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On a Type of Semi-Symmetric Non-Metric Connection on Riemannian Manifolds

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Abstract. The object of the present paper is to characterize a Riemannian manifold admitting a type of semi-symmetric non-metric connection.

1. Introduction

In 1924, Friedmann and Schouten [1] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\bar{\nabla}$ on a differentiable manifold (M^n, g) with Riemannian connection ∇ is said to be a semi-symmetric connection if the torsion tensor T of the connection $\bar{\nabla}$ satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

where η is a 1-form and ξ is a vector field defined by $\eta(X) = g(X, \xi)$, for all vector fields $X \in \chi(M^n)$, $\chi(M^n)$ is the set of all differentiable vector fields on M^n.

In 1932, Hayden [4] introduced the idea of semi-symmetric connections on a Riemannian manifold (M^n, g). A semi-symmetric connection $\bar{\nabla}$ is said to be a semi-symmetric metric connection if

$$\bar{\nabla} g = 0.$$

The study of semi-symmetric metric connection was further developed by Yano [6], Amur and Pujar [7], Chaki and Konar [12], De [17] and many others.

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After long gap the study of a semi-symmetric connection $\bar{\nabla}$ satisfying

$$\bar{\nabla} g \neq 0.$$

was initiated by Prvanović [9] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [14].

In 1992, Agashe and Chafle [13] introduced and studied a semi-symmetric non-metric connection. The semi-symmetric non-metric connections was further developed by several authors such as De and Biswas [18], Biswas, De and Barua [16], De and Kamilya ([20], [21]) and many others.

In 1967, R.N.Sen and M.C.Chaki [15] studied certain curvature restrictions on a certain kind of conformally flat Riemannian space of class one and they obtained the following expression of the covariant derivative of the curvature tensor :

$$K^h_{ijk,l} = 2\lambda_l K^h_{ijk} + \lambda_i K^h_{ljk} + \lambda_j K^h_{ilk} + \lambda_k K^h_{ijl} + \lambda^h K_{lijk}, \tag{1.2}$$

where K^h_{ijk} are the components of the curvature tensor K with respect to the Levi-Civita connection,

$$K_{ijkl} = g_{hl} K^h_{ijk},$$

λ_i is a non-zero covarient vector and "," denotes covarient differentiation with respect to the metric tensor g_{ij}.

Later in 1987, M.C.Chaki [10] called a manifold a pseudo symmetric manifold whose curvature tensor satisfies (1.2). In index free notation this can be stated as follows: A non- flat Riemannian manifold (M^n, g) , $n \geq 2$ is said to be a pseudo symmetric manifold [10] if its curvature tensor K satisfies the condition

$$(D_X K)(Y,Z)W = 2A(X)K(Y,Z)W + A(Y)K(X,Z)W + A(Z)K(Y,X)W$$
$$+A(W)K(Y,Z)X + g(K(Y,Z)W,X)U,$$

where D denotes the operator of covarient differentiation with respect to the metric tensor g. The 1- form A is called the associated 1-form of the manifold . If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan [3]. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. In this connection we can mention the notion of weakly symmetric manifold introduced by *Tamássy* and Binh [8]. Such a manifold was denoted by $(WS)_n$.

In a recent paper De and Gazi [19] introduced a type of non-flat Riemannian manifold (M^n, g), $n \geq 2$ whose curvature tensor K of type (1,3) satisfies the condition

(1.3)

$$(D_X K)(Y,Z)W = [A(X)+B(X)]K(Y,Z)W + A(Y)K(X,Z)W + A(Z)K(Y,X)W$$
$$+A(W)K(Y,Z)X + g(K(Y,Z)W,X)U,$$

where A,U and D have the meaning already mentioned and B is a non-zero 1-form, V is a vector field defined by $B(X) = g(X,V), \quad \forall X$.

Such a manifold was called an almost pseudo symmetric manifold and was denoted by $(APS)_n$.

If $B = A$, then from the definitions it follows that $(APS)_n$ deduces to a $(PS)_n$. In the same paper the authors constructed two non-trivial examples of $(APS)_n$. It may be mentioned that almost pseudo symmetric manifolds is not a particular case of weakly symmetric manifolds.

Let (M^n, g) , $(n > 3)$ be a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric, that is,
(1.4)

$$(\nabla_X T)(Y,Z) = [A(X)+B(X)]T(Y,Z)+A(Y)T(X,Z)+A(Z)T(Y,X)+g(T(Y,Z),X)U,$$

where A and B are defined earlier.

A non- flat Riemannian manifold (M^n, g) , $n \geq 3$ is said to be a quasi-Einstein manifold [11] if its Ricci tensor $\tilde{S}$ of the Levi-Civita connection is of the form

$$\tilde{S}(X,Y) = ag(X,Y) + bA(X)A(Y),$$

where a and b are smooth functions of the manifold.

In the present paper we consider a Riemannian manifold admitting a semi- symmetric non-metric connection whose torsion tensor is almost pseudo symmetric.

The paper is organized as follows:

After preliminaries in section 3, we first obtain the expressions of the curvature tensor and the Ricci tensor of the semi-symmetric non-metric connection. In this section we prove that if a Riemannian Manifold admits a semi-symmetric non-metric connection whose curvature tensor vanishes and the torsion tensor is almost pseudo symmetric with respect to the semi-symmetric non-metric connection, then the manifold becomes a quasi-Einstein manifold. Finally, we deal with a simply connected $(APS)_n, (n > 3)$ addmitting such a semi-symmetric non-metric connection.

2. Preliminaries

Let $\tilde{r}$ denotes the scalar curvature of the manifold with respect to the Levi-Civita connection and L denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor, that is,

$$g(LX,Y) = \tilde{S}(X,Y), \tag{2.1}$$

for any vector field X, Y.

Contracting Y in (1.3), it follows that

$$(D_X\tilde{S})(Z,W) = [A(X)+B(X)]\tilde{S}(Z,W)+A(K(X,Z)W)+A(Z)\tilde{S}(X,W)+A(W)\tilde{S}(X,Z)+g(K(U,Z)W,X). \tag{2.2}$$

Putting $Z = W = e_i$ in (2.2), where $\{e_i\}$, i= 1,2,3...,n; is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, 1 \leq i \leq n$, we get

$$d\tilde{r}(X) = [A(X) + B(X)]\tilde{r} + 4A(LX). \tag{2.3}$$

3. Riemannian Manifolds Admitting a Special Type of the Semi-Symmetric Non-Metric Connection

Theorem 3.1. *If a Riemannian manifold admits a semi-symmetric non-metric connection whose curvature tensor R vanishes and torsion tensor T satisfies (1.4), then the manifold is a quasi-Einstein manifold .*

Proof. Let M be an n-dimensional Riemannian manifold with Riemannian metric g. If ∇ is the semi-symmetric non-metric connection of a Riemannian manifold M, then ∇ is given by [13]

$$\nabla_X Y = D_X Y + A(Y)X. \tag{3.1}$$

Let R be the curvature tensor with respect to semi-symmetric non-metric connection. Then R and K are related by [13]

$$R(X,Y)Z = K(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X, \tag{3.2}$$

for all vector fields X, Y, Z on M, where α is a $(0, 2)$ tensor given by

$$\alpha(X,Z) = (D_X A)(Z) - A(X)A(Z). \tag{3.3}$$

In this section we consider a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor T satisfies (1.4).
From (1.1), contracting over X, we get

$$(C_1^1 T)(Y) = (n-1)A(Y). \tag{3.4}$$

From (3.4), it follows that

$$(\nabla_X C_1^1 T)(Y) = (n-1)(\nabla_X A)(Y). \tag{3.5}$$

Contracting over Z in (1.4) and using (3.4), we obtain

$$(\nabla_X C_1^1 T)(Y) = (2n-3)A(X)A(Y) + (n-1)B(X)A(Y) + A(U)g(X,Y). \tag{3.6}$$

From (3.5) and (3.6) yields

$$(n-1)(\nabla_X A)(Y) = (2n-3)A(X)A(Y) + (n-1)B(X)A(Y) + A(U)g(X,Y). \tag{3.7}$$

Using (3.1) and (3.3), it follows that

$$(\nabla_X A)(Y) = (D_X A)(Y) - A(X)A(Y) = \alpha(X,Y). \tag{3.8}$$

Therefore, from (3.7) and (3.8), we have

$$\alpha(X,Y) = \frac{2n-3}{n-1}A(X)A(Y) + B(X)A(Y) + \frac{1}{n-1}A(U)g(X,Y). \tag{3.9}$$

Using (3.9) in (3.2) yields
(3.10)

$$R(X,Y)Z = K(X,Y)Z + \frac{2n-3}{n-1}A(X)A(Z)Y + B(X)A(Z)Y + \frac{1}{n-1}A(U)g(X,Z)Y$$
$$-\frac{2n-3}{n-1}A(Y)A(Z)X - B(Y)A(Z)X - \frac{1}{n-1}A(U)g(Y,Z)X.$$

From (3.10), we get
(3.11)

$$\tilde{R}(X,Y,Z,W) = \tilde{K}(X,Y,Z,W) + \frac{2n-3}{n-1}A(X)A(Z)g(Y,W) + B(X)A(Z)g(Y,W)$$
$$+\frac{1}{n-1}A(U)g(X,Z)g(Y,W) - \frac{2n-3}{n-1}A(Y)A(Z)g(X,W)$$
$$-\frac{1}{n-1}A(U)g(Y,Z)g(X,W) - B(Y)A(Z)g(X,W),$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$ and $\tilde{K}(X,Y,Z,W) = g(K(X,Y)Z,W)$.

Putting $X = W = e_i$ in (3.11) where $\{e_i\}$, $1 \leq i \leq n$ is an orthonormal basis of the tangent space at any point of the manifold M^n and then summing over i, we

obtain
(3.12)

$$S(Y,Z) = \tilde{S}(Y,Z) - (2n-3)A(Y)A(Z) - (n-1)B(Y)A(Z) - A(U)g(Y,Z),$$

where S be the Ricci tensor with respect to the semi-symmetric non-metric connection.

Proposition 3.1. *If a Riemannian manifold admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric, then*
(i)*the curvature tensor of the semi-symmetric non-metric connection is given by* (3.10),
(ii) *the Ricci tensor of the semi-symmetric non-metric connection is given by* (3.12),
(iii) *the Ricci tensor S is symmetric if and only if $B(Y)A(Z) = B(Z)A(Y)$.*

Suppose

$$R(X,Y)Z = 0.$$

Then from the above equation, we have

$$S(Y,Z) = 0.$$

Hence the equation (3.12) reduces to
(3.13)

$$\tilde{S}(Y,Z) = (2n-3)A(Y)A(Z) + (n-1)B(Y)A(Z) + A(U)g(Y,Z).$$

Since $\tilde{S}$ is symmetric, therefore
Therefore,
(3.14)

$$B(Y)A(Z) = B(Z)A(Y).$$

Putting $Z = U$ in (3.14), it follows that
(3.15)

$$B(Y) = fA(Y).$$

where $f = \frac{B(U)}{A(U)}$
Now using (3.17) in (3.15), we obtain
(3.16)

$$\tilde{S}(Y,Z) = A(U)g(Y,Z) + [(2n-3) + (n-1)f]A(Y)A(Z).$$

Therefore , $\tilde{S}(X,Y) = ag(X,Y) + bA(X)A(Y)$,
where $a = A(U)$ and $b = [(2n-3) + (n-1)f]$.

Hence the proof is completed. □

4. Special Conformally Flat $(APS)_n$ Admitting a Special Type of the Semi-Symmetric Non-Metric Connection

Theorem 4.1. *If a $(APS)_n$ $(n > 3)$ admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes, then the manifold is a particular kind of a special conformally flat manifold, namely a subprojective manifold.*

Proof. Chen and Yano [2] introduced the notion of a special conformally flat manifold which generalizes the notion of a subprojective manifold. A conformally flat manifold is called a special conformally flat manifold if the tensor H of type $(0, 2)$ defined by

$$H(X,Y) = -\frac{1}{n-2}\tilde{S}(X,Y) + \frac{\tilde{r}}{2(n-1)(n-2)}g(X,Y), \tag{4.1}$$

is expressible in the form

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(D_X\alpha)(D_Y\alpha),$$

where α and β are two scalars such that α is positive. In particular, if β is a function of α then the special conformally flat manifold is called a subprojective manifold [5].

Let us consider $(APS)_n$ admitting a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes.
Using (3.16) in (4.1), we get

$$H(X,Y) = \frac{\tilde{r} - 2(n-1)A(U)}{2(n-1)(n-2)}g(X,Y) - \frac{2n-3+f(n-1)}{n-2}A(X)A(Y). \tag{4.2}$$

Now, put

$$\alpha^2 = -\frac{\tilde{r} - 2(n-1)A(U)}{(n-1)(n-2)}. \tag{4.3}$$

From (3.16), we get

$$\tilde{r} = (n-1)(3+f)A(U), n \geq 3. \tag{4.4}$$

Since $\tilde{r} \neq 0$, it follows that α^2 will be positive provided that $\tilde{r} < 0$.
From (3.16) and (2.1), it follows that

$$LY = [2n-3+(n-1)f]A(Y)U + A(U)Y. \tag{4.5}$$

From (4.5), we obtain

$$A(LY) = (n-1)(2+f)A(U)A(Y). \tag{4.6}$$

Using (4.6) and (3.15) in (2.3), we have

$$d\tilde{r}(X) = [(1+f)\tilde{r} + 4(n-1)(2+f)A(U)]A(X). \tag{4.7}$$

Let us take the covariant derivative of both side of (4.3) with respect to X and using (4.7), we obtain

$$D_X\alpha = -\frac{(1+f)\tilde{r} + 4(n-1)(2+f)A(U)}{2(n-1)(n-2)\alpha}A(X). \tag{4.8}$$

From (4.8), we have

$$A(X) = -\frac{2(n-1)(n-2)\alpha}{(1+f)\tilde{r} + 4(n-1)(2+f)A(U)}D_X\alpha. \tag{4.9}$$

Thus, due to (4.3), (4.9) and (4.2) can be expressed in the form

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(D_X\alpha)(D_Y\alpha),$$

where

$$\beta = -\frac{4(2n-3) + (n-1)f)(n-1)^2(n-2)}{[(1+f)\tilde{r} + 4(n-1)(2+f)A(U)]^2}\alpha^2. \tag{4.10}$$

In virtue of (4.10), we deduce that β is a function of α. Thus the theorem is proved. □

Corollary 4.1.([2]) *Every simply connected subprojective space can be isometrically immersed in a Euclidean space as a hypersurface.*

Moreover, using this Corollary, we can also state the following theorem:

Theorem 4.2. *If a simply connected $(APS)_n$ $(n > 3)$ admits a semi-symmetric non-metric connection whose torsion tensor is almost pseudo symmetric and the curvature tensor of the semi-symmetric non-metric connection vanishes, then the manifold can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.*

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