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Strong Convergence Theorems by Modified Four Step Iterative Scheme with Errors for Three Nonexpansive Mappings

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Abstract. The aim of this paper is to prove strong convergence theorem by a modified three step iterative process with errors for three nonexpansive mappings in the frame work of uniformly smooth Banach spaces. The main feature of this scheme is that its special cases can handle both strong convergence like Halpern type and weak convergence like Ishikawa type iteration schemes. Our result extend and generalize the result of S. H. Khan [10], Kim and Xu [12] and many other authors.

1. Introduction

Throughout this paper, let E be a normed linear space, C a nonempty closed convex subset of E, and $T : C \to C$ is a nonlinear mapping.
Recall that T is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote $F(T)$, the set of fixed point, that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is nonexpansive such that $F(T) \neq \phi$.

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In recent years, fixed point iteration process for nonexpansive mappings in Banach space including Halpern type, Mann type, Ishikawa type for one mapping and Das-Debata type for two mappings have been studied extensively by many authors to solve the nonlinear operator equations as well as variational inequalities in Hilbert spaces and Banach spaces. Noor [14] and Xu and Noor [22] suggested and analyzed three step iterative methods for solving different classes of variational inequalities. It has been shown that three step schemes are numerically better than two step and one step methods. Related to the variational inequalities, it is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis.

Let us have a look at these schemes one by one.

In 1967, Halpern [7] first introduced the following iteration scheme (see also Browder [2])

$$\begin{aligned} x_0 &= x \in C, \text{ chosen arbitrarily,} \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)Tx_n, \ n \geq 0 \end{aligned} \tag{1.1}$$

He pointed out that the conditions $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that, if the iteration scheme (1.1) converges to a fixed point of T, then these conditions must be satisfied. It is also an important fact about this scheme is that even if we do not have the strong convergence of a convex combination of Tx_n and x_0.
On the other hand, Mann [13] introduced the following iteration scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \ n \geq 0 \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval [0,1].
The third iteration process is referred to as Ishikawa's iteration process [8]

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)Tx_n \\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n)Ty_n, \ n \geq 0 \end{aligned} \tag{1.3}$$

where the initial guess x_0 is taken in C arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0,1].
It is pointed out that both (1.2)and (1.3) have only weak convergence in general (see [6],[20]for an example). For example, Reich [15] showed that if E is uniformly convex and has a Frechet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to a point in $F(T)$.

In [4], Das and Debata introduced the following scheme for two mappings T and S:

$$\begin{aligned} x_0 &= x \in C, \\ z_n &= \alpha_n^{'} S x_n + \beta_n^{'} x_n, \\ x_{n+1} &= \alpha_n T z_n + \beta_n x_n, \end{aligned} \tag{1.4}$$

where $\alpha_n^{'} + \beta_n^{'} = 1 = \alpha_n + \beta_n$.

Khan [9] introduced the following iteration scheme:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= a_n x + (1 - a_n) y_n, \\ y_n &= b_n x_n + (1 - b_n) T z_n, \\ z_n &= c_n x_n + (1 - c_n) S x_n, \quad n \in \mathbb{N}. \end{aligned} \tag{1.5}$$

On the other hand, Kim and Xu [12] gave the following iteration scheme:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= a_n \omega + (1 - a_n) y_n, \\ y_n &= b_n x_n + (1 - b_n) T x_n, \quad n \in \mathbb{N}; \end{aligned} \tag{1.6}$$

where x_1 and ω are given but fixed usually different elements of C.
Note that the scheme (1.6) is a special case of (1.5) when $S = I$ and x_1 in $x_{n+1} = a_n x + (1 - a_n) y_n$, is taken as ω. However, the converse is not true.

Khan [10] presented a new iteration scheme by generalizing (1.5) to the one with error terms but taking the idea of $x_1 \neq \omega$ in general, from (1.6):

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= a_n \omega + (1 - a_n) y_n, \\ y_n &= \alpha_n T z_n + \beta_n x_n + \gamma_n u_n, \\ z_n &= \alpha_n^{'} S x_n + \beta_n^{'} x_n + \gamma_n^{'} v_n, \quad n \in \mathbb{N}. \end{aligned} \tag{1.7}$$

Motivated by Khan [10], in this paper we introduce a general modified four step iteration process for finding a common fixed point of three nonexpansive mappings. Our purpose in this paper is to establish the strong convergence theorem under some mild conditions for three nonexpansive mappings in the framework of uniformly smooth Banach spaces.

In this paper, we introduce the following modified iteration scheme: Let R, S, $T : C \to C$ be three self mappings, then the modified four step iteration scheme is defined as follows:

$$
\begin{aligned}
x_0 &= x \in C \\
w_n &= \alpha_n^{''} x_n + \beta_n^{''} R x_n + \gamma_n^{''} \delta_n \\
z_n &= \alpha_n^{'} x_n + \beta_n^{'} S w_n + \gamma_n^{'} v_n \\
y_n &= \alpha_n x_n + \beta_n T z_n + \gamma_n u_n \\
x_{n+1} &= a_n \omega + a_n^{'} x_n + a_n^{''} y_n, \ n \geq 0 \qquad (1.8)
\end{aligned}
$$

where the initial guess x_0 is taken in C arbitrarily, $\omega \in C$ is an arbitrary but fixed element in C and sequence $\{\alpha_n\}$, $\{\alpha_n^{'}\}$, $\{\alpha_n^{''}\}$, $\{\beta_n\}$, $\{\beta_n^{'}\}$, $\{\beta_n^{''}\}$, $\{\gamma_n\}$, $\{\gamma_n^{'}\}$, $\{\gamma_n^{''}\}$, $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$, $\{\delta_n\}$ are bounded sequence in C. We prove, under certain appropriate assumption on the control sequences, that $\{x_n\}$ defined by (1.8) converges strongly to a fixed point of R, S, & T.

2. Preliminaries

Let E be a real Banach space and let J denote the duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \ x \in E$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.
The norm of E is said to be Gâteaux differentiable (and E is said to be smooth), if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \qquad (2.1)$$

exist for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth), if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$.
In what follows, we shall make use of the following lemmas:

Lemma 2.1. *If E is uniformly smooth then the duality mapping is single-valued and norm to norm uniformly continuous on each bounded subset of E.*

Lemma 2.2.([12]) *If E is a Banach space, then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle$$

holds where $j(x + y) \in J(x + y)$.

Recall that, if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny [5],[17] provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever

$x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction plays an important role in our argument. They are characterized as follows [1],[5],[17]: if E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality:

$$\langle x - Qx, J(y - Qx)\rangle \le 0, \ \forall x \in C, \ y \in D$$

Reich [16] showed that, if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a sunny nonexpansive retraction from C into D and it can be constructed as follows:

Lemma 2.3. *Let E be a uniformly smooth Banach space and C a nonempty closed convex subset of E. Let T be a nonexpansive mapping from C into itself such that $F(T) \neq \phi$. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ as $Qu = \lim_{t\to0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies:*

$$\langle u - Qu, J(z - Qu)\rangle \le 0, \ u \in C, \ z \in F(T).$$

Lemma 2.4.([21]) *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \gamma_n\sigma_n, \ n \ge 0,$$

where $\{\gamma_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}_{n=0}^{\infty}$ converges to zero.

3. Main Results

Theorem 3.1. *Let E be a uniformly smooth Banach space and C be closed convex subset of E. Let R, S, and T be nonexpansive mappings from C into itself such that $F = F(R) \cap F(S) \cap F(T) \neq \phi$. Furthermore, let $\{x_n\}$ be defined by (1.8), where $\{u_n\}$, $\{v_n\}$, $\{\delta_n\}$ are bounded sequences in C. The parameters α_n, $\alpha_n^{'}$, $\alpha_n^{''}$, β_n, $\beta_n^{'}$, $\beta_n^{''}$, γ_n, $\gamma_n^{'}$, $\gamma_n^{''}$, a_n, $a_n^{'}$ and $a_n^{''}$ in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha_n^{'} + \beta_n^{'} + \gamma_n^{'} = \alpha_n^{''} + \beta_n^{''} + \gamma_n^{''} = a_n + a_n^{'} + a_n^{''} = 1$. Also assume that*

(i) $\sum_{n=0}^{\infty} a_n = \infty$, $\lim_{n\to\infty} a_n = 0$;

(ii) $0 < \liminf_{n\to\infty} a_n^{'} \le \limsup_{n\to\infty} a_n^{'} < 1$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1}^{'} - \alpha_n^{'}| < \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1}^{''} - \alpha_n^{''}| < \infty$;

(iv) $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1}^{'} - \beta_n^{'}| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1}^{''} - \beta_n^{''}| < \infty$;

(v) $\sum_{n=0}^{\infty} \gamma_n < \infty,\ \sum_{n=0}^{\infty} \gamma_n^{'} < \infty,\ \sum_{n=0}^{\infty} \gamma_n^{''} < \infty.$

Then $\{x_n\}$ *converges strongly to a common fixed point of* $R,\ S,$ *and* T.

Proof. First, we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, taking a fixed point $p \in F$. Since $\{u_n\},\ \{v_n\}$ and $\{\delta_n\}$ are bounded sequences in C. So without loss of generality we may assume that there exists $M > 0$ such that

$$\max\{\sup_{k\geq 1}\{\|u_k - q\|, \|v_k - q\|, \|\delta_k - q\|, \|\omega - q\|, \|x_1 - q\|\}\} \leq M.$$

Now by mathematical induction, we shall prove that $\|x_n - q\| \leq M$ for all $n \in N$.The assertion is true for $n = 1$. Suppose that the assertion is true for $n = k$, for some positive integer k, i.e. suppose that $\|x_k - q\| \leq M$ for some positive integer k. We now prove that $\|x_{k+1} - q\| \leq M$. We have

$$\begin{aligned}\|w_k - q\| &= \|\alpha_k^{''} x_k + \beta_k^{''} R x_k + \gamma_k^{''} \delta_k - q\| \\ &\leq \alpha_k^{''}\|x_k - q\| + \beta_k^{''}\|x_k - q\| + \gamma_k^{''}\|\delta_k - q\| \\ &\leq M,\end{aligned}$$

$$\begin{aligned}\|z_k - q\| &= \|\alpha_k^{'} x_k + \beta_k^{'} S w_k + \gamma_k^{'} v_k - q\| \\ &\leq \alpha_k^{'}\|x_k - q\| + \beta_k^{'}\|w_k - q\| + \gamma_k^{'}\|v_k - q\| \\ &\leq M,\end{aligned}$$

and

$$\begin{aligned}\|y_k - q\| &= \|\alpha_k x_k + \beta_k T z_k + \gamma_k u_k - q\| \\ &\leq \alpha_k\|x_k - q\| + \beta_k\|z_k - q\| + \gamma_k\|u_k - q\| \\ &\leq M.\end{aligned}$$

So that

$$\begin{aligned}\|x_{k+1} - q\| &= \|a_k \omega + a_k^{'} x_k + a_k^{''} y_k - q\| \\ &\leq a_k\|\omega - q\| + a_k^{'}\|x_k - q\| + a_k^{''}\|y_k - q\| \\ &\leq M.\end{aligned}$$

This shows that $\{x_n\}$ is a bounded sequence. So are $\{y_n\},\ \{z_n\}$ and $\{w_n\}$.
Next we claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.1}$$

In order to prove, (3.1), put

$$K = \{\|\omega\|, \|Rx_n\|, \|Sw_n\|, \|Tz_n\|, \|x_n\|, \|u_n\|, \|u_n\|, \|v_n\|, \|\delta_n\|\},$$

and consider

$$
\begin{aligned}
\|w_n - w_{n-1}\| &= \|\alpha_n^{''} x_n + \beta_n^{''} R x_n + \gamma_n^{''} \delta_n \\
&\quad - \alpha_{n-1}^{''} x_{n-1} - \beta_{n-1}^{''} R x_{n-1} - \gamma_{n-1}^{''} \delta_{n-1}\| \\
&= \|\alpha_n^{''} x_n - \alpha_n^{''} x_{n-1} + \beta_n^{''} R x_n - \beta_n^{''} R x_{n-1} \\
&\quad - \alpha_{n-1}^{''} x_{n-1} + \alpha_n^{''} x_{n-1} - \beta_{n-1}^{''} R x_{n-1} \\
&\quad + \beta_n^{''} R x_{n-1} + \gamma_n^{''} \delta_n - \gamma_{n-1}^{''} \delta_{n-1}\| \\
&= \|\alpha_n^{''}(x_n - x_{n-1}) + \beta_n^{''}(R x_n - R x_{n-1}) + (\alpha_n^{''} - \alpha_{n-1}^{''}) x_{n-1} \\
&\quad + (\beta_n^{''} - \beta_{n-1}^{''}) R x_{n-1} + \gamma_n^{''} \delta_n - \gamma_{n-1}^{''} \delta_{n-1}\| \\
&\le \alpha_n^{''} \|x_n - x_{n-1}\| + \beta_n^{''} \|x_n - x_{n-1}\| + |\alpha_n^{''} - \alpha_{n-1}^{''}| \|x_{n-1}\| \\
&\quad + |\beta_n^{''} - \beta_{n-1}^{''}| \|R x_{n-1}\| + |\gamma_n^{''}| \|\delta_n\| + |\gamma_{n-1}^{''}| \|\delta_{n-1}\| \\
&\le (1 - \gamma_n^{''}) \|x_n - x_{n-1}\| + \eta^{''} K \\
&\le \|x_n - x_{n-1}\| + \eta^{''} K,
\end{aligned}
$$

where $\eta^{''} = |\alpha_n^{''} - \alpha_{n-1}^{''}| + |\beta_n^{''} - \beta_{n-1}^{''}| + |\gamma_n^{''}| + |\gamma_{n-1}^{''}|$,

$$
\begin{aligned}
\|z_n - z_{n-1}\| &= \|\alpha_n^{'} x_n + \beta_n^{'} S w_n + \gamma_n^{'} v_n \\
&\quad - \alpha_{n-1}^{'} x_{n-1} - \beta_{n-1}^{'} S w_{n-1} - \gamma_{n-1}^{'} v_{n-1}\| \\
&= \|\alpha_n^{'}(x_n - x_{n-1}) + \beta_n^{'}(S w_n - S w_{n-1}) + (\alpha_n^{'} - \alpha_{n-1}^{'}) x_{n-1} \\
&\quad + (\beta_n^{'} - \beta_{n-1}^{'}) S w_{n-1} + \gamma_n^{'} v_n - \gamma_{n-1}^{'} v_{n-1}\| \\
&\le \alpha_n^{'} \|x_n - x_{n-1}\| + \beta_n^{'} \|w_n - w_{n-1}\| + |\alpha_n^{'} - \alpha_{n-1}^{'}| \|x_{n-1}\| \\
&\quad + |\beta_n^{'} - \beta_{n-1}^{'}| \|S w_{n-1}\| + |\gamma_n^{'}| \|v_n\| + |\gamma_{n-1}^{'}| \|v_{n-1}\| \\
&\le \alpha_n^{'} \|x_n - x_{n-1}\| + \beta_n^{'} [\|x_n - x_{n-1}\| + \eta^{''} K] + \eta^{'} K \\
&\le (1 - \gamma_n^{'}) \|x_n - x_{n-1}\| + (\eta^{'} + \eta^{''}) K \\
&\le \|x_n - x_{n-1}\| + (\eta^{'} + \eta^{''}) K,
\end{aligned}
$$

where $\eta^{'} = |\alpha_n^{'} - \alpha_{n-1}^{'}| + |\beta_n^{'} - \beta_{n-1}^{'}| + |\gamma_n^{'}| + |\gamma_{n-1}^{'}|$, and

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \|\alpha_n x_n + \beta_n T z_n + \gamma_n u_n - \alpha_{n-1} x_{n-1} \\
&\quad - \beta_{n-1} T z_{n-1} - \gamma_{n-1} u_{n-1}\| \\
&\leq \alpha_n \|x_n - x_{n-1}\| + \beta_n \|z_n - z_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|T z_{n-1}\| + |\gamma_n| \|u_n\| + |\gamma_{n-1}| \|u_{n-1}\| \\
&\leq \alpha_n \|x_n - x_{n-1}\| + \beta_n [\|x_n - x_{n-1}\| + (\eta^{'} + \eta^{''}) K] + \eta K, \\
&\leq \|x_n - x_{n-1}\| + (\eta + \eta^{'} + \eta^{''}) K,
\end{aligned}
$$

where $\eta = |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n| + |\gamma_{n-1}|$,
and so

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|a_n \omega + a_n^{'} x_n + a_n^{''} y_n - a_{n-1} \omega \\
&\quad - a_{n-1}^{'} x_{n-1} - a_{n-1}^{''} y_{n-1}\| \\
&= \|(a_n - a_{n-1}) \omega + a_n^{'} (x_n - x_{n-1}) + a_n^{''} (y_n - y_{n-1}) \\
&\quad + (a_n^{'} - a_{n-1}^{'}) x_{n-1} + (a_n^{''} - a_{n-1}^{''}) y_{n-1}\| \\
&\leq |a_n - a_{n-1}| \|\omega\| + a_n^{'} \|x_n - x_{n-1}\| + a_n^{''} \|y_n - y_{n-1}\| \\
&\quad + |a_n^{'} - a_{n-1}^{'}| \|x_{n-1}\| + |a_n^{''} - a_{n-1}^{''}| \|y_{n-1}\| \\
&\leq [|a_n - a_{n-1}| + |a_n^{'} - a_{n-1}^{'}| + |a_n^{''} - a_{n-1}^{''}|] K + a_n^{'} \|x_n - x_{n-1}\| \\
&\quad + a_{n-1}^{''} [\|x_n - x_{n-1}\| + (\eta + \eta^{'} + \eta^{''}) K].
\end{aligned}
$$

Applying Lemma 2.4, we get,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$

Observing that,

$$
\begin{aligned}
\|y_n - x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_n \|\gamma f(x_n) - A y_n\| + \alpha_n^{'} \|x_n - y_n\|,
\end{aligned}
$$

and by condition (i) and (ii), we can easily get,

$$\lim_{n \to \infty} \|y_n - x_n\| = 0.$$

Also

$$\begin{aligned}\|Tx_n - x_n\| &\le \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\| + \|y_n - Tx_n\| \\ &\le \|x_{n+1} - x_n\| + a_n\|\omega - y_n\| + a_n^{'}\|x_n - y_n\| + \alpha_n\|Tz_n\| \\ &\quad + \beta_n\|x_n - Tx_n\| + \gamma_n\|u_n\| \\ &\quad \to 0.\end{aligned}$$

Now we claim that $\limsup_{n\to\infty}\langle \omega - p, J(x_n - p)\rangle \le 0$.
By Lemma 2.3, $p = Q\omega = \lim_{t\to 0} p_t$ with p_t being the fixed point of the contraction $x \mapsto tu + (1-t)Tx$.
From that p_t solves the fixed point equation

$$p_t = t\omega + (1-t)Tp_t.$$

We have

$$\|p_t - x_n\| = \|(1-t)(Tp_t - x_n + t(\omega - x_n))\|.$$

It follows from Lemma 2.2, that

$$\begin{aligned}\|p_t - x_n\|^2 &\le (1-t)^2\|Tp_t - x_n\|^2 + 2t\langle \omega - x_n, J(p_t - x_n)\rangle \\ &\le (1-t)^2(\|Tp_t - Tx_n\| + \|Tx_n - x_n\|)^2 + 2t\langle \omega - x_n, J(p_t - x_n)\rangle \\ &\le (1 - 2t + t^2)\|p_t - x_n\|^2 + f_n(t) \\ &\quad + 2t\langle p_t - \omega, J(p_t - x_n)\rangle + 2t\|p_t - x_n\|^2,\end{aligned}$$

where

$$\lim_{n\to\infty} f_n(t) = (2\|p_t - x_n\| + \|x_n - Tx_n\|)\|x_n - Tx_n\| = 0. \tag{3.2}$$

It follows that

$$\langle p_t - \omega, J(p_t - x_n)\rangle \le \frac{t}{2}\|p_t - x_n\|^2 + \frac{1}{2t}f_n(t).$$

Letting $n \to \infty$ and noting (3.2), we obtain

$$\limsup_{n\to\infty}\langle p_t - \omega, J(p_t - x_n)\rangle \le \frac{t}{2}L, \tag{3.3}$$

where $L > 0$ is a constant such that $L \ge \|p_t - x_n\|^2$ for all $t \in (0,1),\ n \ge 1$.
Since $\{p_t - x_n\}$ is a bounded sequence and J is norm-to-norm continuous uniformly on each bounded subset of E by Lemma 2.1 and $p_t \to p$ as $t \to 0$, therefore letting $t \to 0$ in (3.2), we get

$$\langle \omega - p, J(x_n - p)\rangle \le 0.$$

Finally, we show that $x_n \to p$. From Lemma 2.2, we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|a_n^{''}(y_n - p) + a_n^{'}(x_n - p) + a_n(\omega - p)\|^2 \\ &\leq \|a_n^{''}(y_n - p) + a_n^{'}(x_n - p)\|^2 + 2a_n\langle \omega - p, J(x_{n+1} - p)\rangle \\ &\leq (1 - a_n)\|x_n - p\|^2 + 2a_n\langle \omega - p, J(x_{n+1} - p)\rangle\end{aligned}$$

By Lemma 2.4, it is easy to see that $\|x_n - p\|^2 \to 0$ as $n \to \infty$.
This completes the proof of the theorem. □

Our theorem generalizes Theorem 1 of Kim and Xu [12] as follows:

Corollary 3.1. *Let C be a closed and convex subset of a uniformly smooth Banach space E. Let T be a nonexpansive mapping from C into itself such that $F(T) \neq \phi$. Further, let $\{x_n\}$ defined by (1.5) satisfy*

$$0 < a_n < 1,\ a_n \to 0,\ \sum_{n=1}^{\infty} a_n = \infty,\ \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty,$$
$$0 < b_n < 1,\ b_n \to 0,\ \sum_{n=1}^{\infty} b_n = \infty,\ \sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty,$$

for all $n \in N$. Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.1.

1. It is also worth mentioning that unlike Kim and Xu [12], we do not need the condition $\sum_{n=1}^{\infty} b_n = \infty$.

2. The schemes (1.2),(1.3),(1.4) requiring $a_n = 0$ are already covered by Theorem 2 proved by Khan and Fukhar-ud-din [11]. Our scheme (1.8) also generalizes the corresponding results of S. H. Khan [10]

3. Results proved under similar conditions using schemes (1.1), (1.5),(1.6),(1.7) are also covered by our theorem.
4. If $f : C \to C$ is a contraction map and we replace u by $f(x_n)$ in the recursion formula (1.8), we obtain what some authors now call viscosity iteration method. We note that our theorem in this paper carry over trivially to the so-called viscosity process. One simply replace u by $f(x_n)$ and using the fact that f is a contraction map, one can repeat the argument of this paper.

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