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Theoretical Discussion on Mathematical Knowledge for Teaching from Constructivists' Perspective[1]

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In the present paper, we argue any research concerning human knowledge construction, components, or types needs to clarify its epistemological stance regarding 'knowledge' in that such viewpoint might have much influence on the nature of knowledge the researcher sees and the way in which evidence for knowledge development is gathered. Thus, we suggest two alternative research groups who conducted their studies on mathematical knowledge for teaching with an explicit epistemological standpoint. We finalize our discussion by reviewing concrete examples in the previous literature on teacher knowledge of fraction conducted by the two groups.

Keywords: mathematical knowledge for teaching (MKT), teacher knowledge, constructivism, fraction
MESC Classification: B59, D29
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1. INTRODUCTION

The relationship between teacher knowledge and student learning was not conceptualized until Shulman's (1986) seminal work on pedagogical content knowledge. In mathematics education, scholars (e.g., Ball, Thames & Phelps, 2008; Hill, Ball & Schilling, 2008; Thompson & Thompson, 1996) have used the phrase *mathematical knowledge for teaching* (MKT) to stress mathematics teachers' knowledge used in solving problems

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arising in teaching practice, and a plethora of research in mathematics teacher knowledge have been conducted since then. Ball and her colleagues sought to compartmentalize teacher knowledge into finer pieces of knowledge that entail aspects of both subject matter knowledge and pedagogical content knowledge. Their work shed light on the field by suggesting the importance of specialized knowledge for teaching mathematics that many educated adults, unless they teach math, do not necessarily need.

Researchers under this line of study attempt to refine the framework by studying teacher knowledge in specific mathematical topic (e.g., mathematical knowledge for teaching algebra) or by analyzing correlations amongst their MKT, quality of math instruction, and student achievement. Little discussion, however, of their philosophical perspective viewing the nature of knowledge was shared until now. If one's goal is to investigate a sort of knowledge in a specific field and provide with a theory of analyzing such knowledge (in this case, teacher knowledge), it seems natural to start out the discussion by providing his/her epistemological or ontological stance concerning knowledge as it shapes what is true or not. Ball and her colleagues state that they deliberately avoid discussing their perspective of knowledge (Hill, Ball & Schilling, 2008) because they want to rely on empirical evidence regarding what types of knowledge math teachers would need. We, despite of it, believe that it is crucial to definitize one's basic philosophical stance supporting his/her study in that the stance might have far-reaching (explicit or implicit) influence on the whole process of his/her research.

The purpose of this paper is to suggest two alternative perspectives of mathematical knowledge for teaching both of which are grounded in constructivism: teacher knowledge as *interiorized*[2] schemes and teacher knowledge from *knowledge-in-pieces* perspective. We will finalize our discussion by providing review of literatures on teacher knowledge of fraction under these two different perspectives.

2. TEACHER KNOWLEDGE AS INTERIORIZED SCHEME

Scholars have implicitly discussed mathematical knowledge for teaching as interiorized scheme (Cobb & Steffe, 1983; Steffe, 1990; Steffe & Wiegel, 1992; Thompson & Thompson, 1996). Some have advocated teachers' mathematical knowledge for teaching should be grounded in 'conception-based perspective' (Heinz, Kinzel, Simon & Tzur, 2000; Simon, 2000) and others have used 'mathematical knowledge for conceptual teaching' (Thompson & Thompson, 1996). They are not identical in that Simon (1995; 2000) and Tzur (1999a; 1999b) base their view of teacher knowledge from 'emergent

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[2] *Interiorized* schemes are different from *internalized* ones in that the former are further abstracted form of schemes that are stripped of their contextual details (Olive, 2001).

perspective' by Cobb & Yackel (1995), which coordinated von Glasersfeld's 'radical constructivism' and Bauersfeld's 'interactionism', whereas Steffe & Thompson (2000) based their study purely on von Glasersfeld's 'constructivism'. Radical constructivists encourage teachers to be researchers and model builders and to create second order model of children's mathematics (Cobb & Steffe, 1983).

Students do not have a knowledge container so that adult can transfer their knowledge; knowledge is merely a tool for children to understand the world that they are confronting. Students need to be considered as living organisms who are trying to make sense out of their experience. In this perspective, what students learn in the class at any moment is not only influenced by the instruction but also by what students already know and by instruction in which they have participated. Mutually, a teacher's instructional actions at any moment are also not simply a matter of conducting what he or she has prepared for the class, but are influenced by what the teacher understands about what he or she is teaching and by what he or she acknowledges about what students know and about how students productively build upon that knowledge (Thompson & Saldanha, 2003). In other words, teachers' need to build *second order knowledge* of students' *first order knowledge* by hypothesizing *a zone of potential construction* of their child.

First-order models constitute children's own mathematical knowledge which is the network of schemes: schema (Olive & Steffe, 2001). In other words, these are the models that an individual constructs to organize or control their experience; hence these models are inaccessible as long as there is no machine, which can copy children's mental activities. However teachers can construct *second-order models* of their students' knowledge from intensive conceptual analysis of children's notions and operations (Olive & Steffe, 2001). Therefore first-order models are *children's mathematics* which constitutes students' mathematical realities that is independent of adults' realities, and second-order models are *mathematics of children* that are adults' schema for children's schema. In this perspective of learning, school mathematics should consist of mathematics of children so that we can establish *mathematics for children* that consists of mathematical concepts and operations that children might learn (Steffe, 1988). Hence, radical constructivists would support children's algorithm to generate more powerful mathematics and schemes, then, since we are all human, there will be consensual domain (Maturana, 1988) so that we can discuss about the mature form of algorithm.

On the other hand, scholars (e.g., Cobb, Wood & Yackel, 1990; Simon, 2000) under emergent perspective coordinated psychological constructivism with interactionism, and took into account for more sociological issues such as social norms, sociomathematical norms, and classroom mathematical practices. From this perspective, a human organism is not only considered as a biological system but also an ecosocial system (Cobb & Yackel, 1995). They believe that, at some points, teachers need to provide students with such 'mature' forms of algorithms and children's constructed algorithm will serve in the construction of 'mature' forms of algorithms: 'taken-as-shared' knowledge. For the purpose of this paper, we would place them under the same category because their perspectives of teacher knowledge are yet compatible in various ways. First, they all view the importance of building instruction on what students understand rather than on the formal structures of conventional school mathematics. Steffe & Wiegel (1992) pointed out that the most important role of constructivist teachers is to learn mathematical knowledge of their children and to harmonize their teaching methods with the nature of that mathematical knowledge.

Whether they are rooted in radical constructivism or in emergent perspective, both share the following premises of knowledge: first, in contrast to 'representational view of mind' that our mind represents ontological reality, knowledge is created through human activity and it only represent our experiential reality (Cobb et al., 1990); and learning occurs as we adapt (or accommodate) our current conception; hence we, as cognizant entities, are always constrained or promoted by our anticipatory structure (Steffe, 1990). Hence, for learning to take place, students need to actively assimilate into their current scheme and reorganize their experiential situation by reflective abstraction. Subsequently, for the constructivist teacher, the key is to help children hold their own mathematical activity at a distance and take it as its own object: a characteristic of reflective abstraction. In order to encourage children to engage in mathematical activities and reflect on their activities so that their children can abstract the mathematical concept, teachers also need to continuously monitor their activity with students at a distance, and delve into students' conceptions while they do various activities by comparing with their own mathematical knowledge and provide them with activities so that students can function on their current schemes. In this way, the teachers knowledge evolves simultaneously with the growth in the students' knowledge (Simon, 1995).

Furthermore, teachers need to guide students to use their mathematical knowledge to understand students' mathematics, and at the same time, they set aside their own understanding of the mathematics to consider the mathematics of the students (Heinz et al., 2000; Steffe, 1990). Even though it is demanding job, if teachers' mathematical knowledge for teaching is reflectively abstracted from their internalized view of it (which we would like to call it as interiorized scheme), teachers would be handier in guiding students into such ways.

Along with this view, Simon (2000) found that teacher education program (includes both in-service and pre-service education) frequently fostered a perception-based perspective, and less fostered a conception-based perspective:

> Perception-based perspective refers to a comprehensive view of knowledge and learning that affects the teachers' views of mathematics; expectations about how it is learned, approaches

> to teaching, adaptations of new curricular and teaching tools, and interpretations of professional development experiences (p. 216).

Simon (2000), further, provided typical teachers behaviors under such perspectives: for instance, one teacher assimilated the experience of listening to students' explanations to her perception-based perspective and always asked the students to articulate their reasoning for their work. However, rather than using the students' conceptions to guide the following instruction, she used it to see if the students' reasoning match with her one. Similarly, another teacher, who assimilated his experience of using based-ten block to teach place value into his perception-based perspective, decided to use base-ten block to teach place value in the future with his students. They know listening to students and using manipulatives were important, but they do not know that those were not causing students' learning but students' experiences of those embodiments are creating learning (e.g., base-ten blocks are meaningful representation once learners express a conception using them.) At all events, teachers interpret the mathematics education reform as emphasizing the value of learners' first-hand experience to comprehend mathematical relationships, which are taken to exist independent of human activity and to be perceivable by all identical ways. In our opinion, such teachers were the ones who only internalized what they learned from those experiences, and are yet to interiorize. Providing teachers the ideas of teacher educators or mathematics education researchers (cf. studies of Cognitively Guided Instruction) would at best result in the level at internalization of teachers' current conception of students' learning, and is often ineffective. As Cobb & Steffe (1983) pointed out,

> 'we can no more give teachers our counting-type model than we can give children our knowledge that subtraction is the inverse of addition (p. 93)'.

Under the conception based perspective of teacher knowledge, the focus for teachers is not to understand what their students do not understand (e.g., misconception research) but to understand what they know and how they do what they know; similarly teacher educators under this paradigm put more emphasis on how teachers understand, and how teachers do what they do in the classroom not on what teachers do not know (e.g., insufficient mathematical knowledge research). In other words, it is not as much as important to understand if teachers can solve, for example, partitive or quotitive fraction division problems but in what operations and actions they use to solve those problems. In detail, many teachers know that quotitive division is to measure out a dividend by a divisor and the total number they measured out is the answer; however, they may not know the reason that it works because they only internalized from the given instruction but not interiorized. Even though teachers interiorize such mathematical concepts, those are necessary but insufficient to understand children's mathematics. It is hard for them to interiorize those research-based works of students' thinking (cf. Carpenter, Fennema, Peterson & Carey, 1988; Carpenter, Fennma, Peterson, Chiang & Loaf, 1989) unless they work with children to understand it. Thompson & Thompson (1994; 1996) described one teachers' struggle to teach one child a rate concept even though he interiorized, from observers' perspective, proportional reasoning. It is hard to teach conceptually unless teachers get a chance to decenter what they have learned from their teacher education programs, and try to think about the problem in the way their students may solve.

In sum, these researchers are more interested in what types of tasks teachers use to promote students' current schemes and how teachers' knowledge of students' knowledge evolve as they try to understand students' mathematics. Such models of students learning are essentially teachers' interiorized schemes for the following regards; teachers' interactions with students provide teachers with records of experiences upon which they can reflect, allowing them to abstract regularities of students' knowing (e.g., model of their students' learning), and such abstracted regularities (schemes) serve as the basis for constructing zone of potential construction of students' development (or, hypothetical learning trajectories[3] in a classroom setting). Such second order models of students' mathematics, which are based on retrospective analysis of their students (reflective abstraction process), are goal-directed based on teachers' anticipation and have tremendous potential to change as they try to understand what their students know and to develop useful understandings of students' mathematical knowledge.

3. TEACHER KNOWLEDGE FROM KNOWLEDGE-IN-PIECES PERSPECTIVE

Studies under this category examined teacher knowledge in fine-grained size and tried to understand teachers' thinking from their practice or problem solving strategies (e.g., Behr, Khoury, Harel, Post & Lesh, 1997; Izsák, 2008; Izsák, Tillema & Tunç-Pekkan, 2008; Lehrer & Franke, 1992; Post, Harel, Behr & Lesh, 1991). The epistemological stance, as constructivists, under the latter domain has been elaborated by Smith, diSessa,

[3] Simon (1995) provided a framework to analyze teaching in context, Hypothetical Learning Trajectories (HLTs), which consist of the learning goal that defines the direction, the learning activities, the hypothetical learning process, and the teachers' hypotheses of students' knowing and understanding that will evolve as the actual learning trajectory as they work with their children. Simon's model of teaching is different from the traditional view of teaching in that it puts more emphasis on the role that a teacher's awareness of the learners' conceptions plays in generating tasks that are likely to promote transformations of those conception in the learners (Tzur, 1999a, p. 392).

& Rocschelle (1993); they are in various ways parallel to the first group[4]; both have emphasized the role of prior knowledge in learning, and criticized research on students' misconception, which they considered as supportive of the process of students' knowledge refinement. In addition, as the first group emphasized the role of reflective abstraction in students' learning, this group similarly encouraged children to think about their own thinking- 'meta-cognition' (Schoenfeld, 1992). They suggested that the core element in problem solving activity is in monitoring and assessing progress "on line", and acting in response to the assessments of on-line progress (self-regulation). Hence if things appeared to be proceeding well, the problem solver continues along the same path; if they appeared to be problematic, she takes stock of the process and looks for other options. As we want our students to think about their thinking, we also want teachers to constantly monitor what is taking place during instruction and to act on the basis of perceptions of what is taking place. Besides, they argued that much of teacher expertise can be seen as the result of the development of abstractions as means of perceiving and interpreting things that they experienced – *automaticity* in teaching.

> "Expert teachers often develop automaticity for the repetitive operations that are needed to accomplish their goals, they are more sensitive to task demands and social situation when solving problems, and they are more flexible in their teaching than are novices, and they have fast accurate pattern recognition capabilities (Schoenfeld, 1998, p. 27)."

However, even if they show a commonality to some extent, there is a significant disparity between two perspectives. The first group based their study from children's mathematics whereas the second group started from semantic analysis of expert mathematicians. For instance, in fractions study of students' learning, Steffe and his colleagues (e.g., Steffe & Olive, 2010) conducted teaching experiment with children to understand 'mathematics of children' that comprise 'mathematics for children', whereas the Rational Numbers Project (Behr et al., 1997) has analyzed mathematical knowledge for fractions (as a system of rational numbers) in terms of the strategies students use to solve tasks that are representative of a conceptual domain. The Rational Numbers Project based their study of rational numbers on Kieren's (1976) work which breaks the rational number into subconstructs- part-whole, quotient, ratio number, operator, and measure and have suggested that a complete understanding of rational number requires an understanding of each of those subconstructs separately and also an understanding of the relationships among the subconstructs. Furthermore, the perspective from the former group originated from biology in that they try to account human knowledge as viability to fit to the constraints (cf. neurobiology), whereas the one from the latter group originated from physics

[4] From now on, we will call the scholars who studied teacher knowledge as interiorized scheme as the first group, and the scholars who studied under knowledge-in-pieces perspective as the second group.

in that they seem to account human knowledge, also as scheme, but as interconnected sets of nervous systems (cf. neurology or neuroscience). Research methods under the second perspective investigate teachers' mathematical knowledge are diverse and different from the first group as the principle of knowledge in pieces reflects on this group's studies of teacher knowledge; whereas teaching experiment (Steffe & Thompson, 2000) was the only method they used, the latter group used a variety of methods. Some researchers (Behr et al., 1997; Post et al., 1991) provided teachers with tasks fairly representing the range of knowledge and reasoning in specific conceptual domains, and inferred knowledge elements through survey items and interviews by examining teachers' strategies in solving non-routine tasks; others (Gutstein & Mack, 1998; Izsák, 2008; Izsák et al., 2008; Lehrer & Franke, 1992) have concentrated on particular cognitive structure — mathematical knowledge for teaching — among knowledge, goals, and beliefs, and tried to infer knowledge elements in further grain size in the context of classroom teaching or tutoring. Even though their research methods and data were different one another, their ultimate goal seems to provide more detailed knowledge elements for teacher knowledge.

4. FRACTIONS STUDIES UNDER THE TWO PERSPECTIVES

Despite the rise of the research on teaching from the both groups, there is little research that discusses mathematical knowledge for teaching focusing on a specific subject area. Fraction, with whole number, has been received researchers' attention as a site for their studies for MKT because of abundant research results on children's mathematical thinking and learning on the area. Nevertheless, Tzur's (1999a, 1999b, 2003) study is the only research that discusses mathematical knowledge for teaching in fraction content among the first group. He designed the study on the basis of the *Reorganization Hypothesis* (Olive & Steffe, 2010); children can construct/reorganize fraction schemes from whole number schemes. Tzur, as a teacher and researcher, constructed the second order model of two children's construction of improper fraction scheme under teaching experiment (Steffe & Thompson, 2000), and developed a sequence of stages by interacting with children: equi-partitioning scheme, partitive fraction scheme, iterative fraction scheme, and reversible fraction scheme[5]. Moreover, he specified teachers' role in promoting students' construction of improper fraction scheme. He identified one type of teacher knowledge, generating tasks and questions, in engendering students' learning. In the study, he explained three types of tasks – initial, reflective, and anticipatory, which correspond to the three parts of a scheme by von Glasersfeld:

[5] For the purpose of this paper, we will not elaborate the characteristics of each scheme.

> An initial task is established when learners assimilate a teacher's task using the recognition template of their established schemes, set a general goal in the process of assimilation, and execute the scheme's activity to reach their goal. Building on the actions and language from learners' work on initial tasks, the teacher poses reflective tasks that challenge the learners to reprocess mental records of their activities. Finally anticipatory task is to challenge learners to operate mentally instead of to actually carry out the activities, so that they establish an anticipatory scheme (Tzur, 1999a, pp. 412–413).

He further highlighted an advantage of promoting fraction learning via iterating activity prior to splitting, and this provided an important message to traditional mathematics curriculum that have started fraction learning from splitting activity such as paper-folding. Confrey (1994) supported the importance of splitting activities since her teaching experiment with children showed that such activities promoted her students to generate multiplicative reasoning. It is needless to say that splitting is crucial in learning more advanced fraction concept like fraction multiplication. However, the Fractions Project noted that fractions are both additive and multiplicative quantities, and splitting activities are insufficient for generating additive structure and iterative activities serve that role. When students build up fraction only through splitting operations without iteration, for instance, the students may only think of fraction with part-whole relationship, for example, that "4/5" means "four out of five". This understanding will constrain the students' reasoning when they face with improper fractions. Under this perspective, however, we would argue that further research needs to be conducted with an explicit aim of determining teachers' understanding of what they consider to be essential in students' learning. For instance, since studies have found the importance of reasoning with nested unit structure in promoting children's learning, researchers can study whether teachers conceptualize the unit structure into a deeper level (such as two levels of units or three levels of units structure) so that they can understand students' reasoning.

Whereas little study did explicitly deal with teacher knowledge under the first group, there are some studies that tried to examine teachers' deeper unit structure under the second group; Behr et al. (1997) interviewed 30 preservice elementary teachers to explore strategies on working with tasks focused on one of the rational number subconstructs, operator. They provided conceptual unit analysis based on the conceptual units structure (Behr, Harel, Post & Lesh, 1994) and decomposed the operator subconstruct into another subconstructs: duplicator and partition-reducer, stretcher and shrinker, and multiplier and divisor. They found that teachers were most likely to use the first two strategies to solve the task of 3/4 times '8 bundles of 4 sticks'. In duplicator/partition-reducer (DPR), teachers partitioned (similar to sharing interpretation of division) eight by the quantity of denominator four and reduced into one of four partitioned pieces, then duplicated the one that was a unit with size of the numerator, three. On the other hand, in stretcher/shrinker, the teachers measured out (now operator 3/4 has an quotitive effect of division) four sticks in eight bundles by the quantity of the denominator, four and shrinked by taking 1 from each of eight bundles of four groups, by iterating each one by the size of the numerator, three and then measured it back by groups of 4 resulting in the final answer of 6 bundles of 4 sticks (i.e., stretching). For both strategies, orders are not important; i.e., teachers may duplicate first then partition and reduce, and stretch then shrink. DPR differs from SS in that DPR operates on the 'number' of embedded units and SS operates on the 'size' of embedded units. Moreover SS requires distributive reasoning. Behr et al. found that the teachers who attended to the bundles of sticks tried to operate on the number of units in a unit of units. However, the teachers who attended to the sticks in bundles tried to operate on the sizes of the units in a unit of units. Whether they operate on the size or the number of embedded units, such reasoning obviously requires teachers' attention to the three levels of units.

Unlike the previous study that was based on semantic analysis of unit structure that mathematicians formulated, and then examined teachers' knowledge without children, the following two studies examined teachers' knowledge in a classroom context by coordinating their responses with their children's responses. Izsák (2008) pointed out the importance of studying mathematical knowledge for teaching in a classroom context; teachers may use knowledge when responding to interview questions or survey items designed to measure teachers' knowledge, but they may not use that knowledge when responding to their students' thinking in the class. He utilized Steffe's (1988, 1992, 1994) study of unit structure, which is based on children's construction of composite units to analyze teachers' nested units structure. He examined two middle school teachers' knowledge of fraction multiplication as they interpreted students' work with drawn representation. In the study, Izsák found that teachers' attention to three levels of units were necessary but insufficient in adapting linear or area representations in their teaching. Even though one teacher could attend to the three levels of units in working with fraction multiplication tasks, she could not respond to some of her students' representations since she could not flexibly attend to the units distributed across in the students' drawings. Izsák also showed how different types of knowledge elements- teachers' nested unit structure, pedagogical purposes for using drawing representation, and lesson goals- played a significant role in forming teachers' mathematical knowledge for teaching; for instance, a teacher whose purpose was to illustrate an answer of algorithm tended to use it to explain the solution, whereas a teacher whose pedagogical purpose was to deduce a computation procedure tended to find the pattern from the structure of drawing without attending to three levels of unit structures embedded in the drawing.

In addition, Izsák et al. (2008) explored both teacher and student knowledge of fraction addition using number lines in one classroom. Izsák et el. conducted interviews with one teacher and several pairs of students in the class in which they were asked to solve problems and to explain their understanding of different classroom events as prompted by video of the classroom. From detailed analysis of interactions between the teacher and one student, Izsák et al. determined that drawings can lead to miscommunication between the teachers and the students. Each of different perspectives of fractions the teacher and her student used to solve problems confounded each other's understanding of fractions. She concentrated more on the final image showing size of "amount" than on processes for partitioning units she perceived to be fixed. For example, when she used a number line to represent fraction such as 3/5, she first marked '0' and '1' on each edge of the number line and always drew tick marks from left to right with equal size, and circled the location where 3/5 lied. As a result of her pattern of drawing fraction, she often changed the location of the fixed whole whenever her final point, say 5/5, was little off the original whole that she first marked. Her strategy confused one of her students, and she marked 5/5 right beside of 1, and marked 6/5 at 1. Some teachers who perceived numbers on a number line only as location not as length could not make sense of using the number line to represent fraction multiplication. When they were asked to find correct representations showing fraction multiplications on the number line, some said they were confused because they always thought of number line as one indicating location. Ms. Reese's and her student, Sonya's inattention to the unit whole also led them to interpret fraction as something out of something: fraction as part-whole relationship. When students think of fraction only as part-whole, they will face difficulty in dealing with improper fractional quantities.

5. CONCLUSION

Throughout the paper, we have discussed two perspectives of mathematical knowledge for teaching both of which are grounded in constructivism: teacher knowledge as interiorized schemes and teacher knowledge from knowledge-in-pieces perspective. To summarize the major similarities and disparities between the two parties, they both put emphasis on the role of students' prior knowledge in learning and are cautious about calling students' mathematical behaviors as misconceptions in that somehow coherent reasoning exists behind the students' behaviors. To them, those are the knowledge that could be refined and reorganized toward more sophisticated form of the students' mathematical knowledge. Despite that, their focuses on exploring teacher knowledge were different. Those who conceive of teacher knowledge as interiorized schemes focus more on how teachers' mathematical knowledge for teaching evolves as the teachers try to understand the ways in which their students' make sense of problem situations. By comparing their hypothetical model of students' knowledge with that of actual students' one, their mathematical knowledge for teaching becomes interiorized schemes. On the other hand, those who are studying teacher knowledge under the knowledge-in-pieces perspective seek to delineate fine-grained and diverse knowledge elements that account for mathematical knowledge for teaching.

Whether they were under the first group or the second group, similar findings were reported across the studies about teachers' knowledge of fractions; first, 'what it means to understand fraction well' for teachers turned out to be more than just explaining why, for example, 'invert and multiply' algorithm works for fraction division; moreover, coherent fractional reasoning would be supported by the conceptions that are often not associated with fractions (e.g. splitting and iteration, DPR and SS strategies, distributive reasoning); lastly, as constructing conceptual units (or a nested unit structure) is necessary for learners to meaningfully construct fractions, a teacher, as an observer to analyze or construct a model of children's thinking, needs to be able to employ deeper unit structures in tracking the children's mathematical reasoning as well as their own problem solving.

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