HYPERSTABILITY OF THE GENERAL LINEAR FUNCTIONAL EQUATION

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ABSTRACT. We give some results on hyperstability for the general linear equation. Namely, we show that a function satisfying the linear equation approximately (in some sense) must be actually the solution of it.

1. Introduction

Let $X$, $Y$ be normed spaces over fields $F$, $K$, respectively. A function $f: X \to Y$ is linear provided it satisfies the functional equation

\[ f(ax + by) = Af(x) + Bf(y), \quad x, y \in X, \]

where $a, b \in F \setminus \{0\}$, $A, B \in K$. We see that for $a = b = A = B = 1$ in (1) we get the Cauchy equation while the Jensen equation corresponds to $a = b = A = B = \frac{1}{2}$. The general linear equation has been studied by many authors, in particular the results of the stability can be found in [5], [6], [8], [9], [10], [13], [14].

We present some hyperstability results for the equation (1). Namely, we show that, for some natural particular forms of $\varphi$, the functional equation (1) is $\varphi$-hyperstable in the class of functions $f: X \to Y$, i.e., each $f: X \to Y$ satisfying the inequality

\[ \|f(ax + by) - Af(x) - Bf(y)\| \leq \varphi(x, y), \quad x, y \in X, \]

must be linear. In this way we expect to stimulate somewhat the further research of the issue of hyperstability, which seems to be a very promising subject to study within the theory of Hyers-Ulam stability.

The hyperstability results concerning the Cauchy equation can be found in [2], the general linear in [12] with $\varphi(x, y) = \|x\|^p + \|y\|^p$, where $p < 0$. The Jensen equation was studied in [1] and there were received some hyperstability results for $\varphi(x, y) = c\|x\|^p\|y\|^q$, where $c \geq 0$, $p, q \in \mathbb{R}$, $p + q \notin \{0, 1\}$.
The stability of the Cauchy equation involving a product of powers of norms was introduced by J. M. Rassias in [15], [16] and it is sometimes called Ulam-Găvrută-Rassias stability. For more information about Ulam-Găvrută-Rassias stability we refer to [7], [11], [17], [18], [19].

One of the method of the proof is based on a fixed point result that can be derived from [3] (Theorem 1). To present it we need the following three hypothesis:

(H1) $X$ is a nonempty set, $Y$ is a Banach space, $f_1, \ldots, f_k : X \to X$ and $L_1, \ldots, L_k : X \to \mathbb{R}_+$ are given.

(H2) $T : Y^X \to Y^X$ is an operator satisfying the inequality

$$\|T\xi(x) - T\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|, \quad \xi, \mu \in Y^X, \ x \in X.$$

(H3) $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \quad \delta \in \mathbb{R}_+^X, \ x \in X.$$

Now we are in a position to present the above mentioned fixed point theorem.

**Theorem 1.1.** Let hypotheses (H1)–(H3) be valid and functions $\varepsilon : X \to \mathbb{R}_+$ and $\varphi : X \to Y$ fulfill the following two conditions

$$\|T\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in X,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) < \infty, \quad x \in X.$$

Then there exists a unique fixed point $\psi$ of $T$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in X.$$

Moreover,

$$\psi(x) := \lim_{n \to \infty} T^n\varphi(x), \quad x \in X.$$

The next theorem shows that a linear function on $X \setminus \{0\}$ is linear on the whole $X$.

**Theorem 1.2.** Let $X,Y$ be normed spaces over $\mathbb{F}$, $\mathbb{K}$, respectively, $a,b \in \mathbb{F}\setminus\{0\}$, $A,B \in \mathbb{K}$. If a function $f : X \to Y$ satisfies

(2) \hspace{1cm} f(ax + by) = Af(x) + Bf(y), \quad x,y \in X \setminus \{0\},

then there exist an additive function $g : X \to Y$ satisfying conditions

(3) \hspace{1cm} g(bx) = Bg(x) \text{ and } g(ax) = Ag(x), \quad x \in X

and a vector $\beta \in Y$ with

(4) \hspace{1cm} \beta = (A + B)\beta
such that
\( f(x) = g(x) + \beta, \quad x \in X. \)

Conversely, if a function \( f: X \to Y \) has the form (5) with some \( \beta \in Y \), an
additive \( g: X \to Y \) such that (3) and (4) hold, then it satisfies the equation
(1) (for all \( x, y \in X \)).

Proof. Assume that \( f \) fulfills (2). Replacing \( x \) by \( bx \) and \( y \) by \( -ax \) in (2) we
get
\[ f(0) = Af(bx) + Bf(-ax), \quad x \in X \setminus \{0\}. \]

Next with \( x \) replaced by \( bx \) and \( y \) by \( ax \) in (2) we have
\[ f(2abx) = Af(bx) + Bf(ax), \quad x \in X \setminus \{0\}. \]

Let \( f = f_e + f_o \), where \( f_e, f_o \) denote the even and the odd part of \( f \), respectively.
It is obvious that \( f_e, f_o \) satisfy (2), (6) and (7).

First we show that \( f_o \) is additive. According to (6) and (7) for the odd part
of \( f \) we have
\[ Af_o(bx) = Bf_o(ax), \quad x \in X \setminus \{0\}. \]

Thus
\[ f_o(x) = 2Bf_o\left(\frac{x}{2a}\right) = 2Af_o\left(\frac{x}{2a}\right), \quad x \in X. \]

By (8) and (2)
\[ f_o(x) + f_o(y) = 2Af_o\left(\frac{x}{2a}\right) + 2Bf_o\left(\frac{y}{2b}\right) = 2f_o\left(\frac{a x}{2a} + b \frac{y}{2b}\right) \]
\[ = 2f_o\left(\frac{x+y}{2}\right), \quad x, y \in X \setminus \{0\}. \]

Fix \( z \in X \setminus \{0\} \) and write \( X_z := \{pz : p > 0\} \). Then \( X_z \) is a convex set, there
exist an additive map \( g_z: X_z \to Y \) and a constant \( \beta_z \in Y \) such that
\[ f_o(x) = g_z(x) + \beta_z, \quad x \in X_z. \]

We observe that
\[ g_z(pz) + \beta_z = f_o(pz) = f_o\left(\frac{3pz - pz}{2}\right) = \frac{f_o(3pz) - f_o(pz)}{2} \]
\[ = \frac{g_z(3pz) - g_z(pz)}{2} = g_z(pz), \quad p > 0, \]

which means that \( \beta_z = 0 \). Hence
\[ f_o\left(\frac{1}{2}z\right) = g_z\left(\frac{1}{2}z\right) = \frac{1}{2}g_z(z) = \frac{1}{2}f_o(z). \]

Therefore with (9) we obtain
\[ f_o\left(\frac{x+y}{2}\right) = \frac{f_o(x) + f_o(y)}{2}, \quad x, y \in X. \]
and as \( f_o(0) = 0 \), \( f_o \) is additive. Using additivity of \( f_o \) and (8) we obtain
\[
f_o(bx) = 2Bf_o\left(\frac{x}{2}\right) = Bf_o(x), \quad x \in X
\]
and
\[
f_o(ax) = 2Af_o\left(\frac{x}{2}\right) = Af_o(x), \quad x \in X,
\]
which means that (3) holds with \( g = f_o \). Using (6) and (7) for the even part of \( f \) we obtain
\[
f_e(0) = f_e(2abx), \quad x \in X \setminus \{0\},
\]
which means that \( f_e \) is a constant function and (4) holds with \( \beta := f_e(x) \).

For the proof of the converse, assume that a function \( f : X \to Y \) has the form (5) with some \( \beta \in Y \), an additive \( g : X \to Y \) such that (3) and (4) hold. Then for all \( x, y \in X \)
\[
f(ax + by) = g(ax + by) + \beta = g(ax) + g(by) + \beta
\]
\[
= Ag(x) + Bg(y) + (A + B)\beta
\]
\[
= Af(x) + Bf(y),
\]
which finishes the proof. \( \square \)

2. Hyperstability results

**Theorem 2.1.** Let \( X, Y \) be normed spaces over \( \mathbb{F}, \mathbb{K} \), respectively, \( a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K} \setminus \{0\}, c \geq 0, p, q \in \mathbb{R}, p + q < 0 \) and \( f : X \to Y \) satisfies
\[
\|f(ax + by) - Af(x) - Bf(y)\| \leq c\|x\|^{p}\|y\|^{q}, \quad x, y \in X \setminus \{0\}.
\]
Then \( f \) is linear.

**Proof.** First we notice that without loss of generality we can assume that \( Y \) is a Banach space, because otherwise we can replace it by its completion.

Since \( p + q < 0 \), one of \( p, q \) must be negative. Assume that \( q < 0 \). We observe that there exists \( m_0 \in \mathbb{N} \) such that
\[
\left|\frac{1}{2}\right| |a + bm|^{p+q} + \frac{|B|}{2}m^{p+q} < 1 \quad \text{for } m \geq m_0.
\]
Fix \( m \geq m_0 \) and replace \( y \) by \( mx \) in (10). Thus
\[
\|f(ax + bmx) - Af(x) - Bf(mx)\| \leq c\|x\|^{p}\|mx\|^{q}, \quad x \in X \setminus \{0\}
\]
and
\[
\left\|\frac{1}{A}f((a + bm)x) - \frac{B}{A}f(mx) - f(x)\right\| \leq \frac{c}{|A|}m^{q}\|x\|^{p+q}, \quad x \in X \setminus \{0\}.
\]
Write
\[
T\xi(x) := \frac{1}{A}\xi((a + bm)x) - \frac{B}{A}\xi(mx),
\]
\[
\varepsilon(x) := \frac{c}{|A|}m^{q}\|x\|^{p+q}, \quad x \in X \setminus \{0\},
\]
then (12) takes the form
\[ \|Tf(x) - f(x)\| \leq \varepsilon(x), \quad x \in X \setminus \{0\}. \]
Define
\[ \Lambda \varepsilon(x) := \frac{1}{A} |\eta((a + bm)x) + \frac{B}{A} |\eta(mx)|, \quad x \in X \setminus \{0\}. \]
Then it is easily seen that \( \Lambda \) has the form described in (H3) with \( k = 2 \) and \( f_1(x) = (a + bm)x, f_2(x) = mx, L_1(x) = \frac{1}{|A|}, L_2(x) = |\frac{B}{A}| \) for \( x \in X \setminus \{0\} \).
Moreover, for every \( \xi, \mu \in Y^{X \setminus \{0\}}, x \in X \setminus \{0\} \)
\[ \|T\xi(x) - T\mu(x)\| \]
\[ = \left\| \frac{1}{A} \xi((a + bm)x) - \frac{B}{A} \xi(mx) - \frac{1}{A} \mu((a + bm)x) + \frac{B}{A} \mu(mx) \right\| \]
\[ \leq \frac{1}{A} \|\xi - \mu\|((a + bm)x) + \frac{B}{A} \|\xi - \mu\| \]
\[ = \sum_{i=1}^{2} L_i(x)\|\xi - \mu\|(f_i(x)), \]
so (H2) is valid.
By (11) we have
\[ \varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) \]
\[ = \sum_{n=0}^{\infty} \frac{\varepsilon}{|A|} \frac{m^q}{n!} \left( \frac{1}{A} |a + bm|^{p+q} + \frac{B}{A} |m^{p+q}| \right)^n \|x\|^{p+q} \]
\[ = \frac{\varepsilon}{|A|} \frac{m^q}{n!} \|x\|^{p+q} \left( 1 - |\frac{1}{A}| |a + bm|^{p+q} - |\frac{B}{A}| m^{p+q} \right), \quad x \in X \setminus \{0\}. \]
Hence, according to Theorem 1.1 there exists a unique solution \( F: X \setminus \{0\} \rightarrow Y \) of the equation
\[ F(x) = \frac{1}{A} F((a + bm)x) - \frac{B}{A} F(mx), \quad x \in X \setminus \{0\} \]
such that
\[ \|f(x) - F(x)\| \leq \frac{\varepsilon}{|A|} \frac{m^q}{n!} \|x\|^{p+q} \left( 1 - |\frac{1}{A}| |a + bm|^{p+q} - |\frac{B}{A}| m^{p+q} \right), \quad x \in X \setminus \{0\}. \]
Moreover,
\[ F(x) := \lim_{n \rightarrow \infty} (T^n f)(x), \quad x \in X \setminus \{0\}. \]
We show that
\[ \|T^n f(ax + by) - AT^n f(x) - BT^n f(y)\| \]
\[ \leq c \left( \frac{1}{A} |a + bm|^{p+q} + \frac{B}{A} |m^{p+q}| \right)^n \|x\|^{p+q} \|y\|^{p+q}, \quad x, y \in X \setminus \{0\} \]
Theorem 2.2. Let
\[ (10) \quad \{ \]
We present the proof only when
\[ n \to \infty \]
Thus, by induction we have shown that (13) holds for every
\[ n \in \mathbb{N}_0. \]
If \( n = 0 \), then (13) is simply (10). So, take \( r \in \mathbb{N}_0 \) and suppose that (13) holds for \( n = r \). Then
\[
\begin{align*}
\|T^{r+1}f(ax + by) - AT^{r+1}f(x) - BT^{r+1}f(y)\| \\
&= \left\| \frac{1}{A}T^rf((a + bm)(ax + by)) - B \frac{1}{A}T^rf(m(ax + by)) \\
&\quad - B \frac{1}{A}T^rf((a + bm)x) + B \frac{1}{A}T^rf(mx) \\
&\quad - B \frac{1}{A}T^rf((a + bm)y) + B \frac{1}{A}T^rf(my) \right\| \\
&\leq c \left( \frac{1}{A} \|a + bm\|^{p+q} + \frac{B}{A} \|m\|^{p+q} \right)^{r+1} \frac{1}{A} \|a + bm\|\|x\|\|y\|^{q} \\
&\leq c \left( \frac{1}{A} \|a + bm\|^{p+q} + \frac{B}{A} \|m\|^{p+q} \right)^{r+1} \|x\|\|y\|^{q}, \quad x, y \in X \setminus \{0\}.
\end{align*}
\]
Thus, by induction we have shown that (13) holds for every \( n \in \mathbb{N}_0. \)

Letting \( n \to \infty \) in (13), we obtain that
\[ F(ax + by) = AF(x) + BF(y), \quad x, y \in X \setminus \{0\}. \]
In this way, with Theorem 1.2, for every \( m \geq m_0 \) there exists a function \( F \) satisfying the linear equation (1) such that
\[
\|f(x) - F(x)\| \leq \frac{c}{A}m^q\|x\|^{p+q}, \quad x \in X \setminus \{0\}.
\]
It follows, with \( m \to \infty \), that \( F \) is linear.

In similar way we can prove the following theorem.

**Theorem 2.2.** Let \( X, Y \) be normed spaces over \( F, K \), respectively, \( a, b \in F \setminus \{0\}, A, B \in K \setminus \{0\}, c \geq 0, p, q \in \mathbb{R}, p + q > 0 \) and \( f: X \to Y \) satisfies (10). If \( q > 0 \) and \( |a|^{p+q} \neq |A| \), or \( p > 0 \) and \( |b|^{p+q} \neq |B| \), then \( f \) is linear.

**Proof.** We present the proof only when \( q > 0 \) because the second case is similar.
Let \( q > 0 \) and \( \frac{|a|^{p+q}}{|A|} < 1 \). Replacing \( y \) by \( -\frac{a}{bm}x \), where \( m \in \mathbb{N} \), in (10) we get
\[
\|f\left(\left(a - \frac{a}{m}\right)x\right) - Af(x) - Bf\left(-\frac{a}{bm}x\right)\| \leq c\|x\|^p \left|\frac{a}{bm}\right|^q, \quad x \in X \setminus \{0\},
\]
thus
\[
\begin{align*}
\left\| \frac{1}{A}f\left(\left(a - \frac{a}{m}\right)x\right) - B f\left(-\frac{a}{bm}x\right) - f(x) \right\| \\
&\leq \frac{c}{|A|} \left|\frac{a}{bm}\right|^q \|x\|^{p+q}, \quad x \in X \setminus \{0\}.
\end{align*}
\]
For \( x \in X \setminus \{0\} \) we define
\[
T_m \xi(x) := \frac{1}{A} \xi\left(\left(a - \frac{a}{m}\right)x\right) - B \frac{1}{A} \xi\left(-\frac{a}{bm}x\right),
\]
\begin{equation}
\varepsilon_m(x) := \frac{c}{|A|} \left| \frac{a}{bm} \right| \|x\|^{p+q},
\end{equation}

\begin{equation}
\Lambda_m \eta(x) := \left| \frac{1}{A} \right| \eta \left( a - \frac{a}{m} \right) x + \left| \frac{B}{A} \right| \eta \left( -\frac{a}{bm} x \right),
\end{equation}

and as in Theorem 2.1 we observe that (14) takes the form

\[ \|T_m f(x) - f(x)\| \leq \varepsilon_m(x), \quad x \in X \setminus \{0\} \]

and \( \Lambda_m \) has the form described in (H3) with \( k = 2 \) and \( f_1(x) = (a - \frac{a}{m}) x, \)

\[ f_2(x) = -\frac{a}{bm} x, \quad L_1(x) = \frac{a}{m}, \quad L_2(x) = \frac{A}{m} \]

for \( x \in X \setminus \{0\} \). Moreover, for every \( \xi, \mu \in Y \setminus \{0\}, \ x \in X \setminus \{0\} \)

\[ \|T_m \xi(x) - T_m \mu(x)\| \leq 2 \sum_{i=1}^{n} L_i(x) \| (\xi - \mu)(f_i(x))\|, \]

so (H2) is valid.

Next we can find \( m_0 \in \mathbb{N}, \) such that

\[ \frac{|a|^{p+q}}{|A|} 1 - \frac{1}{m} |x|^{p+q} + \frac{|B|}{|A|} \left| \frac{a}{m} \right|^{p+q} \left( \frac{1}{m} \right)^{p+q} < 1 \quad \text{for } m \in \mathbb{N}_{m_0}. \]

Therefore

\begin{align*}
\varepsilon_m^n(x) & := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
& = \frac{c}{|A|} \left| \frac{a}{bm} \right| \|x\|^{p+q} \sum_{n=0}^{\infty} \left( \frac{|a|^{p+q}}{|A|} 1 - \frac{1}{m} |x|^{p+q} + \frac{|B|}{|A|} \left| \frac{a}{m} \right|^{p+q} \left( \frac{1}{m} \right)^{p+q} \right)^n \\
& = \frac{c}{\frac{|a|^{p+q}}{|A|} 1 - \frac{1}{m} |x|^{p+q} - \frac{|B|}{|A|} \left| \frac{a}{m} \right|^{p+q} \left( \frac{1}{m} \right)^{p+q}}, \quad m \in \mathbb{N}_{m_0}, \ x \in X \setminus \{0\}. 
\end{align*}

Hence, according to Theorem 1.1, for each \( m \in \mathbb{N}_{m_0} \) there exists a unique solution \( F_m : X \setminus \{0\} \rightarrow Y \) of the equation

\[ F_m(x) = \frac{1}{A} F_m \left( a - \frac{a}{m} \right) x - \left( \frac{1}{A} \right) F_m \left( -\frac{a}{bm} x \right), \quad x \in X \setminus \{0\} \]

such that

\[ \|f(x) - F_m(x)\| \leq \frac{c}{\frac{|a|^{p+q}}{|A|} 1 - \frac{1}{m} |x|^{p+q} - \frac{|B|}{|A|} \left| \frac{a}{m} \right|^{p+q} \left( \frac{1}{m} \right)^{p+q}}, \quad x \in X \setminus \{0\}. \]

Moreover,

\[ F_m(ax + by) = AF_m(x) + BF_m(y), \quad x, y \in X \setminus \{0\}. \]

In this way we obtain a sequence \( \{F_m\}_{m \in \mathbb{N}_{m_0}} \) of linear functions such that

\[ \|f(x) - F_m(x)\| \leq \frac{c}{\frac{|a|^{p+q}}{|A|} 1 - \frac{1}{m} |x|^{p+q} - \frac{|B|}{|A|} \left| \frac{a}{m} \right|^{p+q} \left( \frac{1}{m} \right)^{p+q}}, \quad x \in X \setminus \{0\}. \]
So, with $m \to \infty$, $f$ is linear on $X \setminus \{0\}$ and by Theorem 1.2 $f$ is linear.

Let $q > 0$ and $\frac{|A|}{|a|^{p+q}} < 1$. Replacing $x$ by $\left(\frac{1}{a} - \frac{1}{am}\right)x$ and $y$ by $\frac{1}{bm}x$, where $m \in \mathbb{N}$, in (10) we get

$$\|f\left(\frac{1}{a} - \frac{1}{am}\right)x + \frac{1}{bm}x\) - A\left(\frac{1}{a} - \frac{1}{am}\right)x - B\left(\frac{1}{bm}x\right)\| \leq c\left(\frac{1}{a} - \frac{1}{am}\right)x\|\|\frac{1}{bm}x\|\|\right|, \quad x \in X \setminus \{0\}.\]

Whence

$$\|f(x) - A\left(\frac{1}{a} - \frac{1}{am}\right)x - B\left(\frac{1}{bm}x\right)\| \leq c\left(\frac{1}{a} - \frac{1}{am}\right)x\|\|\frac{1}{bm}x\|\|\right|, \quad x \in X \setminus \{0\}.\]

For $x \in X \setminus \{0\}$ we define

$$\mathcal{T}_m \xi(x) := A\xi\left(\frac{1}{a} - \frac{1}{am}\right)x + B\xi\left(\frac{1}{bm}x\right),$$

$$\varepsilon_m(x) := c\left(\frac{1}{a}\right)\|\frac{1}{|b|^q}\|1 - \frac{1}{m}\|\|\frac{1}{|m|^q}\|\|x\|\|p+q, \quad \Lambda_m \eta(x) := |A|\eta\left(\frac{1}{a} - \frac{1}{am}\right)x + |B|\eta\left(\frac{1}{bm}x\right),$$

and as in Theorem 2.1 we observe that (14) takes form

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \quad x \in X \setminus \{0\}$$

and $\Lambda_m$ has the form described in (H3) with $k = 2$ and $f_1(x) = \left(\frac{1}{a} - \frac{1}{am}\right)x$, $f_2(x) = \frac{1}{bam}x$, $L_1(x) = |A|$, $L_2(x) = |B|$ for $x \in X \setminus \{0\}$. Moreover, for every $\xi, \mu \in YX\setminus \{0\}, x \in X \setminus \{0\}$

$$\|\mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x)\| \leq \sum_{i=1}^{2} L_i(x)\|\xi - \mu\|f_i\|,$$

so (H2) is valid.

Next we can find $m_0 \in \mathbb{N}$, such that

$$\frac{|A|}{|a|^{p+q}}\left|1 - \frac{1}{m}\right|\|p+q + \frac{|B|}{|b|^p+q}\left|1 - \frac{1}{m}\right|\|p+q < 1 \quad \text{for} \quad m \in \mathbb{N}_{m_0}.\]

Therefore

$$\varepsilon_m(x) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x)$$

$$= \varepsilon_m(x) \sum_{n=0}^{\infty} \left(\frac{|A|}{|a|^{p+q}}\left|1 - \frac{1}{m}\right| + \frac{|B|}{|b|^p+q}\left|1 - \frac{1}{m}\right|\right)^n$$
In this way we obtain a sequence \( (F_m) \) \( m \in \mathbb{N}_m \), \( x \in X \setminus \{0\} \). By induction it is easy to get
\[
\|F_m\| \leq m \in \mathbb{N}_m, \, x \in X \setminus \{0\}.
\]

Hence, according to Theorem 1.1, for each \( m \in \mathbb{N}_m \) there exists a unique solution \( F_m : X \setminus \{0\} \to Y \) of the equation
\[
F_m(x) = AF_m(\frac{1}{a} - \frac{1}{am})x + BF_m(\frac{1}{bm})x, \quad x \in X \setminus \{0\}
\]
such that
\[
\|f(x) - F_m(x)\| \leq \frac{c}{1 - \frac{|a|}{|p| + q}} |1 - \frac{1}{m} - \frac{|b|}{|p| + q} | \|x\|^{p+q}, \quad x \in X \setminus \{0\}. \tag{15}
\]
Moreover,
\[
F_m(ax + by) = AF_m(x) + BF_m(y), \quad x, y \in X.
\]
In this way we obtain a sequence \( (F_m)_{m \in \mathbb{N}_m} \) of linear functions such that (15) holds. It follows, with \( m \to \infty \), that \( f \) is linear. \( \square \)

**Theorem 2.3.** Let \( X, Y \) be normed spaces over \( \mathbb{F}, K \), respectively, \( a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K} \setminus \{0\}, c \geq 0, p, q > 0 \), and \( f : X \to Y \) satisfies
\[
\|f(ax + by) - Af(x) - Bf(y)\| \leq c\|x\|^p\|y\|^q, \quad x, y \in X. \tag{16}
\]
If \( |a|^{p+q} \neq |A| \) or \( |b|^{p+q} \neq |B| \), then \( f \) is linear.

**Proof.** Of course this theorem follows from Theorem 2.2 but as \( p, q \) are positive we can set \( 0 \) in (16) and get an auxiliary equalities. In this way we obtain another proof which we present in the first case.

Assume that \( |a|^{p+q} < |A| \). Setting \( x = y = 0 \) in (16) we get
\[
f(0)(1 - A - B) = 0. \tag{17}
\]
With \( y = 0 \) in (16) we have
\[
f(ax) = Af(x) + bf(0), \quad x \in X
\]
thus
\[
f(x) = Af(\frac{x}{a}) + Bf(0), \quad x \in X.
\]
Using the last equality, (16) and (17) we get
\[
\left\|Af(\frac{ax + by}{a}) - AAF(\frac{x}{a}) - BAF(\frac{x}{a})\right\| \leq c\|x\|^p\|y\|^q, \quad x, y \in X.
\]
Replacing \( x \) by \( ax \), \( y \) by \( ay \) and dividing the last inequality by \( |A| \) we obtain
\[
\|f(ax + by) - Af(x) - Bf(y)\| \leq \frac{|a|^{p+q}}{|A|} \|x\|^p\|y\|^q, \quad x, y \in X.
\]
By induction it is easy to get
\[
\|f(ax + by) - Af(x) - Bf(y)\| \leq c\left(\frac{|a|^{p+q}}{|A|}\right)^n \|x\|^p\|y\|^q, \quad x, y \in X.
\]
Whence, with \( n \to \infty \), \( f(ax + by) = Af(x) + Bf(y) \) for \( x, y \in X \).
In the case $|A| < |a|^{p+q}$, we use the equation $f(x) = \frac{1}{a} f(ax) - \frac{b}{a} f(0)$ together with (16) and (17).

The following examples show that the assumption in the above theorems are essential.

**Example 2.4.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. Then $f$ satisfies

$$|f(x + y) - f(x) - f(y)| \leq 2|x||y|, \quad x, y \in \mathbb{R},$$

but $f$ does not satisfy the Cauchy equation.

**Example 2.5.** More generally a quadratic function $f(x) = x^2$, $x \in \mathbb{R}$ satisfies

$$|f(ax + by) - Af(x) - Bf(y)| \leq 2|ab||x||y|, \quad x, y \in \mathbb{R},$$

where $A = a^2$, $B = b^2$, but $f$ does not satisfy the linear equation (1).

**Example 2.6.** A function $f(x) = |x|$, $x \in \mathbb{R}$ satisfies

$$|f(x + y) - f(x) - f(y)| \leq c, \quad x, y \in \mathbb{R},$$

but it is not linear.

To the end we show simple application of the above theorems.

**Corollary 2.7.** Let $X, Y$ be normed spaces over $\mathbb{F}$, $\mathbb{K}$, respectively, $a, b \in \mathbb{F} \setminus \{0\}, A, B \in \mathbb{K} \setminus \{0\}, c \geq 0, p, q \in \mathbb{R}$, $H : X^2 \to Y$, $H(w, z) \neq 0$ for some $z, w \in X$ and

$$\|H(x, y)\| \leq c \|x\|^p \|y\|^q, \quad x, y \in X \setminus \{0\},$$

where $c \geq 0$, $p, q \in \mathbb{R}$. If one of the following conditions

1. $p + q < 0$,
2. $q > 0$ and $|a|^{p+q} \neq |A|$,
3. $p > 0$ and $|b|^{p+q} \neq |B|$,

holds, then the functional equation

$$h(ax + by) = Ah(x) + Bh(y) + H(x, y), \quad x, y \in X$$

has no solutions in the class of functions $h : X \to Y$.

**Proof.** Suppose that $h : X \to Y$ is a solution to (19). Then (10) holds, and consequently, according to the above theorems, $h$ is linear, which means that $H(w, z) = 0$. This is a contradiction. \qed
Example 2.8. The functions $f: \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$ and $H: \mathbb{R}^2 \to \mathbb{R}$ given by $H(x,y) = 2xy$ satisfy the equation
\[ f(x + y) = f(x) + f(y) + H(x,y), \quad x, y \in \mathbb{R} \]
and do not fulfill any condition (1)–(3) of Corollary 2.7.

Remark 2.9. We notice that our results correspond with the new results from hyperstability, for example in [4] was proved that linear equation is $\varphi$-hyper-stable with $\varphi(x,y) = c \|x\|^p \|y\|^q$, but there was considered only the case when $c, p, q \in [0, +\infty)$ (see Theorem 20).

References

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