)\in Q_1\cap H$. The class of admissible functions $\Phi_{H,1}[\Omega,q]$ consists of those functions $\varphi:\mathbb{C}^3\times U\to\mathbb{C}$
that satisfy the admissibility condition

$$\varphi(u,v,w;z)\notin\Omega,$$

whenever

$$u=q(\zeta),\quad v=\frac{-1+(\beta+\alpha+1)q(\zeta)+\frac{k\zeta q'(\zeta)}{q(\zeta)}}{\beta+\alpha},$$

$$\mathrm{Re}\left\{\frac{v[(\beta+\alpha-1)(w-v)-v+1]}{(\beta+\alpha)v-(\beta+\alpha+1)u+1}+\frac{(\beta+\alpha)v-2(\beta+\alpha+1)u+1}{\beta+\alpha}\right\}\geq\frac{k}{\beta+\alpha}\mathrm{Re}\left\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\},$$

where

$$z\in U,\ \zeta\in\partial U\backslash E(q)\ \text{ and }\ k\geq 1.$$

Theorem 2.5. *Let* $\varphi\in\Phi_{H,1}[\Omega,q]$. *If* $f(z)\in\Sigma$ *satisfies*

$$\left\{\varphi\left(\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)},\frac{Q_\beta^{\alpha}f(z)}{Q_\beta^{\alpha+1}f(z)},\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)};z\right):z\in U\right\}\subset\Omega \tag{2.12}$$

then

$$\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}\prec q(z).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z):=\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}. \tag{2.13}$$

Then

$$\frac{zp'(z)}{p(z)}=\frac{z\left(Q_\beta^{\alpha+1}f(z)\right)'}{Q_\beta^{\alpha+1}f(z)}-\frac{z\left(Q_\beta^{\alpha+2}f(z)\right)'}{Q_\beta^{\alpha+2}f(z)}. \tag{2.14}$$

In view of the relation (1.7), it follows from (2.14) that

$$(\beta+\alpha)\frac{Q_\beta^{\alpha}f(z)}{Q_\beta^{\alpha+1}f(z)}=\frac{zp'(z)}{p(z)}+(\beta+\alpha+1)p(z)-1. \tag{2.15}$$

Differentiating logarthmically (2.15), further computations show that

$$\frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}=\frac{1}{(\beta+\alpha-1)}\left[\frac{zp'(z)}{p(z)}+(\beta+\alpha+1)p(z)-2\right]+$$

$$\frac{\frac{1}{(\beta+\alpha-1)}\left[(\beta+\alpha+1)zp'(z)+\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{z^2p''(z)}{p(z)}\right]}{\frac{zp'(z)}{p(z)}+(\beta+\alpha+1)p(z)-1}. \tag{2.16}$$

Define the transformations $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r,s,t)=r,\ v=\frac{-1+(\beta+\alpha+1)r+\frac{s}{r}}{\beta+\alpha},$$

$$w(r,s,t) = \frac{1}{\beta+\alpha-1}\left[\frac{s}{r}+(\beta+\alpha+1)r-2\right] + \frac{\frac{1}{(\beta+\alpha-1)}\left[\beta+\alpha+1)s+\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{t}{r}\right]}{\frac{s}{r}+(\beta+\alpha+1)r-1}. \tag{2.17}$$

Let

$$\psi(r;s;t;z) = \varphi(u,v,w;z)$$

$$= \varphi\left(r, \frac{-1+(\beta+\alpha+1)r+\frac{s}{r}}{\beta+\alpha}, \frac{1}{\beta+\alpha-1}\left[\frac{s}{r}+(\beta+\alpha+1)r-2\right] + \frac{\frac{1}{(\beta+\alpha-1)}\left[\beta+\alpha+1)s+\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{t}{r}\right]}{\frac{s}{r}+(\beta+\alpha+1)r-1}\right). \tag{2.18}$$

The proof will make use of lemma 1.1. Using equations (2.13), (2.15) and (2.16), it follows from (2.18) that

$$\psi(p(z), zp'(z), z^2p''(z); z) = \varphi\left(\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}, \frac{Q_\beta^{\alpha}f(z)}{Q_\beta^{\alpha+1}f(z)}, \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}; z\right). \tag{2.19}$$

Hence (2.12) implies

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\varphi \in \Phi_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that.

$$\frac{t}{s}+1 = (\beta+\alpha)\left(\frac{v\left[(\beta+\alpha-1)(w-v)-v+1\right]}{(\beta+\alpha)v-(\beta+\alpha+1)u+1} + \frac{(\beta+\alpha)v-2(\beta+\alpha+1)u+1}{\beta+\alpha}\right),$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.1, $p(z) \prec q(z)$ or

$$\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)} \prec q(z) \quad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, with $\Omega = h(U)$, for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_{H,1}[h(U), q]$ is written as $\Phi_{H,1}[h, q]$. □

The following result is an immediate consequence of Theorem (2.5).

Theorem 2.6. *Let $\varphi \in \Phi_{H,1}[h, q]$ with $q(0) = 1$. If $f(z) \in \Sigma$ satisfies*

$$\varphi\left(\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}, \frac{Q_\beta^{\alpha}f(z)}{Q_\beta^{\alpha+1}f(z)}, \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}; z\right) \prec h(z) \ (z \in U), \tag{2.20}$$

then

$$\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)} \prec q(z).$$

In the particular case $q(z) = 1 + Mz, M > 0$, the class of admissible functions $\Phi_{H,1}[\Omega, q]$ becomes the class $\Phi_{H,1}[\Omega, M]$.

Definition 2.4. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition

$$\varphi\left(1 + Me^{i\theta}, 1 + \frac{1}{\beta+\alpha}\left[\frac{(\beta+\alpha+1)(1+Me^{i\theta})+k)}{1+Me^{i\theta}}\right]Me^{i\theta},\right.$$

$$\frac{1}{(\beta+\alpha-1}\left[\frac{kMe^{i\theta}}{1+Me^{i\theta}} + (\beta+\alpha+1)(1+Me^{i\theta}) - 2\right]$$

(2.21)

$$\left. + \frac{(M+e^{-i\theta})\left\{kM\left[(\beta+\alpha+1)(1+Me^{i\theta})+1\right]Le^{-i\theta}\right\} - k^2M^2}{(\beta+\alpha-1)(M+e^{-i\theta})\left\{kM + e^{-i\theta}(1+Me^{i\theta})\left[(\beta+\alpha+1)(1+Me^{i\theta})-1\right]\right\}}; z\right) \notin \Omega$$

whenever $z \in U$, $\mathrm{Re}(Le^{-i\theta}) \geq kM(k-1)$ for all real θ and $k \geq 1$

Corollary 2.5. *Let* $\varphi \in \Phi_{H,1}[\Omega, M]$. *If* $f(z) \in \Sigma$ *satisfies*

$$\varphi\left(\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}, \frac{Q_\beta^{\alpha}f(z)}{Q_\beta^{\alpha+1}f(z)}, \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}; z\right) \in \Omega \ (z \in U),$$

then

$$\left|\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)} - 1\right| < M.$$

In the special case $\Omega = q(U) = \{w : |w-1| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is a simply denoted by $\Phi_{H,1}[M]$, and Corollary 2.5 takes the following form:

Corollary 2.6. *Let* $\varphi \in \Phi_{H,1}[M]$. *If* $f(z) \in \Sigma$ *satisfies*

$$\left|\varphi\left(\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}, \frac{Q_\beta^{\alpha}f(z)}{Q_\beta^{\alpha+1}f(z)}, \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}; z\right) - 1\right| < M \ (z \in U),$$

then

$$\left|\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)} - 1\right| < M.$$

3. Superordination Results Involving the Operator Q_β^α

The dual problem of differential subordination, that is, differential superordination of the operator Q_β^α is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q(z) \in H$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi_H^{'}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\varphi(u, v, w; \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{m(\beta+\alpha)q(z) + zq'(z)}{m(\beta+\alpha)},$$

$$\mathrm{Re}\left\{\frac{w-u}{v-u} - \frac{2\beta+2\alpha-1}{\beta+\alpha-1}\right\} \leq \frac{1}{m(\beta+\alpha-1)}\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\},$$

where $z \in U, \zeta \in \partial U$ and $m \geq 1$.

Theorem 3.1. *Let* $\varphi \in \Phi'_H[\Omega, q]$. *If* $f(z) \in \Sigma$, $zQ_\beta^{\alpha+1}f(z) \in Q_1$ *and*

$$\varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right)$$

is univalent in U, *then*

$$\Omega \subset \left\{\varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right) : z \in U\right\} \tag{3.1}$$

implies

$$q(z) \prec zQ_\beta^{\alpha+1}f(z).$$

Proof. Let $p(z)$ defined by (2.2) and $\psi(z)$ defined by (2.6). Since $\varphi \in \Phi'_H[\Omega, q]$, from (2.7) and (3.1) we have

$$\Omega \subset \left\{\psi(p(z).zp'(z), z^2p''(z); z) : z \in U\right\}.$$

From (2.6), we see that the admissibility condition for $\varphi \in \Phi'_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by lemma 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec zQ_\beta^{\alpha+1}f(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. Then the class $\Phi'_H[h(U), q]$ is written as $\Phi'_{H,1}[h, q]$. Proceeding similarly, as in the previous section, the following result is an immediate consequence of Theorem 3.1. □

Theorem 3.2. *Let* $q(z) \in H$, $h(z)$ *is analytic on* U *and* $\varphi \in \Phi'_H[h,q]$. *If* $f(z) \in \Sigma$, $zQ_\beta^{\alpha+1}f(z) \in Q_1$ *and* $\varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right)$ *is univalent in* U, *then*

$$h(z) \prec \varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right) \ (z \in U), \tag{3.2}$$

implies

$$q(z) \prec zQ_\beta^{\alpha+1}f(z).$$

Theorem 3.1 and Theorem 3.2 can only be used to obtain subordinations of differential superordination of the form (3.1) or (3.2).
The following theorem proves the existence of the best subordinant of (3.2) for certain φ.

Theorem 3.3. *Let* $h(z)$ *be analytic in* U, *and* $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$\varphi\left(p(z), \frac{zp'(z)+(\beta+\alpha)p(z)}{\beta+\alpha}, \frac{z^2p''(z)+2(\beta+\alpha)zp'(z)+(\beta+\alpha)(\beta+\alpha-1)p(z)}{(\beta+\alpha)(\beta+\alpha-1)}; z\right) = h(z) \tag{3.3}$$

has asolution $q(z) \in Q_1$ *if* $\varphi \in \Phi'_H[h,q]$, $f(z) \in \Sigma$, $zQ_\beta^{\alpha+1}f(z) \in Q_1$ *and*

$$\varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right)$$

is univalent in U *then*

$$h(z) \prec \varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right)$$

implies

$$q(z) \prec zQ_\beta^{\alpha+1}f(z),$$

and $q(z)$ *is the best subordinant.*

Proof. the proof is similiar to the proof of Theorem 2.4 and is therefore omitted.
Combining Theorems 2.2 and 3.2, we obtain the following sandwich Theorem. □

Corollary 3.1. *Let* $h_1(z)$ *and* $g_1(z)$ *be analytic functions in* U, $h_2(z)$ *be univalent in* U, $q_2(z) \in Q_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\varphi \in \Phi_H[h_2,q_2] \cap \Phi'_H[h_1,q_1]$. *If* $f(z) \in \Sigma$, $zQ_\beta^{\alpha+1}f(z) \in H \cap Q_1$ *and*

$$\varphi\left(zQ_\beta^{\alpha+1}f(z), zQ_\beta^{\alpha}f(z), zQ_\beta^{\alpha-1}f(z); z\right)$$

is univalent in U, *then*

$$h_1(z) \prec \varphi\left(zQ_\beta^{\alpha+1}f(z),\ zQ_\beta^{\alpha}f(z),\ zQ_\beta^{\alpha-1}f(z); z\right) \prec h_2(z) \quad (z \in U),$$

implies

$$q_1(z) \prec zQ_\beta^{\alpha+1}f(z) \prec q_2(z).$$

Definition 3.2. Let Ω be a set in $\mathbb{C}$ and $q(z) \in H$. with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\varphi(u, v, w; \zeta) \in \Omega,$$

whenever

$$u = q(z),\ v = \frac{-1 + (\beta + \alpha + 1)q(z) + \frac{zq'(z)}{mq(z)}}{\beta + \alpha},$$

$$\mathfrak{Re}\left\{\frac{v[(\beta+\alpha-1)(w-v)-v+1]}{(\beta+\alpha)v-(\beta+\alpha+1)u+1} + \frac{(\beta+\alpha)v-2(\beta+\alpha+1)u+1}{\beta+\alpha}\right\} \leq \frac{1}{m(\beta+\alpha)}\mathfrak{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\},$$

where

$$z \in U, \zeta \in \partial U \quad \text{and} \quad m \geq 1.$$

Now will give the dual result of theorem 2.5 for differential superordination

Theorem 3.4. *Let* $\varphi \in \Phi'_{H,1}[\Omega, q]$. *If* $f(z) \in \Sigma$, $\dfrac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)} \in Q_1$ *and*

$$\varphi\left(\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}, \frac{Q_\beta^{\alpha}f(z)}{Q_\beta^{\alpha+1}f(z)}, \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}; z\right)$$

is univalent in U *then*

$$\Omega \subset \left\{\varphi\left(\frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}, \frac{Q_\beta^{\alpha}f(z)}{Q_\beta^{\alpha+1}f(z)}, \frac{Q_\beta^{\alpha-1}f(z)}{Q_\beta^{\alpha}f(z)}; z\right) : z \in U\right\} \tag{3.4}$$

implies

$$q(z) \prec \frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}.$$

Proof. Let $p(z)$ be defined by (2.13) and ψ by (2.18). Since $\varphi \in \Phi'_{H,1}[\Omega, q]$, from (2.19) and (3.4) that we have

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in U\right\}.$$

From (2.18), the admissibility condition for $\varphi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{Q_\beta^{\alpha+1}f(z)}{Q_\beta^{\alpha+2}f(z)}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case class $\Phi'_{H,1}[h(U), q]$ is written as $\Phi'_{H,1}[h, q]$., the following result is an immediate consequence of Theorem 3.4. □

Theorem 3.5. *Let $q(z) \in H$, $h(z)$ be analytic in U and $\varphi \in \Phi'_{H,1}[\Omega, q]$.*

If $f(z) \in \Sigma$, $\dfrac{Q_\beta^{\alpha+1} f(z)}{Q_\beta^{\alpha+2} f(z)} \in Q_1$ and $\varphi\left(\dfrac{Q_\beta^{\alpha+1} f(z)}{Q_\beta^{\alpha+2} f(z)}, \dfrac{Q_\beta^{\alpha} f(z)}{Q_\beta^{\alpha+1} f(z)}, \dfrac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^{\alpha} f(z)}; z\right)$ is univalent in U, then

$$h(z) \prec \varphi\left(\frac{Q_\beta^{\alpha+1} f(z)}{Q_\beta^{\alpha+2} f(z)}, \frac{Q_\beta^{\alpha} f(z)}{Q_\beta^{\alpha+1} f(z)}, \frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^{\alpha} f(z)}; z\right) (z \in U), \tag{3.5}$$

implies

$$q(z) \prec \frac{Q_\beta^{\alpha+1} f(z)}{Q_\beta^{\alpha+2} f(z)}.$$

Combining Theorems 2.6 and 3.5, we obtain the following Sandwich-type theorem.

Corollary 3.2. *Let* $h_1(z)$ and $q_1(z)$ be analytic functions in U, $h_2(z)$ be univalent function in U, $q_2(z) \in Q_1$ with $q_1(0) = q_2(0) = 1$, and $\varphi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$. If $f(z) \in \Sigma$, $\dfrac{Q_\beta^{\alpha+1} f(z)}{Q_\beta^{\alpha+2} f(z)} \in H \cap Q_1$ and

$$\varphi\left(\frac{Q_\beta^{\alpha+1} f(z)}{Q_\beta^{\alpha+2} f(z)}, \frac{Q_\beta^{\alpha} f(z)}{Q_\beta^{\alpha+1} f(z)}, \frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^{\alpha} f(z)}; z\right)$$

is univalent in U, then

$$h_1(z) \prec \varphi\left(\frac{Q_\beta^{\alpha+1} f(z)}{Q_\beta^{\alpha+2} f(z)}, \frac{Q_\beta^{\alpha} f(z)}{Q_\beta^{\alpha+1} f(z)}, \frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^{\alpha} f(z)}; z\right) \prec h_2(z) (z \in U),$$

implies

$$q_1(z) \prec \frac{Q_\beta^{\alpha+1} f(z)}{Q_\beta^{\alpha+2} f(z)} \prec q_2(z).$$

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