LAGRANGE MULTIPLIER METHOD FOR SOLVING VARIATIONAL INEQUALITY IN MECHANICS

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Abstract. Lagrange multiplier method for solving the contact problem in elasticity is considered. Based on lower semicontinuity of sensitivity functional we prove the convergence of modified dual scheme to corresponding saddle point.

Introduction

Duality methods based on classical schemes for constructing Lagrangian functionals are widely used for solving variational inequalities in mechanics. In general it is not able to prove their convergence to the corresponding saddle point. For coercive variational inequalities the convergence with respect to the primal variable can be shown only. It is provided under assumption that the step size of dual variable is sufficiently small. For semicoercive variational inequalities it is not able to use classical Lagrangian functional because of the quadratic form of the functional to be minimized has a nontrivial null space. To remedy this situation, a modified Lagrangian functionals are examined. In order to prove that duality method based on modified Lagrangian functional converges to a saddle point we show that the corresponding sensitivity functional is a weakly lower semicontinuous in the original Hilbert space.

1. Semicoercive contact problem of elasticity

Consider a two-dimensional contact problem between an trapezoid elastic body Ω and an absolutely rigid support (Figure 1).

The boundary Γ of domain Ω is equal to $\overline{\Gamma}_O \bigcup \overline{\Gamma}_K \bigcup \overline{\Gamma}_P$, where Γ_O, Γ_K and Γ_P are open pair wise disjoint subset of Γ such that mes Γ_O and mes Γ_K are positive.
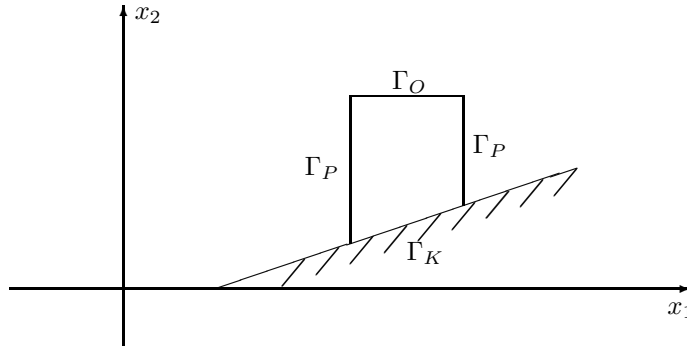
FIGURE 1. Elastic body and rigid support

For the displacement vector $u = (u_1, u_2)$ we define the strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2,$$

and the stress tensor $\sigma_{ij}(u) = c_{ijpm}\varepsilon_{pm}(u)$, where $i, j, p, m = 1, 2$; $c_{ijpm} = c_{jipm} = c_{pmij}$; summation is implied over repeated indexes.

Denote $n = (n_1, n_2)$ is a unit outward normal vector on Γ; $u_n = u \cdot n$; $u_t = u - u_n n$; $\sigma_i(u) = \sigma_{ij}(u)n_j$ for $i = 1, 2$; $\sigma(u) = (\sigma_1(u), \sigma_2(u))$; $\sigma_n(u) = \sigma_{ij}(u)n_i n_j$; $\sigma_t(u) = \sigma(u) - \sigma_n(u)n$; $\sigma_{ij,j}(u) = \frac{\partial \sigma_{ij}(u)}{\partial x_j}$, $i, j = 1, 2$.

For given vector-functions $F = (f_1, f_2)$ and $T = (T_1, T_2)$, consider the boundary value problem [5, 8]

$$
\begin{aligned}
-\sigma_{ij,j}(u) &= f_i \quad \text{in } \Omega, \, i = 1, 2, \\
u_n = 0, \sigma_t(u) &= 0 \quad \text{on } \Gamma_O, \\
\sigma_{ij}(u)n_j &= T_i \quad \text{on } \Gamma_P, \, i = 1, 2, \\
\end{aligned}
$$

(1)

$$u_n \leq 0, \sigma_t(u) \leq 0, u_n \sigma_n(u) = 0, \sigma_t(u) = 0 \quad \text{on } \Gamma_K.$$

Define the set (see Figure 1)

$$K = \{v \in [H^1(\Omega)]^2 \colon v_n = 0 \text{ on } \Gamma_O, v_n \leq 0 \text{ on } \Gamma_K\}.$$

Assume that $c_{ijpm} \in L_\infty(\Omega)$, $i, j, p, m = 1, 2$; $F \in [L_2(\Omega)]^2$; and $T \in [L_2(\Gamma_P)]^2$. Suppose that the nonlinear boundary value problem has a solution $u \in [H^2(\Omega)]^2$. Then it can be shown that u satisfies the variational inequality [5, 8]

$$(2) \quad a(u, v - u) - \int_\Omega f_i(v_i - u_i)d\Omega - \int_{\Gamma_P} T_i(v_i - u_i)d\Gamma \geq 0, \quad \forall v \in K,$$

where $a(u, v) = \int_\Omega c_{ijpm}\varepsilon_{pm}(u)\varepsilon_{ij}(v)d\Omega$.

The variational inequality (2) can be present in the following way

$$(3) \qquad \begin{cases} J(v) = \frac{1}{2}a(v,v) - \int_\Omega f_i v_i d\Omega - \int_{\Gamma_P} T_i v_i d\Gamma - \min_{v \in K}. \end{cases}$$

The kernel R of the bilinear form $a(u,v)$ is not empty in $[H^1(\Omega)]^2$ and consists of the vector function $\rho = (\rho_1, \rho_2)$, where $\rho_1 = a_1 - bx_2$, $\rho_2 = a_2 + bx_1$; and a_1, a_2 and b are arbitrary fixed scalars.

Assuming that there exists a constant $\alpha_0 > 0$ such that

$$c_{ijpm}\varphi_{ij}\varphi_{pm} \geq \alpha_0 \varphi_{ij}\varphi_{ij} \quad \text{on } \Omega$$

(φ_{ij} are arbitrary), the quadratic form $a(v,v)$ is positive definite on the orthogonal complement $R^\perp$ of R with respect to inner product [3]

$$(4) \qquad \langle u, v \rangle = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} d\Omega + \left(\int_\Omega u_i d\Omega \right)\left(\int_\Omega v_i d\Omega \right).$$

Let us define the space

$$W = \{ v \in [H^1(\Omega)]^2 \colon v_n = 0 \text{ on } \Gamma_O \}.$$

The subspace $\widetilde{R} = W \bigcap R$ is a set of virtual rigid displacements (i.e., displacements of Ω as an absolutely rigid body with the strict (two-sides) conditions being preserves). According to Figure 1, the set $\widetilde{R}$ is a one-dimensional set and looks in the following way

$$\widetilde{R} = \{ \rho = (\rho_1, \rho_2) \colon \rho_1 = a, \rho_2 = 0 \},$$

where a is an arbitrary constant.

Since Γ_K is not parallel to Γ_O, a unit outward normal vector n on Γ_K satisfies the condition $n_1 \neq 0$. According to Figure 1, $n_1 > 0$ on Γ_K. Then

$$(5) \qquad \widetilde{R} \cap K = \{ \rho = (a, 0) \colon a \leq 0 \}.$$

It is shown in [5] that the form

$$(6) \qquad \left(\int_\Omega c_{ijpm}\varepsilon_{ij}(v)\varepsilon_{pm}(v) d\Omega + \left(\int_{\Gamma_K} v_n d\Gamma \right)^2 \right)^{\frac{1}{2}}$$

is a norm in the space W. This norm is equivalent to norm corresponding to scalar product (4).

From (5) the solvability condition of problem (3) follows [1, 5]

$$(7) \qquad \int_\Omega F_1 d\Omega + \int_{\Gamma_P} T_1 d\Gamma > 0.$$

2. The duality method

For arbitrary $m \in L_2(\Gamma_K)$ we introduce the set

$$K_m = \{v \in W : v_n \leq m \text{ on } \Gamma_K\}$$

and for all functions $m \in L_2(\Gamma)$, define the sensitivity functional

$$\chi(m) = \begin{cases} \inf_{v \in K_m} J(v), & \text{if } K_m \neq \varnothing, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to see that if a function $m \in L_2(\Gamma)$ is lower bounded on Γ_K, the corresponding set K_m is not empty and $\inf_{v \in K_m} J(v) > -\infty$ [1]. The set K_m can be empty if $m \in L_2(\Gamma_K) \setminus H^{1/2}(\Gamma_K)$ and not lower bounded on Γ_K (see [7, 9]). Then $\chi(m)$ is a proper convex functional on $L_2(\Gamma_K)$, but it's effective domain $\operatorname{dom}\chi = \{m \in L_2(\Gamma_K) : \chi(m) < +\infty\}$ does not coincide with $L_2(\Gamma_K)$. Notice that $\operatorname{dom}\chi$ is a convex but not closed set. In this case, $\overline{\operatorname{dom}\chi} = L_2(\Gamma_K)$.

Remark. In our earlier publications [11, 12, 14], it was erroneously assumed that the effective domain of the sensitivity functional is identical to $L_2(\Gamma_K)$. In this paper, we correct this inaccuracy. All the theorems on the convergence of duality methods presented in those earlier publications remain valid.

We define the following functional on the space $W \times L_2(\Gamma_K) \times L_2(\Gamma_K)$

$$K(v, l, m) = \begin{cases} J(v) + \frac{1}{2r} \int_\Gamma ((l+m)^2 - l^2) d\Gamma, & \text{if } v_n \leq m \text{ on } \Gamma_K, \\ +\infty, & \text{otherwise,} \end{cases}$$

and modified Lagrangian functional $M(v, l)$ on space $W \times L_2(\Gamma_K)$

$$M(v, l) = \inf_m K(v, l, m) = J(v) + \frac{1}{2r} \int_\Gamma \left(((l + rv_n)^+)^2 - l^2 \right) d\Gamma.$$

Here $r > 0$ is a constant, $(l + rv_n)^+ = \max\{0, l + rv_n\}$.

Let us introduce the modified dual functional

$$(8) \quad \underline{M}(l) = \inf_v M(v, l) = \inf_v \left\{ J(v) + \frac{1}{2r} \int_\Gamma \left(((l + rv_n)^+)^2 - l^2 \right) d\Gamma \right\}.$$

Functional $\underline{M}(l)$ has the another presentation [11, 14]

$$(9) \quad \underline{M}(l) = \inf_m \left\{ \chi(m) + \int_\Gamma lm d\Gamma + \frac{r}{2} \int_\Gamma m^2 d\Gamma \right\}.$$

Definition 1. A point $(\overline{v}, \overline{l}) \in W \times L_2(\Gamma_K)$ is called a saddle point of the modified Lagrangian functional $M(v, l)$ if

$$M(\overline{v}, l) \leq M(\overline{v}, \overline{l}) \leq M(v, \overline{l}), \quad \forall v \in W, \ \forall l \in L_2(\Gamma_K).$$

If $(\overline{v}, \overline{l})$ is a saddle point of $M(v, l)$, then $\overline{v}$ is a solution of problem (3) and $\overline{l}$ is a solution of dual problem

$$(10) \qquad \underline{M}(l) - \max_{l \in L_2(\Gamma)},$$

and, moreover, if $\overline{v} \in H^2(\Omega)$, then $\overline{l} = \sigma_n(\overline{v})$ [11, 13].

Theorem 1. *Let* $\overline{m} \in L_2(\Gamma_K)$ *does not belong to* $\operatorname{dom} \chi$. *Then, for every sequence* $\{m^i\} \subset \operatorname{dom} \chi$ *such that* $\lim_{i \to \infty} \|m^i - \overline{m}\|_{L_2(\Gamma_K)} = 0$, *it holds that* $\lim_{i \to \infty} \chi(m^i) = +\infty$.

Proof. Take a function $\overline{m}$ that does not belong to $\operatorname{dom} \chi$ and consider an arbitrary sequence $\{m^i\} \subset \operatorname{dom} \chi$ such that $\lim_{i \to \infty} \|m^i - \overline{m}\|_{L_2(\Gamma_K)} = 0$. We can prove the existence of point $v^i = \operatorname*{argmin}_{v \in K_{m^i}} J(v)$ for $i = 1, 2, \dots$ if condition (7) is fulfilled. In fact, take an arbitrary $v \in W$ and set

$$\overline{v}_1 = \frac{1}{n_1 \operatorname{mes} \Gamma_K} \int_{\Gamma_K} v_n d\Gamma.$$

Consider the vector $\overline{v} = (\overline{v}_1, 0) \in \widetilde{R}$. Let $\widetilde{v} = v - \overline{v} = (v_1 - \overline{v}_1, v_2)$. We have

$$\int_{\Gamma_K} \widetilde{v}_n d\Gamma = \int_{\Gamma_K} \left((v_1 - \overline{v}_1) n_1 + v_2 n_2 \right) d\Gamma$$

$$= \int_{\Gamma_K} (v_1 n_1 + v_2 n_2) d\Gamma - \frac{1}{\operatorname{mes} \Gamma_K} \int_{\Gamma_K} \left(\int_{\Gamma_K} v_n d\Gamma \right) d\Gamma = 0.$$

In this case, norm (6) of $\widetilde{v}$ is

$$\left(\int_{\Omega} c_{ijpm} \varepsilon_{ij}(\widetilde{v}) \varepsilon_{pm}(\widetilde{v}) d\Omega \right)^{\frac{1}{2}} = \left(a(\widetilde{v}, \widetilde{v}) \right)^{\frac{1}{2}}.$$

On the space W, we define the linear functional

$$L_1(v) = \int_{\Omega} F_i v_i d\Omega + \int_{\Gamma_P} T_i v_i d\Gamma.$$

Then, for an arbitrary $v \in K_{m^i}$ it holds that

$$(11) \qquad J(v) = \frac{1}{2} a(\widetilde{v}, \widetilde{v}) - L_1(\widetilde{v}) - L_1(\overline{v})$$

$$= \frac{1}{2} a(\widetilde{v}, \widetilde{v}) - L_1(\widetilde{v}) - \overline{v}_1 \left(\int_{\Omega} F_1 d\Omega + \int_{\Gamma_P} T_1 d\Gamma \right).$$

Since
$$(12)$$
$$\overline{v}_1 = \frac{1}{n_1 \operatorname{mes} \Gamma} \int_{\Gamma_K} v_n d\Gamma \le \frac{1}{n_1 \operatorname{mes} \Gamma} \int_{\Gamma_K} m^i d\Gamma \le \frac{1}{n_1 (\operatorname{mes} \Gamma)^{\frac{1}{2}}} \|m^i\|_{L_2(\Gamma_K)},$$

condition (7) as applied to $v \in K_{m^i}$ implies that

$$J(v) \longrightarrow +\infty \text{ as } \|v\|_{[H^1(\Omega)]^2} \longrightarrow \infty,$$

that is v^i exists.

We show that $\lim\limits_{i\to\infty} \|v^i\|_W = +\infty$. Assume the contrary, that is, there exists a subsequence $\{v^{i_j}\}$ and a constant $c > 0$ such that $\|v^{i_j}\|_W \le C$ for all i_j. Since $[H^1(\Omega)]^2 \subset [H^{1/2}(\Gamma)]^2$, we have $\|v^{i_j}\|_{[H^{1/2}(\Gamma)]^2} \le C_1$, where $c_1 > 0$ is a constant. Moreover, $\{v^{i_j}\}$ is a compact sequence in $[L_2(\Gamma)]^2$. Let $\widehat{v} \in [H^{1/2}(\Gamma)]^2$ be a weak limit point of this sequence. Without loss of generality, we can assume that $\widehat{v}$ is a weak limit of $\{v^{i_j}\}$. Then $\{v^{i_j}\}$ strongly (that is, in the norm) converges to $\widehat{v}$ in $[L_2(\Gamma)]^2$ and, hence, in $[L_2(\Gamma_K)]^2$. Since $v_n^{i_j} \le m^{i_j}$, we have $\widehat{v}_n \le \overline{m}$ on Γ_K, which implies that $K_{\overline{m}} \ne \varnothing$. This contradiction proves that $\lim\limits_{i\to\infty} \|v^i\|_W = +\infty$.

Since $\lim\limits_{i\to\infty} \|m^i - \overline{m}\|_{L_2(\Gamma_K)} = 0$, then it follows from (12), that

$$(13) \qquad \overline{v}_1^i \le c_2, \quad c_2 > 0 \text{ — const}, \quad i = 1, 2, \ldots.$$

Now, the condition $\lim\limits_{i\to\infty} \|v^i\|_W = +\infty$ and (11) imply that $\lim\limits_{i\to\infty} J(v^i) = +\infty$ and $\lim\limits_{i\to\infty} \chi(m^i) = +\infty$. □

Theorem 2. *Let $\overline{m} \in L_2(\Gamma_K)$ belongs to $\operatorname{dom}\chi$. Then, for every sequence $\{m_i\} \subset \operatorname{dom}\chi$ converging to $\overline{m}$ in $L_2(\Gamma_K)$, it holds that $\varliminf\limits_{i\to\infty} \chi(m^i) \ge \chi(\overline{m})$.*

Proof. Let $\{m^i\} \subset \operatorname{dom}\chi$ and $\lim\limits_{i\to\infty} \|m^i - \overline{m}\|_{L_2(\Gamma_K)} = 0$. From the sequence $\{m^i\}$, we extract a subsequence $\{m^{i_j}\}$ for which

$$\lim_{j\to\infty} \chi(m^{i_j}) = \varliminf_{i\to\infty} \chi(m^i).$$

Consider the subsequence $\{v^{i_j}\}$, where $v^{i_j} = \operatorname*{argmin}\limits_{v \in K_{m^{i_j}}} J(v)$. We can suppose that $\{v^{i_j}\}$ is a bounded sequence in $[H^1(\Omega)]^2$ (otherwise, from (11), (13) it follows, that $\lim\limits_{j\to\infty} \chi(m^{i_j}) = +\infty$, and the theorem is proved). Since $[H^1(\Omega)]^2 \subset [H^{1/2}(\Gamma)]^2$, then the sequence $\{v^{i_j}\}$ is also bounded in $[H^{1/2}(\Gamma)]$. It follows that $\{v^{i_j}\}$ is a weakly compact sequence in $[H^{1/2}(\Gamma)]^2$. Let $\widehat{v}$ be its weak limit point. Without loss of generality, we can assume that $\{v^{i_j}\}$ is a weakly converging sequence; that is, $\widehat{v}$ is a weak limit point of $\{v^{i_j}\}$ in $[H^{1/2}(\Gamma)]^2$. Since $[H^{1/2}(\Gamma)]$ is compactly embedded in $[L_2(\Gamma)]$ and $[L_2(\Gamma)]^2 \subset [H^{-1/2}(\Gamma)]^2$, then $\{v^{i_j}\}$ converges to $\widehat{v}$ in $[L_2(\Gamma)]^2$ and, hence, in $[L_2(\Gamma_K)]^2$. We have $m^{i_j} \longrightarrow \overline{m}$ in $L_2(\Gamma_K)$, $v^{i_j} \longrightarrow \widehat{v}$ in $[L_2(\Gamma_K)]^2$, and $v_n^{i_j} \le m^{i_j}$ on Γ_K. Then $\widehat{v}_n \le \overline{m}$ on Γ_K.

Let $\widehat{\widehat{v}} = \operatorname*{argmin}\limits_{v = \widehat{v} \text{ on } \Gamma} J(v)$. We have

$$J(v^{i_j}) - J(\widehat{\widehat{v}}) = a(\widehat{\widehat{v}}, v^{i_j} - \widehat{\widehat{v}}) - \int_\Omega F_s(v_s^{i_j} - \widehat{\widehat{v}}_s)d\Omega - \int_{\Gamma_P} T_s(v_s^{i_j} - \widehat{\widehat{v}}_s)d\Gamma$$

$$+ \frac{1}{2}a(v^{i_j} - \widehat{\widehat{v}}, v^{i_j} - \widehat{\widehat{v}})$$

$$= \langle \mu, v^{i_j} - \widehat{\widehat{v}} \rangle - \int_{\Gamma_P} T_s(v_s^{i_j} - \widehat{\widehat{v}}_s)d\Gamma + \frac{1}{2}a(v^{i_j} - \widehat{\widehat{v}}, v^{i_j} - \widehat{\widehat{v}}),$$

where $\langle \mu, v \rangle = a(\widehat{\widehat{v}}, v) - \int_\Omega F_s v_s d\Omega$ and $\mu \in [H^{-\frac{1}{2}}(\Gamma)]^2$ [7, 9].

Since $\{v^{i_j}\}$ weakly converges to $\widehat{v}$ in $[H^{1/2}(\Gamma)]^2$, we have

$$\lim_{j \to \infty} \langle \mu, v^{i_j} - \widehat{\widehat{v}} \rangle = 0.$$

From this relation and from the convergence of $\{v^{i_j}\}$ to $\widehat{v}$ in $[L_2(\Gamma)]^2$, we conclude that

$$\lim_{j \to \infty} \chi(m^{i_j}) = \lim_{j \to \infty} J(v^{i_j}) \geq J(\widehat{\widehat{v}}) \geq \chi(\overline{m})$$

or

$$\varliminf_{i \to \infty} \chi(m^{i_j}) \geq \chi(\overline{m}). \qquad \Box$$

Theorems 1 and 2 imply that the functional $\chi(m)$ is lower semicontinuous on $L_2(\Gamma_K)$. Since $\chi(m)$ is convex functional, it is also weakly lower semicontinuous on $L_2(\Gamma_K)$.

For an arbitrary $l \in L_2(\Gamma_K)$, we consider the functional

$$F_l(m) = \chi(m) + \int_{\Gamma_K} lm d\Gamma + \frac{r}{2} \int_{\Gamma_K} m^2 d\Gamma,$$

where $r > 0$ is a constant. Then the dual functional $\underline{M}(l)$ has the form

$$\underline{M}(l) = \inf_{m \in L_2(\Gamma_K)} F_l(m).$$

For a fixed $l \in L_2(\Gamma_K)$, we examined the functional $F_l(m)$ for $m \in L_2(\Gamma_K)$.

Theorem 3. *The functional $F_l(m)$ is coercive in $L_2(\Gamma_K)$.*

Proof. Since $\chi(m)$ is a lower semicontinuous functional, then the epigraph of sensitivity functional

$$\text{epi}\,\chi \equiv \{(v, a) \in L_2(\Gamma_K) \times R \colon \chi(v) \leq a\}$$

is a convex closed set in $L_2(\Gamma_K) \times R$. According Mazur separation theorem [4, p. 164] there are $\psi \in L_2(\Gamma_K)$ and $d \in R$, such that

$$\int_{\Gamma_K} \psi m d\Gamma + \chi(m) + d \geq 0, \qquad \forall m \in \text{dom}\,\chi.$$

Hence $F_l(m) \longrightarrow +\infty$ under $\|m\|_{L_2(\Gamma_K)} \longrightarrow \infty$. $\qquad \Box$

It is easy to see that $F_l(m)$ is a weakly lower semicontinuous functional. Then it follows from Theorem 3 that there is unique $m(l) = \underset{m \in L_2(\Gamma_K)}{\text{argmin}}\ F_l(m)$. It is obvious that $m(l) \in \text{dom}\,\chi$. Let us introduce the function

$$Q(l) = \chi(m(l)), \quad \forall l \in L_2(\Gamma_K).$$

Theorem 4. *The function $Q(l)$ is continuous in $L_2(\Gamma_K)$.*

Proof. For a given element $l \in L_2(\Gamma_K)$, the inequality

$$\chi(m(l)) + \int_{\Gamma_K} lm(l)d\Gamma + \frac{r}{2}\int_{\Gamma_K}(m(l))^2d\Gamma + \frac{r}{2}\|m(l) - m\|^2_{L_2(\Gamma_K)}$$
$$\leq \chi(m) + \int_{\Gamma_K} lmd\Gamma + \frac{r}{2}\int_{\Gamma_K} m^2d\Gamma$$

is fulfilled for every $m \in L_2(\Gamma_K)$.

Choose two elements l_1, l_2 on $L_2(\Gamma_K)$. Let $m_1 = m(l_1)$ and $m_2 = m(l_2)$. Last inequality implies the relations

$$(14) \qquad \chi(m_1) + \int_{\Gamma_K} l_1 m_1 d\Gamma + \frac{r}{2}\int_{\Gamma_K} m_1^2 d\Gamma + \frac{r}{2}\|m_1 - m_2\|^2_{L_2(\Gamma_K)}$$
$$\leq \chi(m_2) + \int_{\Gamma_K} l_1 m_2 d\Gamma + \frac{r}{2}\int_{\Gamma_K} m_2^2 d\Gamma,$$

$$(15) \qquad \chi(m_2) + \int_{\Gamma_K} l_2 m_2 d\Gamma + \frac{r}{2}\int_{\Gamma_K} m_2^2 d\Gamma + \frac{r}{2}\|m_1 - m_2\|^2_{L_2(\Gamma_K)}$$
$$\leq \chi(m_1) + \int_{\Gamma_K} l_2 m_1 d\Gamma + \frac{r}{2}\int_{\Gamma_K} m_1^2 d\Gamma.$$

Combining (14) and (15), we find that

$$(16) \qquad r\|m_1 - m_2\|^2_{L_2(\Gamma_K)} \leq \int_{\Gamma_K}(l_1 - l_2)(m_2 - m_1)d\Gamma.$$

From (16), we derive

$$(17) \qquad \|m_1 - m_2\|_{L_2(\Gamma_K)} \leq \frac{1}{r}\|l_1 - l_2\|_{L_2(\Gamma_K)}.$$

Relations (14) and (15) also imply the two-sided inequality

$$\int_{\Gamma_K} l_1(m_1 - m_2)d\Gamma + \frac{r}{2}\int_{\Gamma_K}(m_1^2 - m_2^2)d\Gamma \leq \chi(m_2) - \chi(m_1)$$
$$\leq \int_{\Gamma_K} l_2(m_1 - m_2)d\Gamma + \frac{r}{2}\int_{\Gamma_K}(m_1^2 - m_2^2)d\Gamma.$$

Let l_2 approaches l_1 in $L_2(\Gamma_K)$. The above two-sided inequality and relation (17) lead to the equality $\lim_{l_2 \to l_1} Q(l_2) = Q(l_1)$. $\qquad\square$

From Theorem 4 and inequality (17) it follows that convex functional $(-\underline{M}(l)) = -\inf_{m \in L_2(\Gamma_K)} F_l(m)$ is a continuous functional in $L_2(\Gamma_K)$.
Consequently, its subdifferential is not empty for all $l \in l_2(\Gamma_K)$, that is, $\partial(-\underline{M}(l)) \neq \varnothing$ (see [2]).

Theorem 5. *The dual functional $\underline{M}(l)$ is Gáteaux differentiable in $L_2(\Gamma_K)$ and its derivative $\nabla \underline{M}(l)$ satisfies the Lipschitz condition with the constant $\frac{1}{r}$, that is, for all $l_1, l_2 \in L_2(\Gamma_K)$, it holds that*

$$\|\nabla \underline{M}(l_1) - \nabla \underline{M}(l_2)\|_{L_2(\Gamma_K)} \leq \frac{1}{r}\|l_1 - l_2\|_{L_2(\Gamma_K)}.$$

Proof. Let $l \in L_2(\Gamma_K)$ and $-\tau \in \partial(-\underline{M}(l))$. For an arbitrary $\xi \in L_2(\Gamma_K)$, we have

(18) $$\underline{M}(\xi) \leq \underline{M}(l) + \langle \tau, \xi - l \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L_2(\Gamma_K)$.

Inequality (18) implies that

$$\chi(m(\xi)) + \int_{\Gamma_K} \xi m(\xi) d\Gamma + \frac{r}{2} \int_{\Gamma_K} (m(\xi))^2 d\Gamma$$

$$\leq \chi(m(l)) + \int_{\Gamma_K} l m(l) d\Gamma + \frac{r}{2} \int_{\Gamma_K} (m(l))^2 d\Gamma + \langle \tau, \xi - l \rangle$$

$$\leq \chi(m(\xi)) + \int_{\Gamma_K} l m(\xi) d\Gamma + \frac{r}{2} \int_{\Gamma_K} (m(\xi))^2 d\Gamma + \langle \tau, \xi - l \rangle.$$

Hence

$$\int_{\Gamma_K} m(\xi)(\xi - l) d\Gamma \leq \langle \tau, \xi - l \rangle, \quad \forall \xi \in L_2(\Gamma_K).$$

For arbitrary $h \in L_2(\Gamma_K)$ and $\beta > 0$, we set $\xi = l + \beta h$. Then

$$\int_{\Gamma_K} m(l + \beta h) h d\Gamma \leq \langle \tau, h \rangle, \quad \forall h \in L_2(\Gamma_K).$$

Sending β to zero and using (17), we obtain

(19) $$\int_{\Gamma_K} m(l) h d\Gamma = \langle \tau, h \rangle, \quad \forall h \in L_2(\Gamma_K).$$

In view of the uniqueness of $m(l)$, we conclude that functional $\underline{M}(l)$ is Gáteaux differentiable in $L_2(\Gamma_K)$ and $\nabla \underline{M}(l) = \tau = m(l)$ (see [2]). Inequality (17) completes the proof. $\square$

Consider the dual problem

(20) $$\underline{M}(l) - \max_{l \in L_2(\Gamma_K)}.$$

Assume that a solution u to problem (3) belongs to the class $[H^2(\Omega)]^2$ and mes $\widetilde{\Gamma}_K > 0$, where $\widetilde{\Gamma}_K = \{x \in \Gamma_K : \sigma_n(u) < 0\}$. Then the vector function u is the unique solution to problem (3), while the element $-\sigma_n(u)$ is the unique solution to the dual problem (20) (see [11]).

Since the gradient of the functional $\underline{M}(l)$ satisfies the Lipschitz condition, the dual problem can be solved by using the gradient method for maximizing a functional (see [6, 13])

(21) $$l^{s+1} = l^s + r m(l^s), \quad s = 0, 1, 2, \ldots,$$

where $l^0 \in L_2(\Gamma_K)$ is an arbitrary initial value,

$$m(l^s) = \nabla\underline{M}(l^s) = \underset{m \in L_2(\Gamma_K)}{\operatorname{argmin}} \left\{ \chi(m) + \int_{\Gamma_K} l^s m d\Gamma + \frac{r}{2} \int_{\Gamma_K} m^2 d\Gamma \right\},$$

and r is a constant.

Theorem 6. *The mapping $P(l) = l + rm(l)$ satisfies the following two conditions (see [11, Theorem 4]) :*

    (1) $P(-\sigma_n(u)) = -\sigma_n(u);$
    (2) $\|P(-\sigma_n(u)) - P(l)\|_{L_2(\Gamma_K)} < \| - \sigma_n(u) - l\|_{L_2(\Gamma_K)}, \quad \forall l \neq -\sigma_n(u).$

Theorem 7. *The sequence $\{l^s\}$ constructed by the gradient method (21) satisfies the limit equality $\lim_{s \to \infty} \|m(l^s)\|_{L_2(\Gamma)} = 0$.*

Proof. Since $\nabla\underline{M}(l)$ satisfies the Lipschitz condition, the equality

$$\underline{M}(l + h) - \underline{M}(l) = \int_0^1 \langle m(l + th), h \rangle dt$$

holds for all $h \in L_2(\Gamma_K)$ (see [6]). By analog with [13, p. 31], this implies the assertion of the theorem.

The gradient method (21) can be written as an algorithm for solving problem (3) (see [11]):

(22)

    (i) $u^{s+1} = \underset{v \in [H^1(\Omega)]^2}{\operatorname{argmin}} \left\{ J(v) + \frac{1}{2r} \int_{\Gamma_K} \left(((l^s + rv_n)^+)^2 - (l^s)^2 \right) d\Gamma \right\},$

    (ii) $l^{s+1} = l^s + r \max \left\{ -u_n^{s+1}, -\dfrac{l^s}{r} \right\},$

where $l^0 \in L_2(\Gamma_K)$.

Algorithm (22) converges with respect to the functional; that is,

$$\lim_{s \to \infty} J(u^s) = \min_{v \in K} J(v) = J(u).$$

As before, u is a solution to problem (3).

Indeed, from Theorem 6 it follows that

$$\| - \sigma_n(u) - l^{s+1}\|_{L_2(\Gamma_K)} < \| - \sigma_n(u) - l^s\|_{L_2(\Gamma_K)}.$$

It means that $\{l^s\}$ is a bounded sequence in $L_2(\Gamma_k)$. The functional $\chi(m)$ is weakly lower semicontinuous on $L_2(\Gamma_K)$, which yields

$$\varliminf_{s \to \infty} \left\{ \chi(m(l^s)) + \int_{\Gamma_K} l^s m(l^s) d\Gamma + \frac{r}{2} \int_{\Gamma_K} \left(m(l^s) \right)^2 d\Gamma \right\}$$
$$= \varliminf_{s \to \infty} \chi(m(l^s)) \geq \chi(0) = J(u).$$

On the other hand,

$$\underline{M}(l^s) = \chi(m(l^s)) + \int_{\Gamma_K} l^s m(l^s) d\Gamma + \frac{r}{2} \int_{\Gamma_K} \left(m(l^s) \right)^2 d\Gamma$$

$$= \inf_{m \in L_2(\Gamma_K)} \left\{ \chi(m) + \int_{\Gamma_K} l^s m d\Gamma + \frac{r}{2} \int_{\Gamma_K} m^2 d\Gamma \right\} \leq \chi(0), \quad s = 0, 1, 2, \ldots.$$

Therefore,

$$\overline{\lim_{s \to \infty}} \left\{ \chi(m(l^s)) + \int_{\Gamma_K} l^s m(l^s) d\Gamma + \frac{r}{2} \int_{\Gamma_K} \left(m(l^s) \right)^2 d\Gamma \right\} \leq \chi(0).$$

Consequently, there exists the limit

$$\lim_{s \to \infty} \left\{ \chi(m(l^s)) + \int_{\Gamma_K} l^s m(l^s) d\Gamma + \frac{r}{2} \int_{\Gamma_K} \left(m(l^s) \right)^2 d\Gamma \right\} = \chi(0) = J(u).$$

Now, Theorem 7 implies that

$$\lim_{s \to \infty} J(u^s) = \lim_{s \to \infty} \chi(m(l^s)) = \chi(0) = J(u).$$

The convergence of this algorithm with respect to the argument was examined in [10, 15]. □

3. Conclusion and discussion

The modified Lagrangian functional

$$M(v, l) = J(v) + \frac{1}{2r} \int_{\Gamma} \left\{ \left((l^k + rv_n)^+ \right)^2 - (l^k)^2 \right\} d\Gamma$$

is a convex with respect to v if l is fixed. But it is not strongly convex functional with respect to v because of kernel R of bilinear form $a(u, v)$ is not empty. Therefore it is a problem to find point u^{s+1} on step (i) of method (22). Consider an iterative method for solving problem (3) based on combining the modified Lagrangian functional with proximal regularization.

Choose an arbitrary initial point $(u^0, l^0) \in W \times H^{1/2}(\Gamma_K)$ and generated a sequence $\{(u^s, l^s)\}$ as follows.

(j) At the $(s + 1)th$ iteration step $(s = 0, 1, 2, \ldots)$, construct the functional

$$L_S(v) = M(v, l^s) + \frac{1}{2} \|v - u^s\|^2_{[L_2(\Omega)]^2}, \quad \forall v \in W,$$

and find a point $u^{s+1} \in W$ using the criterion

$$(23) \qquad \|u^{s+1} - \overline{u}^{s+1}\|_{[H^1(\Omega)]^2} \leq \sigma_s,$$

where $\overline{u}^{s+1} = \underset{v \in W}{\operatorname{argmin}} \, L_S(v)$, $\sigma_s > 0$, $\sum_{k=1}^{\infty} \sigma_s < \infty$.

(jj) Modify the dual variable l^{s+1} according to the formula

$$(24) \qquad l^{s+1} = (l^s + ru_n^{s+1})^+.$$

The regularizing addition $\frac{1}{2} \|v - u^s\|^2_{[L_2(\Omega)]^2}$ ensures the strong convexity of the functional $L_S(v)$ under minimization. This guarantees that the auxiliary problems

$$L_S(v, l^s) - \min_{v \in W},$$

are uniquely solvable and efficient optimization methods can be used for solving these problems.

Theorem 8. *Let the modified Lagrangian functional $M(v, l)$ has a nonempty set of saddle points and, moreover, such conditions are fulfilled*

(1) $\overline{u}^s \in [H^2(\Omega)]^2$,

(2) $\|\overline{u}^s\|_{H^2(\Omega)]^2} \leq c, \quad c > 0$ – *const.*

Then, the sequence $\{(u^s, l^s)\}$ converges to a saddle point of $M(v, l)$ in space $[H^1(\Omega)]^2 \times L_2(\Gamma_K)$[13].

Algorithm (23), (24) was examined in detail with the help of finite element method in [10, 16]. It is a very effective method for solving semicoercive variational inequalities in mechanics.

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