ON THE LOWER SEMICONTINUITY OF THE SOLUTION SETS FOR PARAMETRIC GENERALIZED VECTOR MIXED QUASIVARIATIONAL INEQUALITY PROBLEMS

Nguyen Van Hung

ABSTRACT. In this paper, we establish sufficient conditions for the solution set of parametric generalized vector mixed quasivariational inequality problem to have the semicontinuities such as the inner-openness, lower semicontinuity and Hausdorff lower semicontinuity. Moreover, a key assumption is introduced by virtue of a parametric gap function by using a nonlinear scalarization function. Then, by using the key assumption, we establish condition $(H_h(\gamma_0, \lambda_0, \mu_0))$ is a sufficient and necessary condition for the Hausdorff lower semicontinuity, continuity and Hausdorff continuity of the solution set for this problem in Hausdorff topological vector spaces with the objective space being infinite dimensional. The results presented in this paper are different and extend from some main results in the literature.

1. Introduction

A vector variational inequality in a finite-dimensional Euclidean space was introduced first by Giannessi [16]. Later, many authors have investigated vector variational inequality problems in abstract spaces, see [1, 13, 19, 20, 24, 26, 27, 29, 30, 31, 35, 36, 38]. With the development of the theory about vector variational inequality problems, it has been seen that vector variational inequality problems have many important applications in vector optimization problems, see [25, 37], vector equilibria problems, see [2, 3, 4, 5, 6, 7, 8, 10, 11, 15, 17, 21, 22, 28, 32, 33, 34, 39], variational relation problems, see [17, 18, 23] and the references therein.

In 2009, Li and Chen [30] and in 2010, Chen et al. [13] introduced a key assumption by virtue of a parametric gap function and proved that the condition (H_g) is a sufficient condition for the Hausdorff lower semicontinuity of the solution set for vector variational inequalities. Recently, Zhong and Huang

[38, 39] also introduced the condition (H_g') which is similar to the one given in [13, 30] and proved that it is a sufficient and necessary condition for the Hausdorff lower semicontinuity of the solution set for parametric set-valued weak vector variational inequalities and for parametric generalized vector quasiequilibrium problem in Banach spaces. Very recently, Hung [19] also studied a class of parametric generalized vector mixed quasivariational inequality problem in Hausdorff topological vector spaces and established the sufficient conditions for the semicontinuity of the solution set such as upper semicontinuity, closedness, outer-continuity and outer-openness. Moreover, the assumption $(H_h(\gamma_0, \mu_0))$ is a sufficient and necessary condition for the Hausdorff lower semicontinuity, continuity and Hausdorff continuity of the solution set for this problem are also obtained.

Motivated by research works mentioned above, in this paper, we introduce a different kind of parametric generalized vector mixed quasivariational inequality problem in Hausdorff topological vector spaces. Let X, Y be two Hausdorff topological vector spaces and Γ, Λ, M be three topological vector spaces. Let $L(X, Y)$ be the space of all linear continuous operators from X to Y. Let $K : X \times \Gamma \to 2^X, T : X \times M \to 2^{L(X,Y)}$ be two set-valued mappings and C be a closed convex cone in Y with int$C \neq \emptyset$. Let $\eta : X \times X \times \Lambda \to X, \psi : X \times X \times \Lambda \to Y$ be two continuous vector-valued functions satisfying $\eta(x, x, \lambda) = 0$ and $\psi(x, x, \lambda) = 0$ for each $x \in X, \lambda \in \Lambda$. And let $H : L(X, Y) \to L(X, Y)$ be a continuous single-valued mapping. Denoted by $\langle z, x \rangle$ the value of a linear operator $z \in L(X; Y)$ at $x \in X$, we always assume that $\langle \cdot, \cdot \rangle$ is continuous.

For $\gamma \in \Gamma, \lambda \in \Lambda, \mu \in M$ consider the following parametric generalized vector mixed quasivariational inequality problem (in short, (QVIP)).

(QVIP) Find $\bar{x} \in K(\bar{x}, \gamma)$ and $\bar{z} \in T(\bar{x}, \mu)$ such that

$$\langle H(\bar{z}), \eta(y, \bar{x}, \lambda) \rangle + \psi(y, \bar{x}, \lambda) \subseteq -C, \forall y \in K(\bar{x}, \gamma).$$

For each $\gamma \in \Gamma, \lambda \in \Lambda, \mu \in M$ we let $E(\gamma) := \{x \in X \mid x \in K(x, \gamma)\}$ and $\Sigma : \Gamma \times \Lambda \times M \to 2^X$ be a set-valued mapping such that $\Sigma(\gamma, \lambda, \mu)$ is the solution set of (QVIP).

Throughout the paper we assume that $\Sigma(\gamma, \lambda, \mu) \neq \emptyset$ for each (γ, λ, μ) in the neighborhood $(\gamma_0, \lambda_0, \mu_0) \in \Gamma \times \Lambda \times M$.

The structure of our paper is as follows. In the first part of this article, we introduce the model parametric generalized vector mixed quasivariational inequality problem. In Section 2, we recall definitions for later uses and discuss the continuity of parametric gap function. In Section 3, we investigate the inner-openness, the lower semicontinuity and the Hausdorff lower semicontinuity of the solution set for (QVIP). Moreover, we also establish condition $(H_h(\gamma_0, \lambda_0, \mu_0))$ is a sufficient and necessary condition for the Hausdorff lower semicontinuity, the continuity and Hausdorff continuity of the solution set for this problem in Hausdorff topological vector spaces with the objective space being infinite dimensional.

2. Preliminaries

In this section, we recall some basic definitions and their some proper-
ties in [2, 9, 12]. Let X and Z be Hausdorff topological vector spaces, and
$G : X \to 2^Z$ be a multifunction. G is said to be lower semicontinuous (lsc) at
x_0 if $G(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Z$ implies the existence of a neigh-
borhood N of x_0 such that $G(x) \cap U \neq \emptyset, \forall x \in N$. An equivalent formulation
is that: G is lsc at x_0 if $\forall x_\alpha \to x_0$, $\forall z_0 \in G(x_0), \exists z_\alpha \in G(x_\alpha), z_\alpha \to z_0$. G is
called upper semicontinuous (usc) at x_0 if for each open set $U \supseteq G(x_0)$, there is
a neighborhood N of x_0 such that $U \supseteq G(N)$. G is called Hausdorff upper semi-
continuous (H-usc in short; Hausdorff lower semicontinuous, H-lsc, respectively)
at x_0 if for each neighborhood B of the origin in Z, there exists a neighborhood
N of x_0 such that, $G(x) \subseteq G(x_0) + B, \forall x \in N$ $(G(x_0) \subseteq G(x) + B, \forall x \in N)$. G
is called continuous at x_0 if it is both lsc and usc at x_0 and to be H-continuous
at x_0 if it is both H-lsc and H-usc at x_0. G is called closed at x_0 if for each
net $\{(x_\alpha, z_\alpha)\} \subseteq \mathrm{graph}G := \{(x, z) \mid z \in G(x)\}, (x_\alpha, z_\alpha) \to (x_0, z_0)$, z_0 must
belong to $G(x_0)$.

Next, we recall a new limit in [23], the inferior open limit: $\mathrm{Liminfo}_{x \to x_0} G(x)$
$:= \{z \in Z :$ there are open neighborhoods U of x_0 and V of z such that $V \subseteq$
$G(x)$ for all $x \in U, x \neq x_0\}$. G is said that inner-open at x_0 if $\mathrm{liminfo}_{x \to x_0} G(x)$
$\supseteq G(x_0)$. From the condition (3) of Lemma 2.1 in [23], we deduce that

$$\mathrm{liminfo}_{x \to x_0} G(x) = \left[\limsup_{x \to x_0} G^c(x) \right]^c,$$

where $G^c(x) = Z \setminus G(x)$.

Lemma 2.1 ([9, 12])**.** *Let X and Z be two Hausdorff topological vector spaces
and $G : X \to 2^Z$ be a multifunction.*

   (i) *If G is usc at x_0, then G is H-usc at x_0. Conversely if G is H-usc at
x_0 and if $G(x_0)$ is compact, then G is usc at x_0;*

  (ii) *If G is H-lsc at x_0 then G is lsc at x_0. The converse is true if $G(x_0)$
is compact;*

 (iii) *If Z is compact and G is closed at x_0, then G is usc at x_0;*

 (iv) *If G is usc at x_0 and $G(x_0)$ is closed, then G is closed at x_0;*

  (v) *If G has compact values, then G is usc at x_0 if and only if, for each
net $\{x_\alpha\} \subseteq X$ which converges to x_0 and for each net $\{y_\alpha\} \subseteq G(x_\alpha)$,
there are $y \in G(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$.*

Lemma 2.2 ([14])**.** *For any fixed each $e \in \mathrm{int}C, y \in Y, r \in R$ and the nonlinear
scalarization function $\xi_e : Y \to R$ defined by $\xi_e(y) := \min\{r \in R : y \in re - C\}$:*

   (i) *ξ_e is a continuous and convex function in Y;*

  (ii) *$\xi_e(y) \leq r \Leftrightarrow y \in re - C$;*

 (iii) *$\xi_e(y) > r \Leftrightarrow y \notin re - C$.*

Now we suppose that $K(x, \gamma)$ and $T(x, \mu)$ are compact sets for any $(x, \gamma) \in$
$X \times \Lambda$ and $(x, \mu) \in X \times M$. We define function $h : X \times \Gamma \times \Lambda \times M \to R$ as

follows

$$h(x, \gamma, \lambda, \mu) = \max_{z \in T(x,\mu)} \max_{y \in K(x,\gamma)} \xi_e(\langle H(z), \eta(y, x, \lambda) \rangle + \psi(y, x, \lambda)).$$

Since $K(x, \gamma)$ and $T(x, \mu)$ are compact sets, $h(x, \gamma, \lambda, \mu)$ is well-defined.

Lemma 2.3.   (i) $h(x, \gamma, \lambda, \mu) \geq 0$ *for all* $x \in E(\gamma)$;
  (ii) $h(x_0, \gamma_0, \lambda_0, \mu_0) = 0$ *if and only if* $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$.

Proof. We define a function $f : X \times L(X, Y) \to R$ as follows

$$f(x, z) = \max_{y \in K(x,\gamma)} \xi_e(\langle H(z), \eta(y, x, \lambda) \rangle + \psi(y, x, \lambda)), \ x \in E(\gamma), \ z \in T(x, \mu).$$

(i) It is easy to see that $f(x, z) \geq 0$. Suppose to the contrary that there exist $x_0 \in E(\gamma)$ and $z_0 \in T(x_0, \mu)$ such that $f(x_0, z_0) < 0$, then

$$\begin{aligned} 0 > f(x_0, z_0) &= \max_{y \in K(x_0,\gamma)} \xi_e(\langle H(z_0), \eta(y, x_0, \lambda) \rangle + \psi(y, x_0, \lambda)) \\ &\geq \xi_e(\langle H(z_0), \eta(y, x_0, \lambda) \rangle + \psi(y, x_0, \lambda)), \forall y \in K(x_0, \gamma). \end{aligned}$$

When $y = x_0$, we have

$$\begin{aligned} &\xi_e(\langle H(z_0), \eta(x_0, x_0, \lambda) \rangle + \psi(x_0, x_0, \lambda)) \\ &= \xi_e(0) \\ &= \min\{r \in R : 0 \in re - C\} \\ &= \min\{r \in R : -re \in -C\} \\ &= \min\{r \in R : r \geq 0\} = 0, \ e \in \text{int}C, \end{aligned}$$

which is a contradiction. Hence,

$$h(x, \gamma, \lambda, \mu) = \max_{z \in T(x,\mu)} \max_{y \in K(x,\gamma)} \xi_e(\langle H(z), \eta(y, x, \lambda) \rangle + \psi(y, x, \lambda)) \geq 0.$$

(ii) Since H, η, ψ and ξ_e are continuous, $h(x_0, \gamma_0, \lambda_0, \mu_0) = 0$ if and only if there exists $z_0 \in T(x_0, \mu_0)$ such that

$$\max_{y \in K(x_0,\gamma_0)} \xi_e(\langle H(z_0), \eta(y, x_0, \lambda_0) \rangle + \psi(y, x_0, \lambda_0)) = 0, \ x_0 \in E(\gamma_0),$$

if and only if, for any $y \in K(x_0, \gamma_0)$

$$\xi_e(\langle H(z_0), \eta(y, x_0, \lambda_0) \rangle + \psi(y, x_0, \lambda_0)) \leq 0.$$

By Lemma 2.2(ii), if and only if, for any $y \in K(x_0, \gamma_0)$

$$\langle H(z_0), \eta(y, x_0, \lambda_0) \rangle + \psi(y, x_0, \lambda_0) \subseteq -C,$$

i.e., $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$. This completes the proof. □

We may call the function h as a parametric gap function for (QVIP) if the properties of Lemma 2.3 are satisfied.

Example 2.4. Let H be an identity mapping, $X = R$, $Y = \mathbb{R}^2$, $\Gamma = \Lambda = M = [0,1]$, $C = \mathbb{R}^2_+$, $K(x,\gamma) = [0,1]$, $T(x,\gamma) = [\frac{1}{2}, 3\gamma^2 x^2 + x^4]$ and $\eta(y,x,\gamma) = x - y$, $\psi(y,x,\gamma) = 0$. Now we consider the problem (QVIP), finding $x \in K(x,\gamma)$ and $z \in T(x,\gamma)$ such that

$$\langle H(z), \eta(y,x,\gamma) \rangle + \psi(y,x,\gamma) = (\frac{1}{2}(x-y), (3\gamma^2 x^2 + x^4)(x-y)) \subseteq -\mathbb{R}^2_+.$$

It follows from a direct computation $\Sigma(\gamma, \lambda, \mu) = \{0\}$ for all $\gamma \in [0,1]$. Now we show that h is a parametric gap function of (QVIP). Indeed, we taking $e = (1,1) \in \text{int}R^2_+$, we have

$$h(x,\gamma,\lambda,\mu) = \max_{y \in K(x,\gamma)} \max_{1 \le i \le 2} [\langle T(x,\gamma), \eta(y,x,\gamma) \rangle + \psi(y,x,\gamma)]_i$$

$$= \max_{y \in K(x,\gamma)} ((3\gamma^2 x^2 + x^4)(x-y))$$

$$= \begin{cases} 0 & \text{if } x = 0, \\ \gamma^2 x^3 + x^5 & \text{if } x \in (0,1]. \end{cases}$$

Hence, h is a parametric gap function of (QVIP).

The following Lemma 2.5 gives sufficient condition for the parametric gap function h to be continuous.

Lemma 2.5. *Assume for problem* (QVIP) *that*

  (i) *K is continuous with compact values in $X \times \Gamma$;*
  (ii) *T is continuous with compact values in $X \times M$.*

Then h is continuous in $X \times \Gamma \times \Lambda \times M$.

Proof. First we prove that h is lower semicontinuous in $X \times \Gamma \times \Lambda \times M$. Indeed, we let $a \in R$. Suppose that $\{(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha)\} \subseteq X \times \Gamma \times \Lambda \times M$ satisfying

$$h(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) \le a, \quad \forall \alpha$$

and

$$(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) \to (x_0, \gamma_0, \lambda_0, \mu_0) \text{ as } \alpha \to \infty.$$

It follows that

$$h(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) = \max_{z \in T(x_\alpha, \mu_\alpha)} \max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(\langle H(z), \eta(y,x_\alpha,\lambda_\alpha) \rangle + \psi(y,x_\alpha,\lambda_\alpha))$$

$$\le a$$

and so

$$\max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(\langle H(z), \eta(y,x_\alpha,\lambda_\alpha) \rangle + \psi(y,x_\alpha,\lambda_\alpha)) \le a, \forall z \in T(x_\alpha, \mu_\alpha).$$

Since T is lower semincontinuous at (x_0, μ_0), for any $z_0 \in T(x_0, \mu_0)$, there exists $z_\alpha \in T(x_\alpha, \mu_\alpha)$ such that $z_\alpha \to z_0$. Since $z_\alpha \in T(x_\alpha, \mu_\alpha)$, we have

(1) $$\max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(\langle H(z_\alpha), \eta(y,x_\alpha,\lambda_\alpha) \rangle + \psi(y,x_\alpha,\lambda_\alpha)) \le a.$$

Since K is lower semincontinuous at (x_0, γ_0), for any $y_0 \in K(x_0, \gamma_0)$, there exists $y_\alpha \in K(x_\alpha, \gamma_\alpha)$ such that $y_\alpha \to y_0$. Since $y_\alpha \in K(x_\alpha, \gamma_\alpha)$, it follows from (1) that

$$(2) \qquad \xi_e(\langle H(z_\alpha), \eta(y_\alpha, x_\alpha, \lambda_\alpha)\rangle + \psi(y_\alpha, x_\alpha, \lambda_\alpha)) \leq a.$$

From the continuity of H, η, ψ and ξ_e. Take the limit in (2), we have

$$(3) \qquad \xi_e(\langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0)) \leq a.$$

Since $y \in K(x_0, \gamma_0)$ and $z \in T(x_0, \mu_0)$ are arbitrary, it follows from (3) that

$$h(x_0, \gamma_0, \lambda_0, \mu_0) = \max_{z \in T(x_0, \mu_0)} \max_{y \in K(x_0, \gamma_0)} \xi_e(\langle H(z), \eta(y, x_0, \lambda_0)\rangle + \psi(y, x_0, \lambda_0))$$
$$\leq a.$$

This proves that, for $a \in R$, the level set $\{(x, \gamma, \lambda, \mu) \mid h(x, \gamma, \lambda, \mu) \leq a\}$ is closed. Hence, h is lower semicontinuous in $X \times \Gamma \times \Lambda \times M$.

Next, we only prove that h is upper semicontinuous in $X \times \Gamma \times \Lambda \times M$. Indeed, let $a \in R$. Suppose that $\{(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha)\} \subseteq X \times \Gamma \times \Lambda \times M$ satisfying

$$h(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) \geq a, \ \ \forall \alpha$$

and

$$(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) \to (x_0, \gamma_0, \lambda_0, \mu_0) \ \text{ as } \ \alpha \to \infty.$$

It follows that

$$h(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) = \max_{z \in T(x_\alpha, \mu_\alpha)} \max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(\langle H(z), \eta(y, x_\alpha, \lambda_\alpha)\rangle + \psi(y, x_\alpha, \lambda_\alpha))$$
$$\geq a.$$

Now, we define the function $f : X \times L(X, Y) \to R$ by

$$f(x, z) = \max_{y \in K(x, \gamma)} \xi_e(\langle H(z), \eta(y, x, \lambda)\rangle + \psi(y, x, \lambda)), x \in E(\gamma).$$

Since ξ_e is continuous in X and K is continuous with compact values in $X \times \Gamma$, by Proposition 19 in Section 1 of Chapter 3 [9], we can deduce that $f(x, z)$ is a continuous function. Thus, from the compactness of $T(x_\alpha, \mu_\alpha)$, there exists $z_\alpha \in T(x_\alpha, \mu_\alpha)$ such that

$$h(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) = \max_{z \in T(x_\alpha, \mu_\alpha)} \max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(\langle H(z), \eta(y, x_\alpha, \lambda_\alpha)\rangle + \psi(y, x_\alpha, \lambda_\alpha))$$
$$= f(x_\alpha, z_\alpha)$$
$$= \max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(\langle H(z_\alpha), \eta(y, x_\alpha, \lambda_\alpha)\rangle + \psi(y, x_\alpha, \lambda_\alpha)) \geq a.$$

From the compactness of $K(x_\alpha, \gamma_\alpha)$, there exists $y_\alpha \in K(x_\alpha, \gamma_\alpha)$ such that

$$\xi_e(\langle H(z_\alpha), \eta(y_\alpha, x_\alpha, \lambda_\alpha)\rangle + \psi(y_\alpha, x_\alpha, \lambda_\alpha))$$
$$(4) \qquad = \max_{y \in K(x_\alpha, \lambda_\alpha)} \xi_e(\langle H(z_\alpha), \eta(y, x_\alpha, \lambda_\alpha)\rangle + \psi(y, x_\alpha, \lambda_\alpha)) \geq a.$$

Since T is upper semicontinuous with compact values in $X \times M$ and K is upper semicontinuous with compact values in $X \times \Gamma$, there exist $z_0 \in T(x_0, \mu_0)$,

$y_0 \in K(x_0, \gamma_0)$ such that $z_\alpha \to z_0, y_\alpha \to y_0$ (can take subnets $\{z_\beta\}$ of $\{z_\alpha\}$ and $\{y_\beta\}$ of $\{y_\alpha\}$ if necessary) as $\alpha \to \infty$. From the continuity of H, η, ψ and ξ_e. Take the limit in (4), we have

$$\xi_e(\langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0)) \geq a.$$

And so, for any $y \in K(x_0, \gamma_0)$ and $z \in T(x_0, \mu_0)$, we have

$$h(x_0, \gamma_0, \lambda_0, \mu_0)) = \max_{z \in T(x_0, \mu_0)} \max_{y \in K(x_0, \gamma_0)} \xi_e(\langle H(z), \eta(y, x_0, \lambda_0)\rangle + \psi(y, x_0, \lambda_0))$$
$$\geq a.$$

This proves that, for $a \in R$, the level set $\{(x, \gamma, \lambda, \mu) \,|\, h(x, \gamma, \lambda, \mu) \geq a\}$ is closed. Hence, h is upper semicontinuous in $X \times \Gamma \times \Lambda \times M$. $\qquad\square$

Remark 2.6. Note that, our the parametric gap function $h(x, \gamma, \lambda, \mu)$ is different from the parametric gap functions in [13, 19, 30, 38]. Thus, the continuity of the parametric gap function $h(x, \gamma, \lambda, \mu)$ in Lemma 2.5 is new.

Lemma 2.7. *Assume for problem* (QVIP) *that*

 (i) $E(\gamma_0)$ *is a compact set;*
 (ii) $K(\cdot, \gamma_0)$ *is lower semicontinuous in X for all $\gamma_0 \in \Gamma$;*
 (iii) $T(\cdot, \mu_0)$ *is upper semicontinuous with compact values in X for all $\mu_0 \in M$.*

Then $\Sigma(\gamma_0, \lambda_0, \mu_0)$ is a closed set. Moreover, $\Sigma(\gamma_0, \lambda_0, \mu_0)$ is a compact set.

Proof. Take any net $\{x_\alpha\} \subset \Sigma(\gamma_0, \lambda_0, \mu_0)$ such that $x_\alpha \to x_0$ and $x_0 \in E(\gamma_0)$ by $E(\gamma_0)$ is a compact set. We need to prove that $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$. If $x_0 \notin \Sigma(\gamma_0, \lambda_0, \mu_0)$, then $\forall z_0 \in T(x_0, \mu_0), \exists y_0 \in K(x_0, \gamma_0)$ such that

$$(5) \qquad \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0) \not\subseteq -C.$$

The lower semicontinuity of $K(\cdot, \gamma_0)$, there is $y_\alpha \in K(x_\alpha, \gamma_0)$ such that $\{y_\alpha\} \to y_0$ for each α.

Since $x_\alpha \in \Sigma(\gamma_0, \lambda_0, \mu_0)$, there exists $z_\alpha \in T(x_\alpha, \mu_0)$ such that

$$(6) \qquad \langle H(z_\alpha), \eta(y_\alpha, x_\alpha, \lambda_0)\rangle + \psi(y_\alpha, x_\alpha, \lambda_0) \subseteq -C.$$

Since $T(\cdot, \mu_0)$ is upper semicontinuous and with compact values in $X \times M$, one has $z_0 \in T(x_0, \mu_0)$ such that $z_\alpha \to z_0$ (can take a subnet if necessary) and since H, η are continuous. We have,

$$\langle H(z_\alpha), \eta(y_\alpha, x_\alpha, \lambda_0)\rangle \to \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle.$$

It follows from the continuity of ψ that

$$\langle H(z_\alpha), \eta(y_\alpha, x_\alpha, \lambda_0)\rangle + \psi(y_\alpha, x_\alpha, \lambda_0) \to \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0)$$

and so

$$(7) \qquad \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0) \subseteq -C.$$

We see a contradiction between (5) and (7), and so we have $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$. Thus, $\Sigma(\gamma_0, \lambda_0, \mu_0)$ is closed. Moreover, it follows from $\Sigma(\gamma_0, \lambda_0, \mu_0) \subset E(\gamma_0)$ and by compactness of $E(\gamma_0)$ that $\Sigma(\gamma_0, \lambda_0, \mu_0)$ is a compact set. $\square$

3. Main results

In this section, we discuss the inner-openness, lower semicontinuity and Hausdorff lower semicontinuity of the solution set for parametric generalized vector mixed quasivariational inequality problem. Moreover, we establish condition $(\mathrm{H}_h(\gamma_0, \lambda_0, \mu_0))$ is a sufficient and necessary condition for the Hausdorff lower semicontinuity, continuity and Hausdorff continuity for the solution set of this problem.

Theorem 3.1. *Assume for problem* (QVIP) *that*

  (i) *E is inner-open in Γ,*
  (ii) *K is upper semicontinuous with compact values in $X \times \Gamma$,*
  (iii) *T is lower semicontinuous in $X \times M$.*

Then Σ is inner-open in $\Gamma \times \Lambda \times M$.

Proof. Suppose to the contrary that Σ is not inner-open in $\Gamma \times \Lambda \times M$. Then, $\exists x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$, $x_0 \notin \liminf_{\gamma \to \gamma_0, \lambda \to \lambda_0, \mu \to \mu_0} \Sigma(\gamma, \lambda, \mu)$. By

$$\liminf_{\gamma \to \gamma_0, \lambda \to \lambda_0, \mu \to \mu_0} \Sigma(\gamma, \lambda, \mu) = \big[\limsup_{\gamma \to \gamma_0, \lambda \to \lambda_0, \mu \to \mu_0} (\Sigma)^c(\gamma, \lambda, \mu) \big]^c,$$

we have $x_0 \in \limsup_{\gamma \to \gamma_0, \lambda \to \lambda_0, \mu \to \mu_0} (\Sigma)^c(\gamma, \lambda, \mu)$. Therefore, there exist nets $\{\gamma_\alpha\}$, $\gamma_\alpha \neq \gamma_0$ converging to γ_0, $\{\lambda_\alpha\}$, $\lambda_\alpha \neq \lambda_0$ converging to λ_0, $\{\mu_\alpha\}$, $\mu_\alpha \neq \mu_0$ converging to μ_0 and net $\{x_\alpha\}$ with $x_\alpha \in (\Sigma)^c(\gamma_\alpha, \lambda_\alpha, \mu_\alpha)$ converging x_0. Since E is inner-open at γ_0 and $x_0 \in E(\gamma_0)$ which implies $x_0 \in \liminf_{\gamma \to \gamma_0} E(\gamma)$. There exist neighborhoods U of γ_0, V of x_0 such that $x_0 \in V \subseteq E(\gamma)$ for all $\gamma \in U$, $\gamma \neq \gamma_0$. Since $(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) \to (x_0, \gamma_0, \lambda_0, \mu_0)$ and by the above contradiction assumption, there must be a subnet $\{(x_\beta, \gamma_\beta, \lambda_\beta, \mu_\beta)\}$ of net $\{(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha)\}$ such that for all β, $x_\beta \notin \Sigma(\gamma_\beta, \lambda_\beta, \mu_\beta)$, i.e., $\forall z_\beta \in T(x_\beta, \mu_\alpha), \exists y_\beta \in K(x_\beta, \gamma_\beta)$ such that

$$(8) \qquad \langle H(z_\beta), \eta(y_\beta, x_\beta, \lambda_\beta) \rangle + \psi(y_\beta, x_\beta, \lambda_\beta) \nsubseteq -C.$$

As K is usc in $X \times \Gamma$ and $K(x_0, \gamma_0)$ is compact, one has $y_0 \in K(x_0, \gamma_0)$ such that $y_\beta \to y_0$ (taking a subnet if necessary). By the lower semicontinuity of T at (x_0, μ_0), one has $z'_\beta \in T(x_\beta, \mu_\beta)$ such that $z'_\beta \to z_0$. Since H, η and $\langle \cdot, \cdot \rangle$ are continuous. We have,

$$\langle H(z'_\beta), \eta(y_\beta, x'_\beta, \lambda_\beta) \rangle \to \langle H(z_0), \eta(y_0, x_0, \lambda_0) \rangle.$$

It follows from the continuity of ψ that

$$\langle H(z'_\beta), \eta(y_\beta, x'_\beta, \lambda_\beta) \rangle + \psi(y_\beta, x'_\beta, \lambda_\beta) \to \langle H(z_0), \eta(y_0, x_0, \lambda_0) \rangle + \psi(y_0, x_0, \lambda_0).$$

By (8), we have

$$\langle H(z_0), \eta(y_0, x_0, \lambda_0) \rangle + \psi(y_0, x_0, \lambda_0) \nsubseteq -C,$$

which is impossible since $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$. Therefore, Σ is inner-open in $\Gamma \times \Lambda \times M$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following example shows that the inner-openness of E is essential.

Example 3.2. Let $X = Y = [0, 1]$, $\Lambda = \Gamma = M = [0, 1]$, $C = R_-\gamma_0 = 0$, H is an identity mapping and

$$T(x, \gamma) = [0, e^{1+\cos^2 \gamma}],$$
$$\eta(x, y, \gamma) = \{1 + 2\gamma\},$$
$$\psi(x, y, \gamma) = \{e^\gamma\},$$
$$f(x, y, \gamma) = \{\frac{1}{e^{2+\gamma}}\},$$
$$K(x, \gamma) = \begin{cases} (0, 1] & \text{if } \gamma \in (0, 1], \\ [-1, 0] & \text{if } \gamma = 0. \end{cases}$$

We have $E(\gamma) = (0, 1]$ for all $\gamma \in (0, 1]$ and $E(0) = [-1, 0]$. It is not hard to see that the assumptions (ii) and (iii) in Theorem 3.1 are satisfied. But E is not inner-open at 0. Hence, Σ also not inner-open at $(0, 0, 0)$. Thus, Theorem 3.1 cannot be applied. In fact, $\Sigma(0, 0, 0) = [-1, 0]$ and $\Sigma(\gamma, \lambda, \mu) = (0, 1]$ for all $\gamma \in (0, 1]$.

The following example shows that all assumptions of Theorem 3.1 are fulfilled.

Example 3.3. Let $X = Y = \mathbb{R}$, $\Lambda = \Gamma = M = [0, 1]$, $C = R_-\gamma_0 = 0$, H is an identity mapping and

$$f(x, y, \gamma) = [\frac{1}{2}, 1], \ K(x, \gamma) = (0, 1),$$
$$T(x, \gamma) = [0, 1], \ \eta(x, y, \gamma) = \{\gamma^2 + 2\gamma\},$$
$$\psi(x, y, \gamma) = \{2^{1+\gamma}\}.$$

We see that the assumptions of Theorem 3.1 are satisfied. And so, Σ is inner-open at $(0, 0, 0)$. In fact, $\Sigma(\gamma, \lambda, \mu) = (0, 1)$ for all $\gamma \in [0, 1]$.

Theorem 3.4. *Assume for problem* (QVIP) *that*
  (i) *E is lower semicontinuous in Γ,*
  (ii) *$\forall x_0 \in K(x_0, \gamma_0)$, $\forall (x_n, \gamma_n, \lambda_n, \mu_n) \to (x_0, \gamma_0, \lambda_0, \mu_0)$ and $\exists z \in T(x_0, \mu_0)$ such that*

$$\langle H(z), \eta(y, x_0, \lambda_0)\rangle + \psi(y, x_0, \lambda_0) \subseteq -C, \forall y \in K(x_0, \gamma_0)$$

  implies that there exists a positive integer n, such that $\exists z \in T(x_n, \mu_n)$ satisfying

$$\langle H(z), \eta(y, x_n, \lambda_n)\rangle + \psi(y, x_n, \lambda_n) \subseteq -C, \forall y \in K(x_n, \gamma_n).$$

Then Σ is lower semicontinuous in $\Gamma \times \Lambda \times M$.

Proof. Suppose to the contrary that Σ is not lower semicontinuous at $(\gamma_0, \lambda_0, \mu_0)$, i.e., $\exists x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$, $\exists(\gamma_n, \lambda_n, \mu_n) \to (\gamma_0, \lambda_0, \mu_0)$, $\forall x'_n \in \Sigma(\gamma_n, \lambda_n, \mu_n)$, $x'_n \nrightarrow x_0$. Since E is lower semicontinuous at $\gamma_0 \in \Gamma$, there is a net $\{x_n\}$ with $x_n \in E(\gamma_n)$, $x_n \to x_0$. By the above contradiction assumption, without loss of generality, if $x_n \notin \Sigma(\gamma_n, \lambda_n, \mu_n)$, i.e., $\forall z_n \in T(x_n, \mu_n)$, $\exists y_n \in K(x_n, \gamma_n)$ such that

$$(9) \qquad \langle H(z_n), \eta(y_n, x_n, \lambda_n)\rangle + \psi(y_n, x_n, \lambda_n) \nsubseteq -C.$$

Since $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$, $\exists z \in T(x_0, \mu_0)$ such that

$$\langle H(z), \eta(y, x_0, \lambda_0)\rangle + \psi(y, x_0, \lambda_0) \subseteq -C, \forall y \in K(x_0, \gamma_0).$$

Since $(x_n, \gamma_n, \lambda_n, \mu_n) \to (x_0, \gamma_0, \lambda_0, \mu_0)$ and by the condition (ii), there exists n, such that $\forall z \in T(x_n, \mu_n)$

$$\langle H(z), \eta(y, x_n, \lambda_n)\rangle + \psi(y, x_n, \lambda_n) \subseteq -C, \forall y \in K(x_n, \gamma_n),$$

which contradicts (9). Therefore, Σ is lower semicontinuous in $\Gamma \times \Lambda \times M$. $\square$

Remark 3.5. In the special case, if $Y = R$, $C = (-\infty, 0]$, $\Gamma = \Lambda = M$, H is an identity mapping and $\eta(y, x, \lambda) = y - g(x, \lambda)$, $\psi(y, x, \lambda) = 0$, where $g : X \times \Lambda \to X$ be a continuous single-valued mapping. Then, the lower semicontinuity of the solution set in Theorem 3.2 of Khanh-Luu [24] is a particular case of Theorem 3.4.

The following example shows that the lower semicontinuity of E is essential.

Example 3.6. Let $X, Y, \Lambda, \Gamma, M, \gamma_0, H, C$ as in Example 3.3 and let $T(x, \gamma) = [0, 2^{1+\sin^2\gamma}]$, $\eta(x, y, \gamma) = 2 + \cos^2\gamma$, $\psi(x, y, \gamma) = 2^\gamma$ and

$$K(x, \gamma) = \begin{cases} [-1, 1] & \text{if} \quad \gamma = 0, \\ [-\gamma - 1, 0] & \text{otherwise.} \end{cases}$$

We have $E(0) = [-1, 1]$, $E(\gamma) = [-\gamma - 1, 0], \forall \gamma \in (0, 1]$. Hence K is usc and the conditions (ii) and (iii) of Theorem 3.4 are easily seen to be fulfilled. But Σ is not upper semicontinuous at $(0, 0, 0)$. The reason is that E is not lower semicontinuous at 0. In fact, $\Sigma(0, 0, 0) = [-1, 1]$ and $\Sigma(\gamma, \lambda, \mu) = [-\gamma - 1, 0]$, $\forall \gamma \in (0, 1]$.

Theorem 3.7. *Impose the assumption of Theorem 3.4 and the following additional conditions*:

(iii) $K(., \gamma_0)$ *is lower semicontinuous in* X *and* $E(\gamma_0)$ *is compact*;
(iv) $T(., \mu_0)$ *is upper semicontinuous with compact values in* X.

Then Σ *is Hausdorff lower semicontinuous in* $\Gamma \times \Lambda \times M$.

Proof. We first prove that $\Sigma(\gamma_0, \lambda_0, \mu_0)$ is closed. Indeed, we let

$$x_n \in \Sigma(\gamma_0, \lambda_0, \mu_0)$$

such that $x_n \to x_0$. If $x_0 \notin \Sigma(\gamma_0, \lambda_0, \mu_0)$, then $\forall z_0 \in T(x_0, \mu_0)$, $\exists y_0 \in K(x_0, \gamma_0)$ such that

$$(10) \qquad \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0) \not\subseteq -C.$$

By the lower semicontinuity of $K(., \gamma_0)$ at x_0, one has $y_n \in K(x_n, \lambda_0)$ such that $y_n \to y_0$. By the lower semicontinuity of T at (x_0, μ_0), one has $z_n \in T(x_n, \mu_0)$ such that $z_n \to z_0$. Since $x_n \in \Sigma(\gamma_0, \lambda_0, \mu_0)$, we have

$$(11) \qquad \langle H(z_n), \eta(y_n, x_n, \lambda_0)\rangle + \psi(y_n, x_n, \lambda_0) \subseteq -C.$$

Since $(x_n, z_n, y_n) \to (x_0, z_0, y_0)$ and from the continuity of H, η, ψ and (11) yields that

$$(12) \qquad \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0) \subseteq -C,$$

we see a contradiction between (10) and (12). Therefore, $\Sigma(\gamma_0, \lambda_0, \mu_0)$ is a closed set.

On the other hand, since $\Sigma(\gamma_0, \lambda_0, \mu_0) \subseteq E(\gamma_0)$ is compact by $E(\gamma_0)$ compact. Since Σ is lower semicontinuous in $\Gamma \times \Lambda \times M$ and $\Sigma(\gamma_0, \lambda_0, \mu_0)$ is compact. Hence by Lemma 2.1(ii), it follows that Σ is Hausdorff lower semicontinuous in $\Gamma \times \Lambda \times M$. And so we complete the proof. $\square$

The following example shows that the compactness of E is essential.

Example 3.8. Let $X = Y = \mathbb{R}^2$, $\Lambda = \Gamma = M = [0, 1]$, $C = \mathbb{R}_- \gamma_0 = 0$, H is an identity mapping and

$$x = (x_1, x_2) \in \mathbb{R}^2, \ K(x, \lambda) = \{(x_1, \lambda x_1^2)\}.$$
$$T(x, \gamma) = [0, e^{2+2\gamma}], \ \eta(x, y, \gamma) = \{\gamma^2\},$$
$$f(x, y, \gamma) = \{\frac{1}{e^{2+\gamma}}\}, \ \psi(x, y, \gamma) = \{2^{\gamma^2}\}.$$

We have $E(0) = \{x \in \mathbb{R}^2 \,|\, x_2 = 0\}$ and $E(\gamma) = \{x \in \mathbb{R}^2 \,|\, x_2 = \gamma x_1^2\}, \forall \gamma \in (0, 1]$. We shows that the assumptions of Theorem 3.7 are satisfied, but the compactness of $E(0)$ is not satisfied. Direct computations give $\Sigma(0, 0, 0) = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 = 0\}$ and then $\Sigma(\gamma, \lambda, \mu) = \{x \in \mathbb{R}^2 \,|\, x_2 = \gamma x_1^2\}, \forall \gamma \in (0, 1]$ is not Hausdorff lower semicontinuous at $(0, 0, 0)$.

The following example shows that all the assumptions of Theorem 3.7 are satisfied.

Example 3.9. Let $X = Y = \mathbb{R}$, $\Lambda = \Gamma = M = [0, 1]$, $C = \mathbb{R}_- \gamma_0 = 0$, H is an identity mapping and

$$T(x, \gamma) = \{1\},$$
$$K(x, \lambda) = [0, 1], \ \eta(x, y, \gamma) = \psi(x, y, \gamma) = \{\gamma^2 + 1\},$$
$$f(x, y, \lambda) = \{\frac{1}{2^{\gamma^2+\gamma+2}}\}.$$

We also that all the assumptions of Theorem 3.7 are satisfied. So, Σ is Hausdorff lower semicontinuous at $(0,0,0)$. In fact, $\Sigma(\gamma, \lambda, \mu) = [0,1]$ for all $\gamma \in [0,1]$.

Next, we establish condition $(H_h(\gamma_0, \lambda_0, \mu_0))$ is a sufficient and necessary condition for the Hausdorff lower semicontinuity, continuity and Hausdorff continuity of the solution set for parametric generalized vector mixed quasivariational inequality problem.

Motivated by the hypothesis (H_1) in [25, 37], (H_g) in [13, 30], (H'_g) in [38] and the assumption $(H_h(\gamma_0, \mu_0))$ in [19], by virtue of the parametric gap function h. Now, we introduce the following key assumption.

$(\boldsymbol{H_h(\gamma_0, \lambda_0, \mu_0)})$ Given $(\gamma_0, \lambda_0, \mu_0) \in \Gamma \times \Lambda \times M$. For any open neighborhood U of the origin in X, there exist $\alpha > 0$ and a neighborhood $V(\gamma_0, \lambda_0, \mu_0)$ of $(\gamma_0, \lambda_0, \mu_0)$ such that for all $(\gamma, \lambda, \mu) \in V(\gamma_0, \lambda_0, \mu_0)$ and $x \in E(\gamma) \backslash (\Sigma(\gamma, \lambda, \mu) + U)$, one has $h(x, \gamma, \lambda, \mu) \geq \alpha$.

Remark 3.10. Zhao [37], Li and Chen [30] remarked that the above hypothesis $(H_h(\gamma_0, \lambda_0, \mu_0))$ is characterized by a common theme used in mathematical analysis. Such a theme interprets a proposition associated with a set, in terms of other propositions associated with the complement set. Instead of imposing restrictions on the solution set, the hypothesis $(H_h(\gamma_0, \lambda_0, \mu_0))$ lays a condition on the behavior of the parametric gap function on the complement of the solution set.

Geometrically, the hypothesis $(H_h(\gamma_0, \lambda_0, \mu_0))$ means that, given a small open neighborhood U of the origin in X, we can find a small positive number $\alpha > 0$ and a neighborhood $V(\gamma_0, \lambda_0, \mu_0)$ of $(\gamma_0, \lambda_0, \mu_0)$, such that for all (γ, λ, μ) in the neighborhood of $(\gamma_0, \lambda_0, \mu_0)$, if a feasible point x is not in the set $\Sigma(\gamma, \lambda, \mu) + U$, then a "gap" by an amount of at least α will be yielded.

The following Lemma 3.11 is modified from Proposition 3.1 in Kien [25].

Lemma 3.11. *Suppose that all conditions in Lemma 2.5 are satisfied. For any open neighborhood U of the origin in X, let*

$$\Phi(\gamma, \lambda, \mu) := \inf_{x \in E(\gamma) \backslash (\Sigma(\gamma, \lambda, \mu) + U)} h(x, \gamma, \lambda, \mu).$$

Then $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds if and only if for any open neighborhood U of the origin in X one has

$$\liminf_{\gamma \to \gamma_0, \lambda \to \lambda_0, \mu \to \mu_0} \Phi(\gamma, \lambda, \mu) > 0.$$

Proof. If $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds, then for any open neighborhood U of the origin in X, there exist $\alpha > 0$ and a neighborhood $V(\gamma_0, \lambda_0, \mu_0)$ of $(\gamma_0, \lambda_0, \mu_0)$ such that for all $(\gamma, \lambda, \mu) \in V(\gamma_0, \lambda_0, \mu_0)$ and $x \in E(\gamma) \backslash (\Sigma(\gamma, \lambda, \mu) + U)$, one has $h(x, \gamma, \lambda, \mu) \geq \alpha$.

This implies that $\Phi(\gamma, \lambda, \mu) \geq \alpha$, for every $(\gamma, \lambda, \mu) \in V(\gamma_0, \lambda_0, \mu_0)$, hence

$$\liminf_{\gamma \to \gamma_0, \lambda \to \lambda_0, \mu \to \mu_0} \Phi(\gamma, \lambda, \mu) \geq \alpha > 0.$$

Conversely, for any open neighborhood U of the origin in X,

$$\omega = \liminf_{\gamma \to \gamma_0, \lambda \to \lambda_0, \mu \to \mu_0} \Phi(\gamma, \lambda, \mu) > 0$$

then there exists a neighborhood $V(\gamma_0, \lambda_0, \mu_0)$ of $(\gamma_0, \lambda_0, \mu_0)$ such that

$$\Phi(\gamma, \lambda, \mu) \geq \alpha > 0$$

for all $(\gamma, \lambda, \mu) \in V(\gamma_0, \lambda_0, \mu_0)$, where $\alpha := \frac{1}{2}\omega$. Hence, for any $x \in E(\gamma) \setminus (\Sigma(\gamma, \lambda, \mu) + U)$, we have

$$h(x, \gamma, \lambda, \mu) \geq \alpha > 0$$

which shows that $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds. $\square$

Remark 3.12 ([9, 12]). (i) Let a set $A \subset X$, A is said to be balanced if $\lambda A \subset A$ for every $\lambda \in R$, with $|\lambda| \leq 1$;

(ii) For each neighborhood U of the origin in X there exists a balanced open neighborhood B of the origin in X such that $B + B + B \subset U$.

Theorem 3.13. *Suppose that condition $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds and*

    (i) *E is lower semicontinuous with compact values in Γ;*
    (ii) *K is continuous with compact values in $X \times \Gamma$;*
    (iii) *T is continuous with compact values in $X \times M$.*

Then Σ is Hausdorff lower semicontinuous in $\Gamma \times \Lambda \times M$.

Proof. Suppose to the contrary that $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds but Σ is not Hausdorff lower semicontinuous at $(\gamma_0, \lambda_0, \mu_0)$. Then there exist a neighborhood U of the origin in X, a net $\{(\gamma_\alpha, \lambda_\alpha, \mu_\alpha)\} \subset \Gamma \times \Lambda \times M$ with $(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) \to (\gamma_0, \lambda_0, \mu_0)$ and a net $\{x_\alpha\}$ such that

$$(13) \qquad x_\alpha \in \Sigma(\gamma_0, \lambda_0, \mu_0) \setminus (\Sigma(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) + U).$$

By the compactness of Σ, we have can assume that $x_\alpha \to x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$. Since Remark 3.12, there exists a balanced open neighborhood B_0 of the origin in X such that $B_0 + B_0 + B_0 \subset U$. Hence, for any given $\varepsilon > 0$, $(x_0 + \varepsilon B_0) \cap E(\gamma_0) \neq \emptyset$. By E is lower semicontinuous at $\gamma_0 \in \Gamma$, there exists some k_1 such that $(x_0 + \varepsilon B_0) \cap E(\gamma_k) \neq \emptyset$ for all $k \geq k_1$.

For $\varepsilon \in (0, 1]$, suppose that $\xi_k \in (x_0 + \varepsilon B_0) \cap E(\gamma_k)$. We claim that $\xi_k \notin \Sigma(\gamma_k, \lambda_k, \mu_k) + B_0$. Otherwise, there exists $\delta_k \in \Sigma(\gamma_k, \lambda_k, \mu_k)$ such that $\xi_k - \delta_k \in B_0$. Without loss of generality, we may assume that $x_k - x_0 \in B_0$ whenever k is sufficiently large. Consequently, we get

$$x_k - \delta_k = (x_k - x_0) + (x_0 - \xi_k) + (\xi_k - \delta_k) \in B_0 + (-\varepsilon B_0) + B_0$$
$$\subset B_0 + B_0 + B_0 \subset U.$$

This implies that $x_k \in \Sigma(\gamma_k, \lambda_k, \mu_k) + U$, contrary to (13). Thus,

$$\xi_k \notin \Sigma(\gamma_k, \lambda_k, \mu_k) + B_0.$$

By the assumption $(H_h(\gamma_0, \lambda_0, \mu_0))$, there exists $\theta > 0$ such that $h(\xi_k, \gamma_k, \lambda_k, \mu_k) \geq \theta$. By Lemma 2.5, h is upper semicontinuous in $X \times \Gamma \times \Lambda \times M$. So, for any $\sigma > 0$ and for k sufficiently large, we have

$$h(\xi_k, \gamma_k, \lambda_k, \mu_k) - \sigma \leq h(x_0, \gamma_0, \lambda_0, \mu_0).$$

We can take σ such that $\theta - \sigma > 0$. Thus,

$$h(x_0, \gamma_0, \lambda_0, \mu_0) \geq h(\xi_k, \gamma_k, \lambda_k, \mu_k) - \sigma \geq \theta - \sigma > 0$$

and so

$$h(x_0, \gamma_0, \lambda_0, \mu_0) = \max_{z \in T(x_0, \mu_0)} \max_{y \in K(x_0, \gamma_0)} \xi_e(\langle H(z), \eta(y, x_0, \lambda_0)\rangle + \psi(y, x_0, \lambda_0))$$
$$> 0.$$

So, there exist $z \in T(x_0, \mu_0)$ and $y \in K(x_0, \gamma_0)$ such that

$$\xi_e(\langle H(z), \eta(y, x_0, \lambda_0)\rangle + \psi(y, x_0, \lambda_0)) > 0.$$

By Lemma 2.2(iii), we have

$$\langle H(z), \eta(y, x_0, \lambda_0)\rangle + \psi(y, x_0, \lambda_0) \notin -C,$$

which contradicts with $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$. Therefore, Σ is Hausdorff lower semicontinuous in $\Gamma \times \Lambda \times M$. $\qquad\square$

Theorem 3.14. *Suppose that*

    (i) *E is continuous with compact values in Γ;*
    (ii) *K is continuous with compact values in $X \times \Gamma$;*
    (iii) *T is continuous with compact values in $X \times M$.*

Then Σ is Hausdorff lower semicontinuous in $\Gamma \times \Lambda \times M$ if and only if $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds.

Proof. From Theorem 3.13, we only need to prove the necessity. Suppose to the contrary that Σ is Hausdorff lower semicontinuous in $\Gamma \times \Lambda \times M$ but $(H_h(\gamma_0, \lambda_0, \mu_0))$ does not hold. By Lemma 3.11, there exists a neighborhood U of the origin in X, such that

$$\liminf_{\gamma \to \gamma_0, \lambda \to \lambda_0, \mu \to \mu_0} \Phi(\gamma, \lambda, \mu) = 0.$$

Then there exists a net $\{(\gamma_\alpha, \lambda_\alpha, \mu_\alpha)\} \subset \Gamma \times \Lambda \times M$ with $(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) \to (\gamma_0, \lambda_0, \mu_0)$ such that

$$(14) \quad \lim_{\alpha \to \infty} \Phi(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) = \lim_{\alpha \to \infty} \inf_{x \in E(\gamma_\alpha)\backslash(\Sigma(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) + U)} h(x, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) = 0.$$

By $E(\gamma_\alpha)\backslash(\Sigma(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) + U)$ is a compact set and h is continuous from Lemma 2.5, there exists $x_\alpha \in E(\gamma_\alpha) \backslash (\Sigma(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) + U)$ satisfying $\Phi(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) = h(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha)$. Clearly, (14) implies

$$\lim_{\alpha \to \infty} h(x_\alpha, \gamma_\alpha, \lambda_\alpha, \mu_\alpha) = 0.$$

Since E is upper semicontinuous with compact values in Γ, we can assume that $x_\alpha \to x_0$ with $x_0 \in E(\gamma_0)$. By the continuity of h, we have $h(x_0, \gamma_0, \lambda_0, \mu_0) =$

0 and so $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$. For any $\delta \in \Sigma(\gamma_0, \lambda_0, \mu_0)$, since Σ is H-lsc in $\Gamma \times \Lambda \times M$, we can find a net $\{\delta_\alpha\} \subset \Sigma(\gamma_\alpha, \lambda_\alpha, \mu_\alpha)$ such that $\delta_\alpha \to \delta, \forall\alpha$. Since $x_\alpha \in E(\gamma_\alpha) \setminus (\Sigma(\gamma_\alpha, \lambda_\alpha, \mu_\alpha) + U)$, we know that $\delta_\alpha - x_\alpha \not\subset U$. Letting $\alpha \to \infty$ we have $\delta - x_0 \not\subset U, \forall\delta \in \Sigma(\gamma_0, \lambda_0, \mu_0)$. Since $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$, we have a contradicts. Thus, $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds. $\qquad\square$

The following example shows that $(H_h(\gamma_0, \lambda_0, \mu_0))$ in Theorem 3.14 is essential

Example 3.15. Let $\Lambda, \Gamma, M, \gamma_0, H$ as in Example 3.2 and $X = [-1, 1], Y = \mathbb{R}$, $C = \mathbb{R}_+$, $K(x, \gamma) = [-1, 1]$, $T(x, \mu) = \{1\}$, $\eta(y, x, \gamma) = (x - y)(\gamma + x^2)$, $\psi(y, x, \gamma) = 0$. Now we consider the problem (QVIP) of finding $x \in K(x, \gamma)$ and $z \in T(x, \mu)$ such that

$$\langle H(z), \eta(y, x, \gamma)\rangle + \psi(y, x, \gamma) = (\gamma + x^2)(x - y) \subseteq -\mathbb{R}_+.$$

It follows from a direct computation

$$\Sigma(\gamma, \lambda, \mu) = \begin{cases} \{-1, 0\} & \text{if} \quad \gamma = 0, \\ \{-1\} & \text{otherwise}. \end{cases}$$

Hence Σ is not H-lsc in $\Gamma \times \Lambda \times M$. Now we show that condition $(H_h(\gamma_0, \lambda_0, \mu_0))$ does not hold at $(0, 0, 0)$. We taking $e = 1 \in \mathrm{int}\mathbb{R}_+$, we have

$$\begin{aligned} h(x, \gamma, \lambda, \mu) &= \max_{z\in T(x,\mu)} \max_{y\in K(x,\gamma)} \xi_e(\langle H(z), \eta(y, x, \lambda)\rangle + \psi(y, x, \lambda)) \\ &= \max_{y\in K(x,\gamma)} ((\gamma + x^2)(x - y)) \\ &= (\gamma + x^2)(x + 1). \end{aligned}$$

We have h is a parametric gap function of (QVIP). For given $(\gamma_0, \lambda_0, \mu_0) \in \Gamma \times \Lambda \times M$, for any open neighborhood $U_\varepsilon(0) = (-\varepsilon, \varepsilon)$, choose ε such that $0 < \varepsilon < 1$. For any $\alpha > 0$ taking $(\gamma_\beta, \lambda_\beta, \mu_\beta) \to (0, 0, 0)$ with $0 < \gamma_\beta < \alpha$ and $x_\beta = 0 \in E(\gamma_\beta) \setminus (\Sigma(\gamma_\beta, \lambda_\beta, \mu_\beta) + U_\varepsilon(0))$. We have $h(x_\beta, \gamma_\beta, \lambda_\beta, \mu_\beta) = \gamma_\beta < \alpha$. Hence, $(H_h(\gamma_0, \lambda_0, \mu_0))$ does not hold at $(0, 0, 0)$.

The following example shows that all the assumptions of Theorem 3.14 are satisfied.

Example 3.16. Let $X, Y, \Lambda, \Gamma, M, \gamma_0, H$ as in Example 3.15 and let $C = \mathbb{R}_+$, $K(x, \gamma) = [0, \gamma]$, $T(x, \mu) = [0, 1]$, $\eta(y, x, \lambda) = x - y$, $\psi(y, x, \lambda) = 0$. Now we finding $x \in K(x, \gamma)$ and $z \in T(x, \mu)$ such that

$$\langle H(z), \eta(y, x, \lambda)\rangle + \psi(y, x, \lambda) = x - y \subseteq -\mathbb{R}_+.$$

It follows from a direct computation $\Sigma(\gamma, \lambda, \mu) = \{0\}$ for all $\gamma \in [0, 1]$. Hence, Σ is H-lsc in $\Gamma \times \Lambda \times M$. Now we check conditions of $(H_h(\gamma_0, \lambda_0, \mu_0))$, we taking $e = 1 \in \mathrm{int}\mathbb{R}_+$, then

$$h(x, \gamma, \lambda, \mu) = \max_{y\in K(x,\gamma)} \max[\langle T(x, \gamma), \eta(y, x, \gamma)\rangle + \psi(y, x, \gamma)]$$

$$= \max_{y \in K(x,\gamma)} ((x - y))$$

$$= \begin{cases} 0 & \text{if } x = 0 \\ \gamma & \text{if } x = \gamma. \end{cases}$$

We have h is a parametric gap function of (QVIP). For given $(\gamma_0, \lambda_0, \mu_0) \in \Gamma \times \Lambda \times M$, for any open neighborhood $U_\varepsilon(0) = (-\varepsilon, \varepsilon)$ and $0 < \varepsilon \leq \gamma$, taking $\alpha = \varepsilon$ and the neighborhood $V_\delta(\gamma_0, \lambda_0, \mu_0) = [\gamma_0 - \varepsilon, \gamma_0 + \varepsilon]$ with $0 < \delta < \min\{\gamma_0 - \varepsilon, 1 - \gamma_0\}$, we have

$$h(x, \gamma, \lambda, \mu) = \gamma \geq \alpha, \forall (\gamma, \lambda, \mu) \in V_\delta(\gamma_0, \lambda_0, \mu_0), \forall x \in E(\gamma) \backslash (\Sigma(\gamma, \lambda, \mu) + U_\varepsilon(0)).$$

Hence, the assumption $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds.

Theorem 3.17. *Suppose that the following conditions are satisfied:*
   (i) *E is continuous with compact values in Λ;*
   (ii) *K is continuous with compact values in $X \times \Gamma$;*
   (iii) *T is continuous with compact values in $X \times M$.*

Then Σ is both continuous and closed in $\Gamma \times \Lambda \times M$ if and only if $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds.

Proof. From Theorem 3.14 and Lemma 2.1(ii), we only need to prove that Σ is both upper semicontinuous and closed in $\Gamma \times \Lambda \times M$. First we prove that Σ is upper semicontinuous in $\Gamma \times \Lambda \times M$. Indeed, we suppose that Σ is not upper semicontinuous at $(\gamma_0, \lambda_0, \mu_0)$, i.e., there is an open subset U of $\Sigma(\gamma_0, \lambda_0, \mu_0)$ such that for all net $\{(\gamma_\alpha, \lambda_\alpha, \mu_\alpha)\}$ convergent to $(\gamma_0, \lambda_0, \mu_0)$, there is $x_\alpha \in \Sigma(\gamma_\alpha, \lambda_\alpha, \mu_\alpha)$, $x_\alpha \notin U$, $\forall \alpha$. By the upper semicontinuity of E in Γ and the compactness of $E(\gamma_0)$, one can assume that $x_\alpha \to x_0 \in E(\gamma_0)$ (taking a subnet if necessary). Now we show that $x_0 \in \Sigma^{\gamma\lambda\mu}(\gamma_0, \lambda_0, \mu_0)$. If $x_0 \notin \Sigma(\gamma_0, \lambda_0, \mu_0)$, then $\forall z_0 \in T(x_0, \mu_0)$, $\exists y_0 \in K(x_0, \gamma_0)$ such that

$$(15) \qquad \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0) \nsubseteq -C.$$

By the lower semicontinuity of K at (x_0, γ_0), there exists $y_\alpha \in K(x_\alpha, \gamma_\alpha)$ such that $y_\alpha \to y_0$. Since $x_\alpha \in \Sigma(\gamma_\alpha, \lambda_\alpha, \mu_\alpha)$, there exists $z_\alpha \in T(x_\alpha, \mu_\alpha)$ such that

$$(16) \qquad \langle H(z_\alpha), \eta(y_\alpha, x_\alpha, \lambda_\alpha)\rangle + \psi(y_\alpha, x_\alpha, \lambda_\alpha) \subseteq -C.$$

Since T is upper semicontinuous and with compact values in $X \times M$, one has $z_0 \in T(x_0, \mu_0)$ such that $z_\alpha \to z_0$ (can take a subnet if necessary) and since H, η are continuous. We have,

$$\langle H(z_\alpha), \eta(y_\alpha, x_\alpha, \lambda_\alpha)\rangle \to \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle.$$

It follows from the continuity of ψ that

$$\langle H(z_\alpha), \eta(y_\alpha, x_\alpha, \lambda_\alpha)\rangle + \psi(y_\alpha, x_\alpha, \lambda_\alpha) \to \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0)$$

and so

$$(17) \qquad \langle H(z_0), \eta(y_0, x_0, \lambda_0)\rangle + \psi(y_0, x_0, \lambda_0) \subseteq -C.$$

We see a contradiction between (15) and (17), and so we have $x_0 \in \Sigma(\gamma_0, \lambda_0, \mu_0)$ $\subseteq U$, this contradicts to the fact $x_\alpha \notin U$, $\forall \alpha$. Hence, Σ is upper semicontinuous in $\Gamma \times \Lambda \times M$.

Now we prove that Σ is closed in $\Gamma \times \Lambda \times M$. Indeed, we suppose that Σ is not closed at $(\gamma_0, \lambda_0, \mu_0)$, i.e., there is a net $\{(x_n, \gamma_n, \lambda_n, \mu_n)\} \to (x_0, \gamma_0, \lambda_0, \mu_0)$ with $x_n \in \Sigma(\gamma_n, \lambda_n, \mu_n)$ but $x_0 \notin \Sigma(\gamma_0, \lambda_0, \mu_0)$. The further argument is the same as above. And so we have Σ is closed in $\Gamma \times \Lambda \times M$. Hence, Σ is both upper semicontinuous and closed in $\Gamma \times \Lambda \times M$. $\qquad\square$

Corollary 3.18. *Suppose that the following conditions are satisfied*:
- (i) *E is continuous with compact values in Λ;*
- (ii) *K is continuous with compact values in $X \times \Gamma$;*
- (iii) *T is continuous with compact values in $X \times M$.*

Then Σ is both Hausdorff continuous and closed in $\Gamma \times \Lambda \times M$ if and only if $(H_h(\gamma_0, \lambda_0, \mu_0))$ holds.

Remark 3.19. From Remark 2.6 as above. Theorems 3.13-3.17 and Corollary 3.18 are different from some results in [13, 19, 30, 38]. Moreover, our the assumption $(H_h(\gamma_0, \lambda_0, \mu_0))$ is also different from the assumption (H_g) in [13, 30, 38].

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Center of Research and Development
Duy Tan University
K7/25, Quang Trung, Danang, Vietnam
and
Department of Mathematics
Dong Thap University
Dong Thap, Vietnam
E-mail address: ngvhungdhdt@yahoo.com